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ALGEBRAICALLY PARANORMAL OPERATORS ON BANACH SPACES

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Abstract. In this paper we show that a bounded linear operator on a Banach space X is polaroid if and only if $p(T)$ is polaroid for some polynomial p. Consequently, algebraically paranormal operators defined on Banach spaces are hereditarily polaroid. Weyl type theorems are also established for perturbations $f(T+K)$, where T is algebraically paranormal, K is algebraic and commutes with T, and f is an analytic function, defined on an open neighborhood of the spectrum of $T+K$, such that f is nonconstant on each of the components of its domain. These results subsume recent results in this area.

1. Paranormal operators

There is a growing interest concerning paranormal operators, ([12, 14, 19, 7, 23]) and subclasses of paranormal operators ([17]), since the class of paranormal operators properly contains a relevant number of Hilbert space operators.

Paranormal operators are polaroid, where a bounded operator $T \in L(X)$ defined on a Banach space is said to be *polaroid* if every isolated point of the spectrum $\sigma(T)$ is a pole of the resolvent. Polaroid operators have been studied in recent papers in relation with Weyl type theorems, see [16, 15, 3, 6]. In this note we show that algebraically paranormal operators on Banach spaces are *hereditarily polaroid*, extending previous results known for Hilbert space operators. This is a consequence of the following more general result: $T \in L(X)$ is polaroid if and only if $f(T)$ is polaroid for some analytic function f (or equivalently, for some polynomial p), defined on an open neighborhood of $\sigma(T)$, such that f is nonconstant on each of the components of its domain. These results are, in the final

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part, applied for obtaining Weyl type theorems for operators $f(T+K)$, where T is algebraically paranormal and K is an algebraic operator which commutes with T.

We introduce the relevant terminology. A bounded linear operator $T \in L(X)$, X an infinite dimensional complex Banach space, is said to be *paranormal* if

$$\|Tx\| \leq \|T^2x\|\|x\| \quad \text{for all } x \in X.$$

It is known that the property of being paranormal is not translation-invariant by scalars. The *quasi-nilpotent part* of an operator $T \in L(X)$ is the set

$$H_0(T) := \{x \in X : \lim_{n\to\infty} \|T^n x\|^{\frac{1}{n}} = 0\}.$$

Clearly, ker $T^n \subseteq H_0(T)$ for every $n \in \mathbb{N}$.

An operator $T \in L(X)$ is said to have *the single valued extension property* at $\lambda_0 \in \mathbb{C}$ (abbreviated SVEP at λ_0), if for every open disc U of λ_0, the only analytic function $f : U \to X$ which satisfies the equation $(\lambda I - T)f(\lambda) = 0$ for all $\lambda \in U$ is the function $f \equiv 0$.
An operator $T \in L(X)$ is said to have SVEP if T has SVEP at every point $\lambda \in \mathbb{C}$. Note that, that both T and its dual T^* (or in the case of Hilbert space operators, the adjoint T') have SVEP at every isolated point of the spectrum $\sigma(T) = \sigma(T^*)$. Furthermore, the SVEP is inherited by the restrictions to closed invariant subspaces, i.e. if $T \in L(X)$ has the SVEP at λ_0 and M is a closed T-invariant subspace then $T\ |M$ has SVEP at λ_0.

The quasi-nilpotent part of an operator generally is not closed. We have

$$H_0(\lambda I - T) \text{ closed} \Rightarrow T \text{ has SVEP at } \lambda, \tag{1.1}$$

see [5].

The following result is well-known, see [12, Corollary 2.10] and [7, p. 2445].

Theorem 1.1. *Every paranormal operator on a separable Banach space has SVEP. Paranormal operators on Hilbert spaces have SVEP.*

It is known that every paranormal operator $T \in L(X)$ is *normaloid*, i.e. $\|T\|$ is equal to the spectral radius of T. Consequently, if $T \in L(X)$

$$T \quad \text{quasi-nilpotent paranormal} \Rightarrow T = 0. \tag{1.2}$$

An operator $T \in L(X)$ for which there exists a complex nonconstant polynomial h such that $h(T)$ is paranormal is said to be *algebraically paranormal*. Note that algebraic paranormality is preserved under translation by scalars and under restriction to closed invariant subspaces.

