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ON THE BOUNDEDNESS AND COMPACTNESS OF A CERTAIN INTEGRAL OPERATOR

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Abstract. Let $\alpha > 0$ and $\beta > 1$. In the present work, the necessary and sufficient conditions for the boundedness and compactness of the integral operator of the form

$$L_{\alpha,\beta} f(x) := v(x) \int_0^x \frac{\ln^{\beta-1}(\frac{x}{y}) f(y) u(y) dy}{(x-y)^{1-\alpha}}, \quad x > 0,$$

from $L^p \to L^q$, with locally integrable non-negative weight functions u and v, in the case $0 < p, q < \infty, p > \max(1/\alpha, 1)$, provided u is non-increasing on $\mathbb{R}^+ := [0, \infty)$ are found.

1. Introduction

For $0 < p < \infty$ we denote $L^p := L^p(\mathbb{R}^+)$ the set of all measurable functions such that $\|f\|_p := \left(\int_0^\infty |f(x)|^p dx\right)^{1/p} < \infty$. Let $\alpha > 0$ and

$$L_{\alpha,\beta} f(x) := v(x) \int_0^x \frac{\ln^{\beta-1}(\frac{x}{y}) f(y) u(y) dy}{(x-y)^{1-\alpha}}, \quad x > 0. \tag{1.1}$$

If $v(x) = u(x) = 1$ and $\beta = 1$, the operator (1.1) coincides with the classical Riemann–Liouville fractional operator ([4], § 9.9). We study the problem of necessary and sufficient conditions for the inequality

$$\|L_{\alpha,\beta} f\|_q \le C \|f\|_p, \tag{1.2}$$

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to hold with a constant C independent on $f \in L^p$ which we assume to be least possible. Boundedness and compactness criteria for the case $0 < \beta \leq 1$ in [16] was found. Also in [7] criteria for operators with power-logarithmic kernels were studied. If $\alpha = \beta = 1$ the inequality (1.2) was completely characterized (see, for instance, [14, 21]). The cases $\alpha > 1, \beta = 1$ and $\alpha \in (0,1), \beta = 1$ were solved with further generalizations in [13, 22, 23, 1, 12], [15, 19, 20, 18, 9, 10] .

Throughout the paper uncertainties of the form $0 \cdot \infty$ are taken to be zeros. The relations $A \ll B$ and $B \gg A$ means that $A \leq cB$, where the constant c depends only on p, q, α, β and may be different in different places. If both $A \ll B$ and $A \gg B$, then we write $A \approx B$. $\mathbb{Z}$ stands for the set of all integers, χ_E is the characteristic function of E. The symbol $p' := \frac{p}{p-1}, p \neq 1$ denotes the conjugate numbers of p, and the symbol $\square$ marks the end of a proof.

2. Preliminaries

Definition 2.1. Let $k(x, y) \geq 0$ be the kernel of the operator of the form

$$Kf(x) := v(x) \int_c^x k(x,y) f(y) u(y) dy, \quad 0 \leq c \leq x \leq d \leq \infty.$$

If there exists a constant $D \geq 1$ such that

$$D^{-1} k(x,y) \leq k(x,z) + k(z,y) \leq Dk(x,y), \quad 0 \leq c \leq y \leq z \leq x \leq d \leq \infty, \tag{2.1}$$

then we call a kernel $k(x, y)$ from *Oinarov's class* and denote $k(x, y) \in \mathcal{O}$ [17].

Standard examples of a kernel $k(x, y) \geq 0$ satisfying (2.1) are

(i) $k(x, y) = (x - y)^{\alpha}, \quad \alpha \geq 0,$
(ii) $k(x, y) = \ln^{\beta}(1 + x - y), k(x, y) = \ln^{\beta}(\frac{x}{y}); \beta \geq 0,$

and their combinations. Let $b : [c, d] \to [0, \infty)$ be a strictly increasing differentiable function and let

$$K_b : L_p(b(c), b(d)) \to L_q(c, d),$$

be an operator of the form

$$K_b f(x) := v(x) \int_{b(c)}^{b(x)} k(x,y) f(y) u(y) dy, \quad 0 \leq c \leq x \leq d \leq \infty, \tag{2.2}$$

where a non-negative kernel $k(x, y)$ satisfies the following definition.

