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BAUMSLAG–SOLITAR GROUP C^*-ALGEBRAS FROM INTERVAL MAPS

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Abstract. We yield operators U and V on Hilbert spaces that are parameterized by the orbits of certain interval maps that exhibit chaotic behavior and obey the (deformed) Baumslag–Solitar relation

$$UV = e^{2\pi i\alpha}VU^n, \qquad \alpha \in \mathbb{R},\ n \in \mathbb{N}.$$

We then prove that the scalar $e^{2\pi i\alpha}$ can be removed whilst retaining the isomorphism class of the C^*-algebra generated by U and V. Finally, we simultaneously unitarize U and V by gluing pairs of orbits of the underlying noninvertible dynamical system and investigate these unitary representations under distinct pairs of orbits.

1. Introduction and preliminaries

In [6, 7, 8, 10] we use symbolic dynamics and yield representations of Cuntz, Cuntz–Krieger, subshift C^*-algebras determined by orbits of nonlinear systems – in particular iterated maps of the interval, and Markov systems. These representations has allowed us to get a clearer relationship between the structure of these algebras and the underlying nonlinear dynamics. The studied systems are non-invertible and the symbolic dynamics is based on one-sided sequences. We obtained operators that are partial isometries, generating the referred algebras. In the present paper, we will be able to obtain unitary operators (leading to

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representations of group C^*-algebras of amenable groups) by inducing invertible nonlinear systems.

The underlying groups we consider are the Baumslag–Solitar groups [1]:

$$\mathrm{BS}(1,n) := \langle u, v | uv = vu^n \rangle. \tag{1.1}$$

There is a rich structure relating the representation theory of the Baumslag–Solitar groups and wavelet representations [4, 5] which goes back to the classical translation operator $Tf(x) = f(x-1)$ and the dilation operator $Uf(x) = \frac{1}{\sqrt{n}} f(\frac{x}{n})$, with $f \in L^2(\mathbb{R})$ and $x \in \mathbb{R}$ satisfying the Baumslag–Solitar relation $UTU^{-1} = T^n$ (with $n \in \mathbb{N}$ fixed).

Using symbolic dynamics tools, we constructed in [7] operators U and V acting on a Hilbert space H_x (related to the orbit of a point x) and satisfying the relation

$$UV = e^{2\pi i \alpha} VU^n \tag{1.2}$$

for $\alpha \in \mathbb{R}$ and $n \in \mathbb{N}$. So (1.2) can be thought of as a deformation of the relation

$$UV = VU^n\,, \tag{1.3}$$

which encodes a unitary representation of the Baumslag–Solitar group $\mathrm{BS}(1,n)$, provided U and V in (1.3) are unitary operators. One natural question to ask is how the C^*-algebra generated by U and V satisfying (1.2) is sensible to the change of the parameter. Note that an isomorphism between the C^*-algebra generated by U and V obeying (1.2) and the C^*-algebra generated by U and V obeying (1.3) can only possibly hold if $n > 1$ since when $n = 1$ the C^*-algebra generated by (1.2) is the rotation algebra [14], while the one attached to (1.3) is commutative (if $n = 1$).

In this paper we show that one can indeed remove the parameter $e^{2\pi i\alpha}$ in the relation (1.2) provided $n \neq 1$, i.e. given U and V operators acting on a Hilbert space H satisfying the relation (1.2) then we can find new operators U' and V' acting on the same Hilbert space H satisfying the relation (1.3) with the C^*-algebras $C^*(U,V)$ and $C^*(U',V')$ being isomorphic (see Lemma 3.3).

In the setup of [7], the operators U and V satisfying (1.2) act on a Hilbert space H_x that depends on the orbit of x under the interval map

$$f(x) = nx + \alpha \pmod 1.$$

Besides, the parameter is given by $e^{2\pi i\alpha}$, the integer number in the relation (1.2) is precisely the slope n of f, the operator U is clearly unitary while V fails to be unitary (as the underlying dynamical system is noninvertible). Lemma 3.3 implies in particular that it is enough to consider the representations obtained from (1.2) with $\alpha = 0$ and $n \in \mathbb{N}$.