Two classical quantities associated with a linear operator T are the *ascent* $p := p(T)$, defined as the smallest non-negative integer p (if it does exist) such that ker $T^p =$ ker T^{p+1}, and the *descent* $q := q(T)$, defined as the smallest non-negative integer q (if it does exists) such that $T^q(X) = T^{q+1}(X)$. It is well-known that if $p(\lambda I - T)$ and $q(\lambda I - T)$ are both finite then $p(\lambda I - T) = q(\lambda I - T)$ and λ is a pole of the the function resolvent $\lambda \to (\lambda I - T)^{-1}$, in particular an isolated point of the spectrum $\sigma(T)$, see Proposition 38.3 and Proposition 50.2 of Heuser [18].

Recall that an invertible operator $T \in L(X)$ is said to be *doubly power-bounded* if $\sup\{\|T^n\| : n \in \mathbb{Z}\} < \infty$.

The following result, for Hilbert spaces operators, has been proved in [7, Theorem 2.4] , but the argument used in the proof is not correct (indeed, paranormality is not translation invariant). Now we give a correct proof of this result in the more general case of Banach space operators.

Lemma 1.2. *Suppose that $T \in L(X)$ is algebraically paranormal and quasi-nilpotent. Then T is nilpotent.*

Proof. Suppose that h is a polynomial for which $h(T)$ is paranormal. From the spectral mapping theorem we have

$$\sigma(h(T)) = h(\sigma(T)) = \{h(0)\}.$$

We claim that $h(T) = h(0)I$. To see that let us consider the two possibilities: $h(0) = 0$ or $h(0) \neq 0$.

If $h(0) = 0$ then $h(T)$ is quasi-nilpotent, so from the implication (1.2), we deduce that $h(T) = 0$, hence the equality $h(T) = h(0)I$ trivially holds.

Suppose the other case $h(0) \neq 0$, and set $h_1(T) := \frac{1}{h(0)} h(T)$. Clearly, $h_1(T)$ has spectrum $\{1\}$ and $\|h_1(T)\| = 1$. Moreover, $h_1(T)$ is invertible and also its inverse $h_1(T)^{-1}$ has norm 1. The operator $h_1(T)$ is then doubly power-bounded and by a classical theorem due to Gelfand, see [20, Theorem 1.5.14] for a proof, it then follows that $h_1(T) = I$, and hence $h(T) = h(0)I$, as claimed.

Now, from the equality $h(0)I - h(T) = 0$, we see that there exist some natural $n \in \mathbb{N}$ and $\mu \in \mathbb{C}$ for which

$$0 = h(0)I - h(T) = \mu\, T^m \prod_{i=1}^{n} (\lambda_i I - T) \quad \text{with } \lambda_i \neq 0,$$

where all $\lambda_i I - T$ are invertible. This obviously implies that $T^m = 0$, so T is nilpotent. □

Recall first that if $T \in L(X)$, the *analytic core* $K(T)$ is the set of all $x \in X$ such that there exists a constant $c > 0$ and a sequence of elements $x_n \in X$ such that $x_0 = x, Tx_n = x_{n-1}$, and $\|x_n\| \leq c^n \|x\|$ for all $n \in \mathbb{N}$.

Theorem 1.3. *If $T \in L(X)$ is algebraically paranormal then every isolated point of the spectrum $\sigma(T)$ is a pole of the resolvent; i.e. T is polaroid.*

Proof. We show that for every isolated point λ of $\sigma(T)$ we have $p(\lambda I - T) = q(\lambda I - T) < \infty$. Let λ be an isolated point of $\sigma(T)$, and denote by P_λ denote the spectral projection associated with $\{\lambda\}$. Then $M := K(\lambda I - T) = \ker P_\lambda$ and $N := H_0(\lambda I - T) = P_\lambda(X)$, see [1, Theorem 3.74]. Therefore, $H = H_0(\lambda I - T) \oplus K(\lambda I - T)$. Furthermore, since $\sigma(T|N) = \{\lambda\}$, while $\sigma(T|M) = \sigma(T) \setminus \{\lambda\}$, so the restriction $\lambda I - T|N$ is quasi-nilpotent and $\lambda I - T|M$ is invertible. Since $\lambda I - T|N$ is algebraically paranormal then Lemma 1.2 implies that $\lambda I - T|N$is nilpotent. In other worlds, $\lambda I - T$ is an operator of Kato Type, see [1, Chapter

1] for details.