Definition 2.2. $k(x, y) \in \mathcal{O}_b$ if there exists a constant $D \geq 1$ such that

$$D^{-1} k(x,y) \leq k(x, b(z)) + k(z,y) \leq Dk(x,y), \quad \begin{cases} 0 \leq c \leq z \leq x \leq d \leq \infty, \\ 0 \leq b(c) \leq y \leq b(z). \end{cases} \tag{2.3}$$

Now we consider the operator of the form

$$\mathcal{K} f(x) := v(x) \int_{a(x)}^{b(x)} k(x,y) f(y) u(y) dy, \tag{2.4}$$

where the boundaries $a(x)$, $b(x)$ satisfy the following conditions:

(i) $a(x)$ and $b(x)$ are differentiable and strictly increasing on $(0,\infty)$;
(ii) $a(0)=b(0)=0$, $a(x)<b(x)$ for $0<x<\infty$, $a(\infty)=b(\infty)=\infty$.

Definition 2.3. $k(x,y)\in\mathcal{O}_{ab}$ if there exists a constant $D\geq 1$ such that

$$D^{-1}k(x,y)\leq k(x,b(z))+k(z,y)\leq Dk(x,y),\quad z\leq x,\ a(x)\leq y\leq b(z). \tag{2.5}$$

The following theorems are taken from [30]. Theorem 2.2 is closely related to the results of [2, 3, 5], [25, 26, 28, 29].

Theorem 2.1. *Let the operator K_b be an operator given by (2.2) with a strictly increasing differentiable function $b(x)\geq 0$ and $k(x,y)\in\mathcal{O}_b$.*
(a) If $1<p\leq q<\infty$, then

$$\|K_b\|_{L_p(b(c),b(d))\to L_q(c,d)}\approx A_{b,0}+A_{b,1},$$

where

$$A_{b,0}:=\sup_{c\leq t\leq d}\left(\int_t^d k^q(x,b(t))v^q(x)dx\right)^{1/q}\left(\int_{b(c)}^{b(t)}u^{p'}(y)dy\right)^{1/p'},$$

$$A_{b,1}:=\sup_{c\leq t\leq d}\left(\int_t^d v^q(x)dx\right)^{1/q}\left(\int_{b(c)}^{b(t)}k^{p'}(t,y)u^{p'}(y)dy\right)^{1/p'},$$

(b) If $1<q\leq p<\infty$, then

$$\|K_b\|_{L_p(b(c),b(d))\to L_q(c,d)}\approx B_{b,0}+B_{b,1},$$

where

$$B_{b,0}:=\left(\int_{b(c)}^{b(d)}\left[\int_{b^{-1}(t)}^d k^q(x,t)v^q(x)dx\right]^{r/q}\left[\int_{b(c)}^t u^{p'}(y)dy\right]^{r/q'}u^{p'}(t)dt\right)^{1/r},$$

$$B_{b,1}:=\left(\int_c^d\left[\int_t^d v^q(x)dx\right]^{r/p}\left[\int_{b(c)}^{b(t)}k^{p'}(t,y)u^{p'}(y)dy\right]^{r/p'}v^q(t)dt\right)^{1/r}.$$

Theorem 2.2. *For the operator defined by (2.4), we take a sequence of points $\{\xi_k\}_k\in\mathbb{Z}\subset(0,\infty)$ such that*

$$\xi_0=1,\quad \xi_k=(a^{-1}\circ b)^k(1),\quad k\in\mathbb{Z},$$

and put

$$\eta_k=a(\xi_k)=b(\xi_{k-1}),\quad \Delta_k=[\xi_k,\xi_{k+1}),\ \ \delta_k=[\eta_k,\eta_{k+1}),\ \ k\in\mathbb{Z}.$$

If $1<p\leq q<\infty$, then

$$\|\mathcal{K}\|_{L_p\to L_q}\approx\mathcal{A}:=\mathcal{A}_0+\mathcal{A}_1,$$

where

$$\begin{aligned}\mathcal{A}_0 &:= \sup_{t>0} \mathcal{A}_0(t)\\ &= \sup_{t>0} \sup_{s\in[b^{-1}(a(t)),t]} \left(\int_s^t k^q(x,b(s))v(x)^q dx\right)^{1/q} \left(\int_{a(t)}^{b(s)} u^{p'}(y)dy\right)^{1/p'},\end{aligned}$$

$$\begin{aligned}\mathcal{A}_1 &:= \sup_{t>0} \mathcal{A}_1(t)\\ &= \sup_{t>0} \sup_{s\in[b^{-1}(a(t)),t]} \left(\int_s^t v(x)^q dx\right)^{1/q} \left(\int_{a(t)}^{b(s)} k^{p'}(s,y)u^{p'}(y)dy\right)^{1/p'}.\end{aligned}$$