In this manner we prove that it is possible to unitarize these operators (meaning that we obtain unitary operators $\mathbf{U}$ and $\mathbf{V}$ obeying the relation (1.3)).

This unitarization is achieved by some sort of gluing the orbits (tensor product of Hilbert spaces) of the underlying noninvertible dynamical system, implicitly inducing an invertible 2-dimensional dynamical system. Namely we slightly modify the construction of U and V given in [7], and construct new operators $\mathbf{U}$ and

$\mathbf{V}$, see Eq. (3.3), acting on the Hilbert space tensor product $H_x \otimes H_y$ and prove that $\mathbf{U}$ and $\mathbf{V}$ are both unitary operators and satisfy the same relation of the original ones U and V. We finally study these new representations in terms of the orbits of x and y by studying the spectrum of $\mathbf{U}$ and $\mathbf{V}$ (see Theorem 3.6). In particular, they are shown to be *-representations of the group C^*-algebra $C^*(\mathrm{BS}(1,n))$ as in [9] since $\mathrm{BS}(1,n)$ is an amenable group.

We review in Section 2 some necessary material from the Baumslag-Solitar groups, operator algebras and symbolic dynamics. In Section 3, we prove the main results as already described above.

2. The Baumslag-Solitar group and its group C^*-algebra

The group $\mathrm{BS}(1,n)$ defined in (1.1) is amenable because it can be written as a crossed product $\mathrm{BS}(1,n) \cong G \rtimes \mathbb{Z}$ by an abelian group G (thus amenable). Indeed as shown in [5], G is the group of n-adic numbers $\mathbb{Z}[\frac{1}{n}] := \bigcup_{k\geq 0} n^{-k}\mathbb{Z}$, and

$$\alpha_i(\frac{k}{n^p}) = n^i \frac{k}{n^p}, \qquad i \in \mathbb{Z}, \frac{k}{n^p} \in \mathbb{Z}[\frac{1}{n}]$$

defines an action of $\mathbb{Z}$ on G and the crossed product structure is given by

$$(i,b)(j,c) = (i+j, \alpha_i(c)+b), \qquad (j,k \in \mathbb{Z}, b,c \in G). \tag{2.1}$$

Note that $\mathbb{Z}[\frac{1}{n}]$ is a discrete abelian group and thus it is amenable. Also the elements $\frac{k}{n^p}$ in $\mathbb{Z}[\frac{1}{n}]$ correspond to $v^{-p}u^k v^p$ and the elements i in $\mathbb{Z}$ correspond to v^{-i} in $\mathrm{BS}(1,n)$. Set $u_{k/n^p} := v^{-p}u^k v^p$. In this way the multiplication rule in (2.1) maybe written as follows:

$$(v^i u_d)(v^{i'} u_{d'}) = v^{i+i'} u_{n^{-i'}d+d'}.$$

We note that $\mathrm{BS}(1,n) \cong \mathrm{BS}(1,n')$ if and only if $n = n'$, see [11]. A map $\pi : \mathrm{BS}(1,n) \to B(H)$ is a unitary representation of the group $\mathrm{BS}(1,n)$ on a Hilbert space H (where $B(H)$ denotes the algebra of bounded linear operators on H) if π i s a group homomorphism such that $\pi(g^{-1}) = \pi(g)^*$.