Now, both T and its dual T^* have SVEP at λ, since λ is isolated in $\sigma(T) = \sigma(T^*)$, and this implies, by Theorem 3.16 and Theorem 3.17 of [1], that both $p(\lambda I - T)$ and $q(\lambda I - T)$ are finite. Therefore, λ is a pole of the resolvent. □

The concept of Drazin invertibility has been introduced in a more abstract setting than operator theory. In the case of the Banach algebra $L(X)$, $T \in L(X)$ is said to be *Drazin invertible* (with a finite index) if and only if $p(T) = q(T) < \infty$.

Definition 1.4. $T \in L(X)$ is said to be *left Drazin invertible* if $p := p(T) < \infty$ and $T^{p+1}(X)$ is closed, while $T \in L(X)$ is said to be *right Drazin invertible* if $q := q(T) < \infty$ and $T^q(X)$ is closed.

Clearly, $T \in L(X)$ is both right and left Drazin invertible if and only if T is Drazin invertible. In fact, if $0 < p := p(T) = q(T)$ then $T^p(X) = T^{p+1}(X)$ is the kernel of the spectral projection associated with the spectral set $\{0\}$, see [18, Prop. 50.2].

The concepts of left or right Drazin invertibility lead to the concepts of left or right pole. Let us denote by $\sigma_{\mathrm{a}}(T)$ the classical *approximate point spectrum* and by $\sigma_{\mathrm{s}}(T)$ the *surjectivity spectrum*. It is well known that $\sigma_{\mathrm{a}}(T^*) = \sigma_{\mathrm{s}}(T)$ and $\sigma_{\mathrm{s}}(T^*) = \sigma_{\mathrm{a}}(T)$.

Definition 1.5. Let $T \in L(X)$, X a Banach space. If $\lambda I - T$ is left Drazin invertible and $\lambda \in \sigma_{\mathrm{a}}(T)$ then λ is said to be a *left pole* of the resolvent of T. A left pole λ is said to have *finite rank* if $\alpha(\lambda I - T) < \infty$. If $\lambda I - T$ is right Drazin invertible and $\lambda \in \sigma_{\mathrm{s}}(T)$ then λ is said to be a *right pole* of the resolvent of T.A right pole λ is said to have *finite rank* if $\beta(\lambda I - T) < \infty$.

Evidently, λ is a pole of T if and only if λ is both a left and a right pole of T. Moreover, λ is a pole of T if and only if λ is a pole of T'. In the case of Hilbert space operators, λ is a pole of T' if and only if $\overline{\lambda}$ is a pole of T^*.

Definition 1.6. Let $T \in L(X)$. Then

(i) T is said to be *left polaroid* if every isolated point of $\sigma_{\mathrm{a}}(T)$ is a left pole of the resolvent of T.

(ii) $T \in L(X)$ is said to be *right polaroid* if every isolated point of $\sigma_{\mathrm{s}}(T)$ is a right pole of the resolvent of T.

(iii) $T \in L(X)$ is said to be *a-polaroid* if every isolated point of $\sigma_{\mathrm{a}}(T)$ is a pole of the resolvent of T.