Moreover, $\mathcal{K}$ *is compact if and only if* $\mathcal{A} < \infty$ *and* $\lim_{t\to 0} \mathcal{A}_i(t) = \lim_{t\to\infty} \mathcal{A}_i(t) = 0$, $i = 0, 1$. *If* $1 < q < p < \infty$, *then*

$$\|\mathcal{K}\|_{L_p\to L_q} \approx \mathcal{B} := \left(\sum_{k\in\mathbb{Z}} \left[\mathcal{B}_{k,1}^r + \mathcal{B}_{k,2}^r + \mathcal{B}_{k,3}^r + \mathcal{B}_{k,4}^r\right]\right)^{1/r},$$

$$\mathcal{B}_{k,1} := \left\{\int_{a(\xi_k)}^{a(\xi_{k+1})} \left(\int_{\xi_k}^{a^{-1}(t)} k^q(x,b(\xi_k))v(x)^q dx\right)^{r/q} \times \left(\int_t^{a(\xi_{k+1})} u^{p'}(y)dy\right)^{r/q'} u^{p'}(t)dt\right\}^{1/r},$$

$$\mathcal{B}_{k,2} := \left\{\int_{\xi_k}^{\xi_{k+1}} \left(\int_{\xi_k}^t v(x)^q dx\right)^{r/p} \times \left(\int_{a(t)}^{a(\xi_{k+1})} k^{p'}(\xi_k,y)u^{p'}(y)dy\right)^{r/p'} v^q(t)dt\right\}^{1/r},$$

$$\mathcal{B}_{k,3} := \left\{\int_{b(\xi_k)}^{b(\xi_{k+1})} \left(\int_{b^{-1}(t)}^{\xi_{k+1}} k^q(x,t)v(x)^q dx\right)^{r/q} \times \left(\int_{b(\xi_k)}^t u^{p'}(y)dy\right)^{r/q'} u^{p'}(t)dt\right\}^{1/r},$$

$$\mathcal{B}_{k,4} := \left\{\int_{\xi_k}^{\xi_{k+1}} \left(\int_{\xi_k}^t v(x)^q dx\right)^{r/p} \left(\int_{b(\xi_k)}^{b(t)} k^{p'}(t,y)u^{p'}(y)dy\right)^{r/p'} v^q(t)dt\right\}^{1/r},$$

and the operator $\mathcal{K}$ *is compact if and only if* $\mathcal{B} < \infty$.

In the proof of Theorem 3.1 below, we apply the *Chebyshev inequality*: if $F(x) \geq 0$ is non-increasing and $G(x) \geq 0$ is non-decreasing on $(a,b) \subset \mathbb{R}$, then

$$\int_a^b F(x)G(x)dx \leq \frac{1}{b-a}\int_a^b F(x)dx \int_a^b G(x)dx. \tag{2.6}$$

In the section 4, we need the following theorem from ([8], Theorem 5.8).

Theorem 2.3. *Each regular linear integral operator L acting from L_p to L_q, where $0 < q < p < \infty$ and $p \geq 1$, is compact.*

Observe, that every bounded integral operator with a non-negative kernel is regular.

3. Boundedness

Let $\mathfrak{M}^+$ be the class of all measurable functions $f\colon [0,\infty) \to [0,+\infty]$. Without a loss of generality we may and shall restrict the inequality (1.2) on $f \in \mathfrak{M}^+$.