Remark 2.1. One non-trivial representation of the Baumslag–Solitar group $B(1,n)$ in 2×2 matrices is given by

$$u \to \begin{pmatrix} \frac{1}{n} & 0 \\ 0 & 1 \end{pmatrix}, \qquad v \to \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

2.1. **Operator algebras input.** A representation of a $*$-algebra $\mathcal{A}$ on a complex Hilbert space H is a $*$-homomorphism $\pi : \mathcal{A} \to B(H)$ into the $*$-algebra $B(H)$ of bounded linear operators on H. Usually representations are studied up to unitary equivalence. Two representations $\pi : \mathcal{A} \to B(H)$ and $\rho : \mathcal{A} \to B(K)$ are (unitarily) equivalent if there is a unitary operator $W : H \to K$ (i.e., W is a surjective isometry) such that

$$W\pi(a) = \rho(a)W, \text{ for every } a \in \mathcal{A}.$$

A representation $\pi : \mathcal{A} \to B(H)$ of some $*$-algebra is said to be *irreducible* if there is no non-trivial subspace of H invariant with respect to all operators $\pi(a)$

with $a \in \mathcal{A}$. A well known result, see e.g. [13, Proposition 3.13.2], says that π is irreducible if and only if

$$x \in B(H) : x\pi(a) = \pi(a)x, \text{ for all } a \in \mathcal{A} \implies x = \lambda \mathbf{1}, \tag{2.2}$$

for some complex number λ, where $\mathbf{1}$ denotes the identity of $B(H)$. By the very definition of comutant, (2.2) can be restated as follows: $\pi(\mathcal{A})' = \mathbb{C}\mathbf{1}$. Equivalently, π is an irreducible representation if $\overline{\pi(A)\xi} = H$ for all non-zero vector $\xi \in H$, where $\pi(A)\xi$ is the span of $\{\pi(a)\xi : a \in \mathcal{A}\}$. The representation is called *faithful* if it is injective. We will be interested in some classes of C*-algebras (Banach *-algebras such that $||aa^*|| = ||a||^2$ holds for all a, see e.g. [13]). Besides, if we have a representation $\pi : \mathcal{A} \to B(H)$ of a C*-algebra A, then π being a *-homomorphism implies that $||\pi(a)|| \leq ||a||$ for all $a \in \mathcal{A}$ an thus in particular π is automatically continuous, see also e.g. [13, Section 1.5.7].

For a discrete group G, the full group C*-algebra C*(G) of G is the C*-enveloping algebra of $l^1(G)$, i.e. the completion of $l^1(G)$ with respect to the largest C^*-norm:

$$||f||_{C^*(G)} := \sup_{\pi} \|\pi(f)\|,$$

where π ranges over all non-degenerate $*$-representations of $l^1(G)$ on Hilbert spaces. The reduced C^*-algebra ${C^*}_{\text{red}}(G)$ is the C*-algebra generated by the (image of) left regular representation $\lambda : G \to B(l^2(G))$ so that $(\lambda_g(f))(h) = f(g^{-1}h)$. The left regular representation gives rise to a *natural* C*-morphism $C^*(G) \to C^*_{\text{red}}(G)$ which is an isomorphism if and only if G is amenable [9]. Note that

$$\Big\{ \sum_{g \in \mathcal{F}} c_g \lambda(g) : c_g \in \mathbb{C},\ \mathcal{F} \text{ finite subset of } G \Big\}$$

is a dense *-subalgebra of $C^*(G)$, see e.g. [9]. In general, for $f \in l^1(G)$, we have:

$$||f||_{C^*_{\text{red}}(G)} \leq ||f||_{C^*(G)} \leq ||f||_{l^1(G)}.$$

We remark that the unit representation provides a morphism $C^*(G) \to \mathbb{C}$, therefore $C^*(G)$ is never a simple C*-algebra for any (non-trivial) group G. Let $\pi : G \to B(H)$ be a unitary representation of a discrete group G on a Hilbert space H. Then we can uniquely extend π to a C^*-representation (i.e. *-homomorphism) $\tilde{\pi}$ of the C^*-algebra $C^*(G)$ on the same Hilbert space as in [9]. Of course we may restrict a given C^*-representation of $C^*(G)$ to the group G. The irreducibility is preserved [9]:

$$\pi \text{ irred. unitary representation of } G \iff \tilde{\pi} \text{ irred. representation of } C^*(G).$$