Let $\operatorname{iso}\sigma(T)$ denote the set of all isolated points of $\sigma(T)$. The condition of being polaroid may be characterized as follows:

Theorem 1.7. [6, Theorem 2.2] *Suppose that $T \in L(X)$. Then we have:*

(i) *T is polaroid if and only if for every $\lambda \in \operatorname{iso}\sigma(T)$, there exists $\nu := \nu(\lambda I - T) \in \mathbb{N}$ such that $H_0(\lambda I - T) = \ker(\lambda I - T)^\nu$.*

(ii) *Suppose that T is left polaroid. Then, for every $\lambda \in \operatorname{iso}\sigma_{\mathrm{a}}(T)$, there exists $\nu := \nu(\lambda I - T) \in \mathbb{N}$ such that $H_0(\lambda I - T) = \ker(\lambda I - T)^\nu$.*

Note that the concepts of left and right polaroid are dual each other, see [3]. If $T \in L(X)$ then the following implications hold:

$$T\ a\text{-polaroid} \Rightarrow T \text{ left polaroid} \Rightarrow T \text{ polaroid}.$$

Furthermore, if T is right polaroid then T is polaroid. The first implication is clear, since a pole is always a left pole. Assume that T is left polaroid and let $\lambda \in \operatorname{iso}\sigma(T)$. It is known that the boundary of the spectrum is contained in $\sigma_{\mathrm{a}}(T)$, in particular every isolated point of $\sigma(T)$, thus $\lambda \in \operatorname{iso}\sigma_{\mathrm{a}}(T)$ and hence λ is a left pole of the resolvent of T. By part (ii) of Theorem 1.7, then there exists a natural $\nu := \nu(\lambda I - T) \in \mathbb{N}$ such that $H_0(\lambda I - T) = \ker(\lambda I - T)^{\nu}$. But λ is isolated in $\sigma(T)$, so T is polaroid, by part (i) of Theorem 1.7.

To show the last assertion suppose that T is right polaroid. Then T^* is left polaroid and hence, by the first part, T^* is polaroid, or equivalently T is polaroid.

2. Weyl type theorems for perturbations of paranormal operators

Recall that an operator $T \in L(X)$ is said to be *Weyl* ($T \in W(X)$), if T is *Fredholm* (i.e. $\alpha(T) := \dim\ker T$ and $\beta(T) := \operatorname{codim} T(X)$ are both finite) and the *index* $\operatorname{ind} T := \alpha(T) - \beta(T) = 0$. The *Weyl spectrum* of $T \in L(X)$ is defined by

$$\sigma_{\mathrm{w}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin W(X)\}.$$

Following Coburn [13], we say that *Weyl's theorem holds* for $T \in L(X)$ if

$$\sigma(T) \setminus \sigma_{\mathrm{w}}(T) = \pi_{00}(T), \tag{2.1}$$

where

$$\pi_{00}(T) := \{\lambda \in \operatorname{iso}\,\sigma(T) : 0 < \alpha(\lambda I - T) < \infty\}.$$

The concept of Fredholm operators has been generalized by Berkani ([10]) in the following way: for every $T \in L(X)$ and a nonnegative integer n let us denote by $T_{[n]}$ the restriction of T to $T^n(X)$ viewed as a map from the space $T^n(X)$ into itself (we set $T_{[0]} = T$). $T \in L(X)$ is said to be *B-Fredholm* if for some integer $n \geq 0$ the range $T^n(X)$ is closed and $T_{[n]}$ is a Fredholm operator. In this case $T_{[m]}$ is a Fredholm operator for all $m \geq n$ ([10]). This enables one to define the index of a Fredholm as $\operatorname{ind} T = \operatorname{ind} T_{[n]}$. A bounded operator $T \in L(X)$ is said to be *B-Weyl* ($T \in BW(X)$) if for some integer $n \geq 0$ $T^n(X)$ is closed and $T_{[n]}$ is Weyl. The *B-Weyl spectrum* $\sigma_{\mathrm{bw}}(T)$ is defined

$$\sigma_{\mathrm{bw}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin BW(X)\}.$$

Another version of Weyl's theorem has been introduced by Berkani and Koliha ([11] as follows: $T \in L(X)$ is said to verify *generalized Weyl's theorem*, (abbreviated, (gW)), if

$$\sigma(T) \setminus \sigma_{\mathrm{bw}}(T) = E(T), \tag{2.2}$$

where

$$E(T) := \{\lambda \in \operatorname{iso}\,\sigma(T) : 0 < \alpha(\lambda I - T)\}.$$

Note the generalized Weyl's theorem entails Weyl's theorem.