Theorem 3.1. *Let $\max(\frac{1}{\alpha},1) < p \leq q < \infty$, $\beta > 1$. Let $v \in \mathfrak{M}^+$ and $u \in \mathfrak{M}^+$ is non-increasing on $[0,\infty)$.*
I) If $\alpha+\beta > 2$ then the inequality

$$\left(\int_0^\infty (L_{\alpha,\beta}f(x))^q\,dx\right)^{1/q} \leq C\left(\int_0^\infty f(x)^p dx\right)^{1/p}, \quad f \in \mathfrak{M}^+, \tag{3.1}$$

holds if and only if $A+B<\infty$, where

$$\begin{aligned}
A_0(\alpha,\beta) &:= \sup_{t>0} A_0(t)\\
&= \sup_{t>0}\left(\int_t^\infty \frac{v(x)^q(\ln\frac{2x}{t})^{(\beta-1)q}dx}{x^{(1-\alpha)q}}\right)^{1/q}\left(\int_0^{\frac{t}{2}} u^{p'}(y)dy\right)^{1/p'},\\
A_1(\alpha,\beta) &:= \sup_{t>0} A_1(t)\\
&= \sup_{t>0}\left(\int_t^\infty \frac{v(x)^q dx}{x^{(1-\alpha)q}}\right)^{1/q}\left(\int_0^{\frac{t}{2}} (\ln\frac{t}{y})^{(\beta-1)p'}u^{p'}(y)dy\right)^{1/p'}, \qquad (3.2)
\end{aligned}$$

$$A := A_0(\alpha,\beta) + A_1(\alpha,\beta),$$

and

$$\begin{aligned}
B_0(\alpha,\beta) &:= \sup_{t>0} B_0(t)\\
&= \sup_{t>0}\sup_{s\in[\frac{t}{2},t]}\left(\int_s^t v(x)^q(x-s)^{(\alpha+\beta-2)q}dx\right)^{1/q}\left(\int_{\frac{t}{2}}^s \frac{u^{p'}(y)dy}{y^{(\beta-1)p'}}\right)^{1/p'},\\
B_1(\alpha,\beta) &:= \sup_{t>0} B_1(t)\\
&= \sup_{t>0}\sup_{s\in[\frac{t}{2},t]}\left(\int_s^t v(x)^q dx\right)^{1/q}\left(\int_{\frac{t}{2}}^s (s-y)^{(\alpha+\beta-2)p'}\frac{u^{p'}(y)dy}{y^{(\beta-1)p'}}\right)^{1/p'},
\end{aligned}$$

$$B := B_0(\alpha,\beta) + B_1(\alpha,\beta).$$

Moreover, $C \approx A + B$.
II) If $1 < \alpha+\beta < 2$ *then the inequality* (3.1) *holds if and only if* $A + D < \infty$, *where*

$$D := \sup_{k\in\mathbb{Z}} D_k = \sup_{k\in\mathbb{Z}} \sup_{t\in(2^k,2^{k+1}]} \left(\int_t^{2^{k+1}} \frac{v(s)^q ds}{s^{(2-\alpha-\beta)q}}\right)^{1/q} \left(\int_{2^{k-1}}^t \frac{u^{p'}(s)ds}{s^{(\beta-1)p'}}\right)^{1/p'}. \quad (3.3)$$

Moreover, $C \approx A + D$.

Proof. (I) $(\alpha+\beta>2)$. We have

$$L_{\alpha,\beta} f(x) = v(x)\int_0^{\frac{x}{2}} \frac{\ln^{\beta-1}(\frac{x}{y}) f(y)u(y)dy}{(x-y)^{1-\alpha}} + v(x)\int_{\frac{x}{2}}^{x} \frac{\ln^{\beta-1}(\frac{x}{y}) f(y)u(y)dy}{(x-y)^{1-\alpha}}$$

$$:= L_1 f(x) + L_2 f(x), \quad f \in \mathfrak{M}^+.$$

If $y \in (0,\frac{x}{2})$, then

$$L_1 f(x) \approx \frac{v(x)}{x^{1-\alpha}} \int_0^{\frac{x}{2}} (\ln(\frac{x}{y}))^{\beta-1} f(y)u(y)dy.$$

We see that, for $\beta > 1$, $(\ln(\frac{x}{y}))^{\beta-1}$ satisfies (2.3). Since

$$\ln\left(\frac{x}{y}\right) = \ln\left(\frac{x}{z}\right) + \ln\left(\frac{z}{y}\right) \le \ln\left(\frac{2x}{z}\right) + \ln\left(\frac{z}{y}\right), \quad 0<z<x, 0<y<z/2,$$

and

$$\ln\left(\frac{x}{y}\right) \ge \frac{1}{2}\left(\ln\left(\frac{2x}{z}\right) + \ln\left(\frac{z}{y}\right)\right) \Leftrightarrow \ln\left(\frac{x}{y}\right) \ge \frac{1}{2}\ln\left(\frac{2x}{y}\right) = \ln\left(\sqrt{\frac{2x}{y}}\right)$$

$$\Leftrightarrow \sqrt{\frac{x}{y}} \ge \sqrt{2} \Leftrightarrow y \le \frac{x}{2},$$

so

$$\ln\left(\frac{x}{y}\right) \approx \ln\left(\frac{2x}{z}\right) + \ln\left(\frac{z}{y}\right),$$