3. Hilbert spaces from Interval maps

Let $f : I \to I$ be a piecewise monotone map of the interval $I = [0,1]$ into itself, that is, there is a minimal partition of open sub-intervals of I, $\mathcal{I} = \{I_1, \cdots, I_m\}$ such that $\overline{\cup_{j=1}^m I_j} = I$ and $f_{|I_j}$ is continuous monotone, for every $j = 1, \cdots, m$, see [12]. We define $f_j := f_{|I_j}$. The inverse branches are denoted by $f_j^{-1} : f(I_j) \to$

I_j. Let χ_{I_i} be the characteristic function on the interval I_i. The following are naturally satisfied

$$f \circ f_i^{-1}(x) = x,\ x \in f(I_i), \quad \text{and} \quad f_i^{-1} \circ f_{|I_i}(x) = x,\ x \in I_i$$

Let $\{1, \cdots, n\}$ be the alphabet associated to some partition $\mathcal{P} = \{I_1, \cdots, I_n\}$ of open sub-intervals of I so that $\overline{\cup_{j=1}^n I_j} = I$, not necessarily $\mathcal{I}$. The *address map,* is defined by

$$ad : \bigcup_{j=1}^n I_j \to \{1, \cdots, n\}, \qquad ad(x) = i \text{ if } x \in I_i.$$

We define

$$\Omega_f := \{x \in I : f^k(x) \in \cup_{j=1}^m I_j \text{ for all } k = 0, 1, \cdots\}.$$

Note that $\overline{\Omega}_f = I$. The *itinerary map* $it : \Omega_f \to \{1, \cdots, n\}^{\mathbb{N}}$ is defined by

$$it(x) = ad(x)ad(f(x))ad(f^2(x)) \cdots$$

and let $\Sigma_f = it(\Omega_f)$. The space Σ_f is invariant under the *shift map*

$$\sigma : \{1, \cdots, n\}^{\mathbb{N}} \to \{1, \cdots, n\}^{\mathbb{N}} \quad \text{defined by} \quad \sigma(i_1 i_2 \cdots) = (i_2 i_3 \cdots),$$

and we have $it \circ f = \sigma \circ it$. We will use σ meaning in fact $\sigma_{|\Sigma_f}$. A sequence in $\{1, \cdots, n\}^{\mathbb{N}}$ is called *admissible*, with respect to f, if it occurs as an itinerary for some point x in I, that is, if it belongs to Σ_f. An *admissible word* is a finite sub-sequence of some admissible sequence. The set of admissible words of size k is denoted by $W_k = W_k(f)$. Given $i_1 \cdots i_k \in W_k$, we define $I_{i_1 \cdots i_k}$ as the set of points x in Ω_f which satisfy

$$ad(x) = i_1, \cdots, ad(f^k(x)) = i_k.$$

As in [6], we consider the following equivalence relation on the set Ω_f,

$$R_f = \{(x, y) :\ f^n(x) = f^m(y) \text{ for some } n, m \in \mathbb{N}_0\}.$$

We write $x \sim y$ whenever $(x, y) \in R_f$. Consider the equivalence class $R_f(x)$ and set H_x the Hilbert space

$$H_x := l^2(R_f(x))$$

with canonical orthonormal basis $\{|y\rangle : y \in R_f(x)\}$, in the Dirac notation. Note that $H_x = H_y$ (are the same Hilbert spaces) whenever $x \sim y$. The inner product $(\cdot, \cdot)$ is given by

$$\langle y|z\rangle = (|y\rangle, |z\rangle) = \delta_{y,z}, \quad \text{with } y, z \in R_f(x).$$

For each $i = 1, \cdots, n$, let us define an operator S_i on H_x, with respect to some partition $\mathcal{P} = \{I_1, \cdots, I_n\}$ of I, as follows:

$$S_i |y\rangle = \chi_{f(I_i)}(y) \left| f_i^{-1}(y) \right\rangle.$$

Note that $\chi_{f(I_i)}(x) = 1$ if and only if there is a pre-image of x in I_i. We have $S_i^* |y\rangle = \chi_{I_i}(y) |f(y)\rangle$. In fact

$$\langle y| S_i |z\rangle\rangle = \left\langle y|f_i^{-1}(z)\right\rangle = \delta_{y, f_i^{-1}(z)}.$$