The following result shows that in presence of SVEP the polaroid condition entails Weyl's type theorems.

Theorem 2.1. *Let $T \in L(X)$ be polaroid and suppose that either T or T^* has SVEP. Then both T and T^* satisfy generalized Weyl's theorem.*

Proof. If T is polaroid also T^* is polaroid, and Weyl's theorem and generalized Weyl's theorem for T, or T^*, are equivalent, see [3, Theorem 3.7]. The assertion then follows from [3, Theorem 3.3]. □

As an immediate consequence of Theorem 2.1 we obtain that, for every algebraically paranormal operator T defined on a separable Banach space, or defined on a Hilbert space (in this case, the dual T^* may be replaced by the Hilbert adjoint T'), then both T and T^* satisfy generalized Weyl's theorem. This result, for algebraically paranormal operators on Hilbert spaces, has been proved in [14]. It should be noted that if T is paranormal on a Banach space X then Weyl's theorem holds for T and T^*, without assuming separability on X, see [12, Theorem 2.12].

Let $\mathcal{H}_{nc}(\sigma(T))$ denote the set of all analytic functions, defined on an open neighborhood of $\sigma(T)$, such that f is nonconstant on each of the components of its domain. Define, by the classical functional calculus, $f(T)$ for every $f \in \mathcal{H}_{nc}(\sigma(T))$.

The proof of the following results may be found in Lemma 1.76 and Lemma 3.101 of [1].

Lemma 2.2. *Let $\{\lambda_1, \dots, \lambda_k\}$ be a finite subset of $\mathbb{C}$, with $\lambda_i \neq \lambda_j$ for $i \neq j$. If $\{\nu_1, \dots \nu_k\} \subset \mathbb{N}$ and $p(\lambda) := \Pi_{i=1}^{k} (\lambda_i - \lambda)^{\nu_i}$ then*

$$\ker p(T) = \bigoplus_{i=1}^{k} \ker (\lambda_i I - T)^{\nu_i}.$$

Furthermore, if $p(\lambda_0) \neq 0$ for some $\lambda_0 \in \mathbb{C}$ then $H_0(\lambda_0 I - T) \cap \ker p(T) = \{0\}$.

Remark 2.3. It is easy to check from the definition of a quasi-nilpotent part the following properties:

(i) $H_0(T) \subseteq H_0(T^k)$, for all $k \in \mathbb{N}$.

(ii) If $T, U \in L(X)$ commutes and $S = TU$ then $H_0(T) \subseteq H_0(S)$.

We are now ready for the main result of this section.

Theorem 2.4. *For an operator $T \in L(X)$ the following statements are equivalent.*

(i) T is polaroid;

(ii) $f(T)$ is polaroid for every $f \in \mathcal{H}_{nc}(\sigma(T))$;

(iii) there exists a non-trivial polynomial p such that $p(T)$ is polaroid;

(iv) there exists $f \in \mathcal{H}_{nc}(\sigma(T))$ such that $f(T)$ is polaroid.

Proof. The implication (i) ⇒ (ii) has been proved in [6, Theorem 2.5]. The implications (ii) ⇒ (iii) ⇒ (iv) are obvious.

(iv) $\Rightarrow$ (i) Suppose $f(T)$ polaroid for some $f \in \mathcal{H}_{nc}(\sigma(T))$ and let $\lambda_0 \in \operatorname{iso}\sigma(T)$ be arbitrary. Then $\mu_0 := f(\lambda_0) \in f(\operatorname{iso}\sigma(T))$. It is easily seen that $\mu_0 \in \operatorname{iso} f(\sigma(T))$. Indeed, suppose that μ_0 is not isolated in $f(\sigma(T))$. Then there exists a sequence $(\mu_n) \subset f(\sigma(T))$ of distinct scalars such that $\mu_n \to \mu_0$ as $n \to +\infty$. Let $\lambda_n \in \sigma(T)$ such that $\mu_n = f(\lambda_n)$ for all n. Clearly, $\lambda_n \neq \lambda_m$ for $n \neq m$, and since $\mu_n = f(\lambda_n) \to \mu_0 = p(\lambda_0)$ then $\lambda_n \to \lambda_0$, and this is impossible since, by assumption, $\lambda_0 \in \operatorname{iso}\sigma(T)$. By the spectral mapping theorem then $\mu_0 \in \operatorname{iso} f(\sigma(T) = \operatorname{iso}\sigma(f(T))$. Now, since $f(T)$ is polaroid, the part (i) of Theorem 1.7 entails that there exists a natural ν such that