Therefore, the inequality (3.1) implies

$$\left(\int_0^\infty \frac{v(x)^q}{x^{(1-\alpha)q}} \left(\int_0^{\frac{x}{2}} (\ln(\frac{x}{y}))^{\beta-1} f(y)u(y)dy\right)^q dx\right)^{1/q} \le C_0 \left(\int_0^\infty f(x)^p dx\right)^{1/p}, \quad (3.4)$$

for $f \in \mathfrak{M}^+$, with $C_0 \le C$ and it follows from Theorem 2.1, that $A \approx C_0$. On the other hand, if $y \in [\frac{x}{2}, x]$,then $\frac{x}{y} - 1 \in [0,1]$. By using $\ln(1+\gamma) \approx \gamma, (\gamma \in [0,1])$, we can write the following

$$\ln(\frac{x}{y}) = \ln(1 + \frac{x-y}{y}) \approx \frac{x-y}{y}.$$

So, we obtain

$$L_2f(x) \approx v(x)\int_{\frac{x}{2}}^{x}(x-y)^{\alpha+\beta-2}f(y)\frac{u(y)}{y^{\beta-1}}dy.$$

The kernel $(x-y)^{\alpha+\beta-2}$, for $\alpha+\beta>2$, satisfies (2.5). Therefore, the inequality (3.1) implies

$$\left(\int_0^\infty v(x)^q\left(\int_{\frac{x}{2}}^{x}(x-y)^{\alpha+\beta-2}f(y)\frac{u(y)}{y^{\beta-1}}dy\right)^q dx\right)^{1/q} \le C_1\left(\int_0^\infty f(x)^p dx\right)^{1/p}, \tag{3.5}$$

for $f\in\mathfrak{M}^+$,with $C_1\le C$ and it follows from Theorem 2.2, that $B\approx C_1$. Moreover, (3.1), is equivalent to (3.4) and (3.5), so that $C\approx A+B$.

(II) $(1<\alpha+\beta<2)$ Now we continue the proof of theorem for second case. We have the same arguments to the proof of part (I) for $L_1f(x)$. However, with the condition on α,β, operator $L_2f(x)$ coincides with the Riemann–Liouville fractional operator. The inequality (3.1) implies

$$\left(\int_0^\infty v(x)^q\left(\int_{\frac{x}{2}}^{x}(x-y)^{\alpha+\beta-2}f(y)\frac{u(y)}{y^{\beta-1}}dy\right)^q dx\right)^{1/q} \le D_0\left(\int_0^\infty f(x)^p dx\right)^{1/p}, \tag{3.6}$$

for $f\in\mathfrak{M}^+$. Moreover, (3.1), is equivalent to (3.4) and (3.6), so that $C\approx A+D_0$. We show, that $D_0\ll D\ll C$ which implies $C\approx A+D$. To this end we construct a new operator and apply the block-diagonal method. Put $\Delta_k:=(2^k,2^{k+1}]$ and define

$$L_k^{(1)}f(x) := v(x)\chi_{\Delta_k}(x)\int_{2^k}^{x}(x-y)^{\alpha+\beta-2}f(y)\frac{u(y)}{y^{\beta-1}}dy,$$

$$L_k^{(2)}f(x) := v(x)\chi_{\Delta_k}(x)\int_{2^{k-1}}^{2^k}(x-y)^{\alpha+\beta-2}f(y)\frac{u(y)}{y^{\beta-1}}dy,$$

$$L_k := L_k^{(1)}+L_k^{(2)},\ \ L^{(1)}:=\sum_{k\in\mathbb{Z}}L_k^{(1)},\ \ L^{(2)}:=\sum_{k\in\mathbb{Z}}L_k^{(2)},\ \ L:=L^{(1)}+L^{(2)}.$$