On the other hand we have

$$\langle y|\, S_i^* \,|z\rangle\rangle = \chi_{I_i}(y)\, \langle f(y)|z\rangle = \chi_{I_i}(y)\delta_{f(y),z}.$$

Since $\delta_{y,f_i^{-1}(z)} = \chi_{I_i}(y)\delta_{f(y),z}$ we have shown that the operators S_i, S_i^* are adjoint of each other. We further remark that S_i is a partial isometry: namely, S_i is an isometry on its restriction to $\mathrm{span}\{|y\rangle : y \in f(I_i)\} \cap H_x$ and vanishes in the remaining part of H_x.

For $\mu = \mu_1 \cdots \mu_k \in W_k$ we define $S_\mu = S_{\mu_1} \cdots S_{\mu_k}$. Also $S_\mu^* = S_{\mu_k}^* \cdots S_{\mu_1}^*$. Thus $S_\mu S_\mu^* \,|y\rangle = \chi_{I_\mu}(y)$.

3.1. **Linear mod 1 interval maps.** Now, let us consider the family of maps

$$f(x) = \beta x + \alpha \pmod 1 \tag{3.1}$$

with $\beta \geq 1$ and $\alpha \in [0, 1[$. Let us consider the partition $\mathcal{I} = \{I_0, \cdots, I_{n-1}\}$ of the interval I, with

$$I_0 =]0, (1-\alpha)/\beta[\quad , \cdots, \quad I_j =](j-\alpha)/\beta, (j+1-\alpha)/\beta[\,, \ \cdots,$$
$$\cdots \quad I_{n-1} =](n-1-\alpha)/\beta, 1[\,,$$

which is the minimal partition of monotonicity for f, and $\{c_j\}$ the set of discontinuity points of f. The set $\{0, 1, \cdots, n-1\}$ will be the alphabet. Set $\xi_{\beta,\alpha} = (\xi_i)_{i\in\mathbb{N}} = it_f(0)$ and consider the σ-invariant compact subset $\Sigma_{\beta,\alpha} = it(I)$ of $\{0, 1, \cdots n-1\}^{\mathbb{N}}$. Depending on the parameters α, β the orbit of 0 can be finite, in which case we obtain a Markov partition, see [10], which is a refinement of the above partition $\mathcal{I}$.

Let ι be the function defined as follows:

$$\iota\,(x) := \begin{cases} 0 \ \text{ if } \ x \in \left[0, \frac{1-\alpha}{\beta}\right[, \\ 1 \ \text{ if } \ x \in \left]\frac{1-\alpha}{\beta}, \frac{2-\alpha}{\beta}\right[, \\ \cdots \\ [\beta]-1 \ \text{ if } \ x \in \left]\frac{n-1-\alpha}{\beta}, 1\right[, \end{cases}$$

where $[\beta]$ denotes the integral part of β and $n = [\beta]+1$. Note that $\iota\,(x)$ is always a natural number in the set $\{0, 1, \cdots, [\beta]-1\}$. Eq. (3.1) can therefore be rewritten as follows

$$f(x) = \beta x + \alpha - \iota(x). \tag{3.2}$$

In order to lift the map f to a circle map, we need the condition $f(0) = f(1)$, see [7]. This implies that $\beta = n$ must be a positive integer number. Then we may define bounded operators U and V (or U_x and V_x if confusion arises) acting on the Hilbert space H_x as follows

$$V|x\rangle = (S_1^* + \cdots + S_n^*)|x\rangle = |f(x)\rangle, \qquad U\,|y\rangle = e^{2\pi i y}\,|y\rangle\,.$$

Then we prove in [7, Theorem 3.4] that for $n = 1$, these operators give rise to an irreducible representation of the irrational rotation algebra $\mathcal{A}_\alpha$. For generic n we have the following generalization.