$$H_0(\mu I - f(T)) = \ker (\mu I - f(T))^\nu. \tag{2.3}$$

Let $g(\lambda) := \mu_0 - f(\lambda)$. Trivially, λ_0 is a zero of g, and g may have only a finite number of zeros. Let $\{\lambda_0, \lambda_1, \dots, \lambda_n\}$ be the set of all zeros of g, with $\lambda_i \neq \lambda_j$, for all $i \neq j$. Define $p(\lambda) := \Pi_{i=1}^n (\lambda_i - \lambda)^{\nu_i}$, where ν_i is the multiplicity of λ_i. Then we can write, for some $k \in \mathbb{N}$,

$$g(\lambda) = (\lambda_0 - \lambda)^k \, p(\lambda) \, h(\lambda),$$

where $h(\lambda)$ is an analytic function which does not vanish in $\sigma(T)$. Consequently,

$$g(T) = \mu_0 I - f(T) = (\lambda_0 I - T)^k p(T) h(T),$$

where $h(T)$ is invertible, and hence

$$H_0(\mu_0 I - f(T)) = H_0((\lambda_0 I - T)^k p(T) h(T)) = H_0((\lambda_0 I - T)^k p(T)).$$

By Remark 2.3, we then have

$$\begin{aligned} H_0(\lambda_0 I - T) &\subseteq H_0((\lambda_0 I - T)^k) \subseteq H_0((\lambda_0 I - T)^k p(T)) \\ &= H_0(\mu_0 I - f(T)), \end{aligned}$$

and, evidently,

$$\ker\, g(T) = \ker\, [(\lambda_0 I - T)^k p(T)].$$

By Lemma 2.2, we also have

$$\ker\, g(T) = \ker\, (\mu_0 I - f(T)) = \ker\, [(\lambda_0 I - T)^k \oplus \ker\, p(T).$$

and hence, from (2.3),

$$H_0(\mu_0 I - f(T)) = \ker\, (\lambda_0 I - T)^{k\nu} \oplus \ker\, p(T)^k.$$

Therefore,

$$H_0(\lambda_0 I - T) \subseteq \ker\, (\lambda_0 I - T)^{k\nu} \oplus \ker\, p(T)^k.$$

Since, by Lemma 2.2, we have $H_0(\lambda_0 I - T) \cap \ker\, p(T)^k = \{0\}$, we then conclude that $H_0(\lambda_0 I - T) \subseteq \ker\, (\lambda_0 I - T)^{k\nu}$. The opposite of the latter inclusion also holds, so we have $H_0(\lambda_0 I - T) = \ker\, (\lambda_0 I - T)^{k\nu}$. Theorem 1.7 then entails that T is polaroid. $\square$

A natural question is if the analogous of Theorem 2.4 holds for left polaroid operators. The implication

$$T \text{ left polaroid} \Rightarrow f(T) \text{ left polaroid},$$

holds for every $f \in \mathcal{H}_{nc}(\sigma(T))$, see [3, Lemma 3.11]. Denote by $\mathcal{H}^i_{nc}(\sigma(T))$ the subset of all $f \in \mathcal{H}_{nc}(\sigma(T))$ such that f is injective.

Theorem 2.5. *For an operator $T \in L(X)$ the following statements are equivalent.*

(i) T is left polaroid;

(ii) $f(T)$ is left polaroid for every $f \in \mathcal{H}^i_{nc}(\sigma(T))$;

(iii) there exists $f \in \mathcal{H}^i_{nc}(\sigma(T))$ such that $f(T)$ is left polaroid.