Since the operators $L^{(1)}$ and $L^{(2)}$ are block-diagonal, then by ([27], Lemma 1) we have for $p\le q$

$$\|L\| := \|L\|_{L^p\to L^q}\approx\sup_{k\in\mathbb{Z}}\|L_k\|_{L^p(2^{k-1},2^{k+1}]\to L^q(2^k,2^{k+1}]} =: \sup_{k\in\mathbb{Z}}\|L_k\|. \tag{3.7}$$

Observe, that

$$\left(\int_0^\infty v(x)^q\left(\int_{\frac{x}{2}}^{x}(x-y)^{\alpha+\beta-2}f(y)\frac{u(y)}{y^{\beta-1}}dy\right)^q dx\right)$$

$$\le \ \ \|Lf\|_q \le \left(\int_0^\infty (L_{\alpha,\beta}f(x))^q\,dx\right)^{1/q},\quad f\in\mathfrak{M}^+, \tag{3.8}$$

and it trivially follows from the left side of (3.8), that $D_0\le\|L\|$. Fix $k\in\mathbb{Z}$ and put $v_k:=v\chi_{\Delta_k}$. If $x\in\Delta_k$ and $y\in[2^{k-1},x)$ then $\frac{1}{x-y}\ge\frac{4}{3x}$. Hence, the inequality

$$\|L_kf\|_{L^q[\Delta_k]}\le\|L_k\|\|f\chi_{[2^{k-1},2^{k+1}]}\|_p,\quad f\in\mathfrak{M}^+,$$

implies the Hardy inequality

$$\left(\int_{2^{k-1}}^{2^{k+1}} \frac{v_k(x)^q dx}{x^{(2-\alpha-\beta)q}} \left(\int_{2^{k-1}}^{x} f(y)\frac{u(y)}{y^{\beta-1}}dy\right)^q dx\right)^{1/q} \ll \|L_k\| \|f\chi_{[2^{k-1},2^{k+1}]}\|_p$$

for all $f \in \mathfrak{M}^+$. Then, applying ([30], Lemma 2.1), the lower bound $\|L\| \gg D$ follows for $p \leq q$ from (3.7). Hence, from the right hand side of (3.8) we obtain $D \ll \|L\| \ll C$. Thus, the lower bound $C \gg A + D$ is proved.
The opposite estimate $C \ll A + D$ will be established, if we show that $\|L\| \ll D$. Denote

$$J := \int_{2^{k-1}}^{x} (x-y)^{\alpha+\beta-2} f(y)\frac{u(y)}{y^{\beta-1}}dy.$$

To this end we prove first that for $x \in \Delta_k$

$$J \ll \frac{1}{x^{2-\alpha-\beta}} \left(\int_{2^{k-1}}^{x} f(y)^p \left[\int_{2^{k-1}}^{y} \frac{u(t)^{p'}dt}{t^{(\beta-1)p'}}\right]^{\frac{1}{p'}} dy\right)^{\frac{1}{p}} \left(\int_{2^{k-1}}^{x} \frac{u(y)^{p'}dy}{y^{(\beta-1)p'}}\right)^{\frac{1}{p'^2}}. \quad (3.9)$$

Set

$$h(\alpha,\beta) := \left[\int_{2^{k-1}}^{y} \frac{u(t)^{p'}dt}{(x-t)^{(2-\alpha-\beta)p'}t^{(\beta-1)p'}}\right],$$

and write

$$J = \int_{2^{k-1}}^{x} \left\{f(y)h(\alpha,\beta)^{\frac{1}{pp'}}\right\} \left\{h(\alpha,\beta)^{-\frac{1}{pp'}}(x-y)^{\alpha+\beta-2}\frac{u(y)}{y^{\beta-1}}\right\} dy$$

(applying Hölder's inequality)

$$\leq \left(\int_{2^{k-1}}^{x} f(y)^p h(\alpha,\beta)^{\frac{1}{p'}} dy\right)^{\frac{1}{p}}$$

$$\times \left(\int_{2^{k-1}}^{x} \frac{u(y)^{p'}}{(x-y)^{(2-\alpha-\beta)p'}y^{(\beta-1)p'}} h(\alpha,\beta)^{-\frac{1}{p}} dy\right)^{\frac{1}{p'}}$$

(calculating the second factor)

$$\approx \left(\int_{2^{k-1}}^{x} f(y)^p h(\alpha,\beta)^{\frac{1}{p'}} dy\right)^{\frac{1}{p}} \left(\int_{2^{k-1}}^{x} \frac{u(y)^{p'}dy}{(x-y)^{(2-\alpha-\beta)p'}y^{(\beta-1)p'}}\right)^{\frac{1}{p'^2}}.$$