Proposition 3.1. *The operators U and V satisfy the relation* (1.2)*, with the parameter given by $e^{2\pi i\alpha}$.*

Proof. We have

$$UV\,|y\rangle = e^{2\pi i f(y)}\,|f(y)\rangle = e^{2\pi i\alpha}e^{2\pi iny}\,|f(y)\rangle\,.$$

On the other hand

$$VU^n\,|y\rangle = e^{2\pi iny}\,|f(y)\rangle\,.$$

Therefore, $UV = e^{2\pi i\alpha}VU^n$. □

This relation appears in [7, Proposition 3.6] under the condition that f is a Markov map, but in Proposition 3.1 we obtain the same relation even in the non Markov cases. We note that for $n \neq 1$, the operator V is not a unitary (in fact it is an partial isometry), even for $\alpha = 0$, because the underlying dynamical system is noninvertible. Clearly, the operator U is always unitary.

Furthermore, we have the following irreducibility criterion for $C^*(U,V)$ as a subalgebra of $B(H_x)$.

Proposition 3.2. *If $\alpha \notin \mathbb{Q}$ then $C^*(U,V)' = \mathbb{C}I$.*

Proof. Let $T \in B(H_x)$ so that it commutes with the generators U and V of $C^*(U,V)$. Since $U|f^j(x)\rangle = e^{2\pi i(x+j\alpha)}|f^j(x)\rangle$, where $\{|f^j(x)\rangle\}$ is the canonical orthonormal basis of H_x with $j \in \mathbb{Z}$, and the eigenvalues $e^{2\pi i(x+j\alpha)}$ of U are all distinct, we conclude that $T|f^j(x)\rangle = c_j|f^j(x)\rangle$ for some scalar c_j. On the other hand, $V|f^j(x)\rangle = |f^{j+1}(x)\rangle$, so the commutation $VT = TV$ gives us $c_j = c_{j+1}$. Therefore T is a scalar multiple of the identity operator I. □

Lemma 3.3. *The C^*-algebra generated by two operators U and V satisfying the relation $UV = e^{2\pi\alpha i}VU^n$ is isomorphic to the C^*-algebra generated by U' and V' satisfying $U'V' = V'U'^n$, for $n > 1$.*

Proof. Given $\mu \in \mathbb{R}$, let $W = e^{2\pi\mu i}U$. Then

$$WV = e^{2\pi\mu i}UV = e^{2\pi\mu i}e^{2\pi\alpha i}VU^n =$$

$$= e^{2\pi(\alpha+\mu)i}VU^n = e^{2\pi(\alpha+\mu)i}Ve^{-2n\pi\mu i}VW^n = e^{2\pi(\alpha+(1-n)\mu)i}VW^n.$$

If $\mu = -\alpha/(1-n)$ then we have $WV = VW^n$ and the algebra generated by U and V is the same generated by U and W. Now set $U' := W, V' := V$. □

It is clear from the above proof that if U and V are unitary operators so are the new operators U' and V'. Now Lemma 3.3 implies that the operators U' and V' do satisfy the relation (1.3) thus $C^*(U,V) \cong C^*(U',V')$ as subalgebras of $B(H_x)$.

Remark 3.4. Lemma 3.3 does not hold for $n = 1$. Indeed, the relation $UV = e^{2\pi i\alpha}VU$ is the famous defining relation of the (universal) rotation C^*-algebra $\mathcal{A}_\alpha$, see [14]. It is the the C^*-algebra generated by two unitaries u and v satisfying the relation

$$uv = e^{2\pi i\alpha}vu,$$

which is non-commutative, whereas the C^*-algebra generated by the unitaries u and v satisfying $uv = vu$ is commutative and in fact isomorphic to $C(S^1 \times S^1)$.