Proof. We have only to show that (iii) $\Rightarrow$ (i). Let λ_0 be an isolated point of $\sigma_{\mathrm{a}}(T)$ and let $\mu_0 := f(\lambda_0)$ As in the proof of Theorem 2.4 it then follows that $\mu_0 \in \operatorname{iso}\sigma_{\mathrm{a}}(f(T))$, so μ_0 is a left pole of $f(T)$. By Theorem 2.9 of [9] there exists a left pole η of T such that $f(\eta) = \mu_0$ and since f is injective then $\eta = \lambda_0$. Therefore, T is left polaroid. □

A bounded operator $T \in L(X)$ is said to be *hereditarily polaroid*, i.e. any restriction to an invariant closed subspace is polaroid. This class of operators has been first considered in [16]. Examples of hereditarily polaroid operators are $H(p)$-operators (i.e. operators on Banach spaces for which for every $\lambda \in \mathbb{C}$ there exists a natural $p := p(\lambda)$ such that $H_0(\lambda I - T) = \ker(\lambda I - T)^p$). Property $H(p)$ is satisfied by every generalized scalar operator, see [20] for details of this class of operators), and in particular for p-hyponormal, log-hyponormal or M-hyponormal operators on Hilbert spaces, see [21]. An example of polaroid operator which is not hereditarily polaroid may be found in [16, Example 2.6].

Corollary 2.6. *Algebraically paranormal operators on Banach spaces are hereditarily polaroid.*

Proof. Let $T \in L(X)$ be algebraically paranormal and M a closed T-invariant subspace of X. By assumption there exists a nontrivial polynomial h such that $h(T)$ is paranormal. The restriction of any paranormal operator to an invariant closed subspace is also paranormal, so $h(T|M) = h(T)|M$ is paranormal and hence polaroid, by Theorem 1.3. From Theorem 2.4 we then conclude that $T|M$ is polaroid. □

Recall that a bounded operator $K \in L(X)$ is said to be *algebraic* if there exists a non-constant polynomial h such that $h(K) = 0$. Trivially, every nilpotent operator is algebraic and it is well-known that if $K^n(X)$ has finite dimension for some $n \in \mathbb{N}$ then K is algebraic. In [4] it is shown that if T is hereditarily polaroid and has SVEP, and K is an algebraic operator which commutes with T then $T + K$ is polaroid and $T^* + K^*$ is a-polaroid.

Theorem 2.7. *Let $T \in L(X)$ be an algebraically paranormal operator on a separable Banach space X, and let $K \in L(X)$ be an algebraic operator commuting with T. Then both $f(T + K)$ and $f(T^* + K^*)$ satisfies (gW) for every $f \in \mathcal{H}_{nc}(\sigma(T+K))$. An analogous result holds if T is an algebraically paranormal operator on a Hilbert space.*

Proof. Suppose that $T \in L(X)$ is algebraically paranormal operator, and let h be a non-trivial polynomial for which $h(T)$ is paranormal, and hence has SVEP, since T has SVEP. From Theorem [1, Theorem 2.40] it the follows that also T has SVEP. Now, by Corollary 2.6 T is hereditarily polaroid. By Theorem 2.15

of [4] then $T + K$ is polaroid and $T^* + K^*$ is a-polaroid (and hence polaroid). By Theorem 2.4 then $f(T + K)$ is polaroid. Moreover, $T + K$ has SVEP, by [8, Theorem 2.14] and hence $f(T + K)$ has SVEP, again by [1, Theorem 2.40]. The assertions then follows by Theorem 2.1.
The last assertion is proved with the same argument, since T has SVEP. □

Theorem 2.7 considerably improves the results of Theorem 2.4 of [14] proved for algebraically paranormal operators defined on a separable Hilbert spaces H, and also improves Theorem 2.5 of [7], proved in the case of paranormal operators on Hilbert spaces. Observe that, always in the situation of Theorem 2.7, the fact that $f(T+K)$ is polaroid entails that all Weyl type theorems (as properties (gw) and (gaW)) hold for $f(T^* + K^*)$, see [3] for definitions and details, in particular Theorem 3.10.

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