Let $x \in \Delta_k, y \in (2^{k-1}, x)$. Since

$$\frac{1}{(x-t)^{(2-\alpha-\beta)p'}},$$

is increasing with respect to $t \in (2^{k-1}, y)$ and

$$\frac{u(t)^{p'}}{t^{(\beta-1)p'}},$$

is decreasing, by Chebyshev's inequality (2.6) and an elementary inequality,

$$b^\gamma - a^\gamma \approx b^{\gamma-1}(b-a), \ b > a > 0, \ \gamma > 0,$$

we find that

$$\int_{2^{k-1}}^{y} \frac{u(t)^{p'}dt}{(x-t)^{(2-\alpha-\beta)p'}t^{(\beta-1)p'}} \leq \frac{1}{y-2^{k-1}} \int_{2^{k-1}}^{y} \frac{u(t)^{p'}}{t^{(\beta-1)p'}}dt \int_{2^{k-1}}^{y} \frac{dt}{(x-t)^{(2-\alpha-\beta)p'}}$$

$$\approx \frac{1}{y-2^{k-1}} \left((x-2^{k-1})^{1-(2-\alpha-\beta)p'} - (x-y)^{1-(2-\alpha-\beta)p'}\right) \left[\int_{2^{k-1}}^{y} \frac{u(t)^{p'}dt}{t^{(\beta-1)p'}}\right]$$

$$\approx \frac{1}{(x-2^{k-1})^{(2-\alpha-\beta)p'}} \left[\int_{2^{k-1}}^{y} \frac{u(t)^{p'}dt}{t^{(\beta-1)p'}}\right] \approx \frac{1}{x^{(2-\alpha-\beta)p'}} \left[\int_{2^{k-1}}^{y} \frac{u(t)^{p'}dt}{t^{(\beta-1)p'}}\right], \ x \in \Delta_k.$$

So, (3.9) is proved. From the definition D we have

$$\left[\int_{2^{k-1}}^{x} \frac{u(t)^{p'}dt}{t^{(\beta-1)p'}}\right]^{\frac{1}{p'}} \leq D \left[\int_{x}^{2^{k+1}} \frac{v_k(s)^q ds}{s^{(2-\alpha-\beta)q}}\right]^{-\frac{1}{q}}, \qquad x \in (2^{k-1}, 2^{k+1}]. \tag{3.10}$$

Applying (3.9), Minkowskii's inequality and (3.10) we write

$$\int_{\Delta_k} v(x)^q \left(\int_{2^{k-1}}^{x} (x-y)^{\alpha+\beta-2} f(y) \frac{u(y)}{y^{\beta-1}} dy\right)^q dx$$

$$\leq \int_{\Delta_k} \frac{v_k(x)^q}{x^{(2-\alpha-\beta)q}} \left(\int_{2^{k-1}}^{x} f(y)^p \left(\int_{2^{k-1}}^{y} \frac{u(t)^{p'}dt}{t^{(\beta-1)p'}}\right)^{\frac{1}{p'}} dy\right)^{\frac{q}{p}} \left(\int_{2^{k-1}}^{x} \frac{u(t)^{p'}dt}{t^{(\beta-1)p'}}\right)^{\frac{q}{p'^2}} dx$$

$$\leq \left(\int_{2^{k-1}}^{2^{k+1}} f(y)^p \left(\int_{2^{k-1}}^{y} \frac{u(t)^{p'}dt}{t^{(\beta-1)p'}}\right)^{\frac{1}{p'}}\right.$$

$$\times \left.\left(\int_{y}^{2^{k+1}} \frac{v_k(x)^q}{x^{(2-\alpha-\beta)q}} \left(\int_{2^{k-1}}^{x} \frac{u(t)^{p'}dt}{t^{(\beta-1)p'}}\right)^{\frac{q}{p'^2}} dx\right)^{\frac{p}{q}} dy\right)^{\frac{q}{p}}$$

$$\ll D^{\frac{q}{p'}} \left(\int_{2^{k-1}}^{2^{k+1}} f(y)^p \left(\int_{2^{k-1}}^{y} \frac{u(t)^{p'}dt}{t^{(\beta-1)p'}}\right)^{\frac{1}{p'}} \left(\int_{y}^{2^{k+1}} \frac{v_k(x)^q dx}{x^{(2-\alpha-\beta)q}}\right)^{\frac{1}{q}} dy\right)^{\frac{q}{p}}$$

$$\leq D^q \left(\int_{2^{k-1}}^{2^{k+1}} f^p\right)^{\frac{q}{p}}$$

and the upper bound $\|L\| \ll D$ follows by Jensen's inequality and the required $C \ll A + D$ is proved. □