3.2. **Unitarization.** We propose in this section to enlarge the Hilbert H_x and define new linear operators $\mathbf{U}$ and $\mathbf{V}$ so that the relation (1.3) holds with the advantage of $\mathbf{U}$ and $\mathbf{V}$ being both unitary operators.

For every $x \in I$, let H_x (see (3)) be the Hilbert space associated to the generalized orbit of x under the interval map (3.2) with $\alpha = 0$:

$$f(x) = nx - \iota(x).$$

Now for every $x, y \in I$, let us consider the Hilbert space $H_x \otimes H_y$. The basis is given by $\{|z\rangle \otimes |w\rangle$ with $z \in R_f(x)$ and $w \in R_f(y)\}$. Next, consider the operators $\mathbf{U}_{x,y}, \mathbf{V}_{x,y} \in B(H_x \otimes H_y)$ defined as follows

$$\mathbf{U}_{x,y} |z\rangle \otimes |w\rangle := e^{2\pi i z} |z\rangle \otimes |w\rangle, \quad \mathbf{V}_{x,y} |z\rangle \otimes |w\rangle := |f(z)\rangle \otimes \left| f^{-1}_{\iota(z)}(w) \right\rangle. \quad (3.3)$$

If no confusion arises, we shall denote $\mathbf{U}_{x,y}$ and $\mathbf{V}_{x,y}$ by $\mathbf{U}$ and $\mathbf{V}$, respectively. The operator $\mathbf{U}$ is clearly unitary, and in fact:

$$\mathbf{U}^* |z\rangle \otimes |w\rangle = e^{-2\pi i z} |z\rangle \otimes |w\rangle.$$

The adjoint of $\mathbf{V}^*$ is given by

$$\mathbf{V}^* |z\rangle \otimes |w\rangle = \left| f^{-1}_{\iota(w)}(z) \right\rangle \otimes |f(w)\rangle.$$

The operator $\mathbf{V}$ is unitary because

$$\begin{aligned} \mathbf{V}\mathbf{V}^* |z\rangle \otimes |w\rangle &= \mathbf{V} \left| f^{-1}_{\iota(w)}(z) \right\rangle \otimes |f(w)\rangle = \\ &= \left| f\left(f^{-1}_{\iota(w)}(z) \right) \right\rangle \otimes \left| f^{-1}_{\iota\left(f^{-1}_{\iota(w)}(z) \right)}(f(w)) \right\rangle = \\ &= |z\rangle \otimes \left| f^{-1}_{\iota(w)}(f(w)) \right\rangle = |z\rangle \otimes |w\rangle, \end{aligned}$$

since $\iota\left(f^{-1}_{\iota(w)}(z) \right) = \iota(w)$. On the other hand,

$$\begin{aligned} \mathbf{V}^*\mathbf{V} |z\rangle \otimes |w\rangle &= |f(z)\rangle \otimes \left| f^{-1}_{\iota(z)}(w) \right\rangle = \\ &= \left| f^{-1}_{\iota\left(f^{-1}_{\iota(z)}(w) \right)}(f(z)) \right\rangle \otimes \left| f\left(f^{-1}_{\iota(z)}(w) \right) \right\rangle = \\ &= \left| f^{-1}_{\iota(z)}(f(z)) \right\rangle \otimes |w\rangle = |z\rangle \otimes |w\rangle. \end{aligned}$$

We have the following relations between $\mathbf{U}$ and $\mathbf{V}$

$$\begin{aligned} \mathbf{U}\mathbf{V} |z\rangle \otimes |w\rangle &= e^{2\pi i f(z)} |f(z)\rangle \otimes \left| f^{-1}_{\iota(z)}(w) \right\rangle = \\ &= e^{2\pi i n z} |f(z)\rangle \otimes \left| f^{-1}_{\iota(z)}(w) \right\rangle \end{aligned}$$

and

$$\mathbf{V}\mathbf{U} |z\rangle \otimes |w\rangle = e^{2\pi i z} |f(z)\rangle \otimes \left| f^{-1}_{\iota(z)}(w) \right\rangle,$$

$$\mathbf{V}\mathbf{U}^n |z\rangle \otimes |w\rangle = e^{2\pi i n z} |f(z)\rangle \otimes \left| f^{-1}_{\iota(z)}(w) \right\rangle.$$

Therefore we easily obtain

$$\mathbf{U}\mathbf{V} = \mathbf{V}\mathbf{U}^n.$$

In particular, we have proven the following.

Theorem 3.5. *The map* $u \to \mathbf{U}, v \to \mathbf{V}$ *gives rise to a unitary representation*

$$\pi_{x,y}: \ \mathrm{BS}(1,n) \to B(H_x \otimes H_y)$$

of the Baumslag–Solitar group $\mathrm{BS}(1,n)$ *on te Hilbert space* $H_x \otimes H_y$*.*

Since the group $BS(n,1)$ is discrete and $\pi_{x,y}$ is a unitary representation, it can be lifted to a *-representation of the group C^*-algebra $C^*(BS(n,1))$.

We remark that the representation $\pi_{x,y}$ is not irreducible. Indeed, if we let $T \in B(H_x \otimes H_y)$ commuting with $\mathbf{U}$, then for every $z \in [x]$ and $w \in [y]$ we conclude (see proof of Proposition 3.2) that:

$$T(z \otimes w) = \sum_{w' \in [y]} c_{z,w'} \ z \otimes w'$$

for some complex numbers $c_{z,w'}$ (with an extra freedom in the second variable, unlike the case in proof of Proposition 3.2). If we further impose that T commutes with $\mathbf{V}$ then we easily see that in $T \notin \mathbb{C}I$ in general. Hence $\pi_{x,y}(C^*(BS(n,1)))' \neq \mathbb{C}I$. Since $C^*(BS(n,1)) \cong C^*_{\mathrm{red}}(BS(n,1))$ as $BS(n,1)$ is an amenable group, we conclude that $\lambda(BS(n,1))' \neq \mathbb{C}I$, where λ is the left regular representation. Note that

$$\pi_{x,y}(C^*_{\mathrm{red}}(BS(n,1)))' \subseteq \pi_{x,y}(C^*_{\mathrm{red}}(BS(n,1))').$$

Theorem 3.6.

(1) *If* $\pi_{x,y}$ *and* $\pi_{x',y'}$ *are unitarily equivalent, then* $x \sim x'$*.*
(2) *If* $x \sim x'$ *and* $y \sim y'$*, then* $\pi_{x,y}$ *and* $\pi_{x',y'}$ *are unitarily equivalent.*

Proof. We first prove (1). Since $\pi_{x,y}$ and $\pi_{x',y'}$ are unitarily equivalent, there exists a surjective isometry $W : H_x \otimes H_y \to H_{x'} \otimes H_{y'}$ such that

$$\pi_{x',y'}(a) = W\pi_{x,y}(a)W^*, \qquad \text{for all } a \in C^*(BS(n,1)).$$

Hence the spectrum $\sigma(\mathbf{U}_{x,y})$ of $\mathbf{U}_{x,y}$ equals the spectrum $\sigma(\mathbf{U}_{x',y'})$ of $\mathbf{U}_{x',y'}$. However $\sigma(\mathbf{U}_{x,y}) = \{e^{2\pi i z} : z \sim x\}$ and $\sigma(\mathbf{U}_{x',y'}) = \{e^{2\pi i w} : w \sim x'\}$. Therefore $x \sim x'$.

We now justify 2). If $x \sim x'$ and $y' \sim y'$ then $H_x = H_{x'}$ and $H_y = H_{y'}$. Moreover $\mathbf{U}_{x,y} = \mathbf{U}_{x',y'}$ and $\mathbf{V}_{x,y} = \mathbf{V}_{x',y'}$. Therefore $\pi_{x,y}$ and $\pi_{x',y'}$ are unitarily equivalent. □

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