Theorem 3.2. *Let* $p > \max(\frac{1}{\alpha}, 1), 0 < q < p < \infty$ *and* $\frac{1}{r} := \frac{1}{q} - \frac{1}{p}$. *Let* $v \in \mathfrak{M}^+$ *and* $u \in \mathfrak{M}^+$ *is monotone decreasing on* $[0, \infty)$.

I) If $\alpha+\beta>2$ then the inequality (3.1) holds if and only if $\mathbb{A}+\mathbb{B}<\infty$, where

$$\mathbb{A}_0(\alpha,\beta) := \left\{\int_0^\infty \left(\int_t^\infty \frac{v(x)^q(\ln\frac{x}{t})^{(\beta-1)q}dx}{x^{(1-\alpha)q}}\right)^{r/q} \times \left(\int_0^t u^{p'}(y)dy\right)^{r/q'} u^{p'}(t)dt\right\}^{1/r},$$

$$\mathbb{A}_1(\alpha,\beta) := \left\{\int_0^\infty \left(\int_t^\infty \frac{v(x)^q dx}{x^{(1-\alpha)q}}\right)^{r/p} \times \left(\int_0^{\frac{t}{2}} \left(\ln\frac{t}{y}\right)^{(\beta-1)p'} u^{p'}(y)dy\right)^{r/p'} \frac{v(t)^q dt}{t^{(1-\alpha)q}}\right\}^{1/r},$$

$$\mathbb{A} := \mathbb{A}_0(\alpha,\beta)+\mathbb{A}_1(\alpha,\beta),$$

$$\begin{aligned}
\mathbb{B}_{k,0}(\alpha,\beta) &:= \left\{\int_{2^{k-1}}^{2^k} \left(\int_{2^k}^{2t} v(x)^q(x-2^k)^{(\alpha+\beta-2)q}dx\right)^{r/q}\right. \\
&\quad\times \left.\left(\int_t^{2^k} \frac{u^{p'}(y)dy}{y^{(\beta-1)p'}}\right)^{r/q'} \frac{u^{p'}(t)dt}{t^{(\beta-1)p'}}\right\}^{1/r}, \\
\mathbb{B}_{k,1}(\alpha,\beta) &:= \left\{\int_{2^k}^{2^{k+1}} \left(\int_{2^k}^{t} v(x)^q dx\right)^{r/p}\right. \\
&\quad\times \left.\left(\int_{\frac{t}{2}}^{2^k} \frac{(2^k-y)^{(\alpha+\beta-2)p'}u^{p'}(y)dy}{y^{(\beta-1)p'}}\right)^{r/p'} v(t)^q dt\right\}^{1/r}, \\
\mathbb{B}_{k,2}(\alpha,\beta) &:= \left\{\int_{2^{k-1}}^{2^k} \left(\int_{t}^{2^{k+1}} v(x)^q(x-t)^{(\alpha+\beta-2)q}dx\right)^{r/q}\right. \\
&\quad\times \left.\left(\int_{2^{k-1}}^{t} \frac{u^{p'}(y)dy}{y^{(\beta-1)p'}}\right)^{r/q'} \frac{u^{p'}(t)dt}{t^{(\beta-1)p'}}\right\}^{1/r}, \\
\mathbb{B}_{k,3}(\alpha,\beta) &:= \left\{\int_{2^k}^{2^{k+1}} \left(\int_{t}^{2^{k+1}} v(x)^q dx\right)^{r/p}\right. \\
&\quad\times \left.\left(\int_{2^k}^{t} \frac{(t-y)^{(\alpha+\beta-2)p'}u^{p'}(y)dy}{y^{(\beta-1)p'}}\right)^{r/p'} v(t)^q dt\right\}^{1/r},
\end{aligned}$$

$$\mathbb{B} := \left(\sum_{k\in\mathbb{Z}} \left(\mathbb{B}^r_{k,0}(\alpha,\beta) + \mathbb{B}^r_{k,1}(\alpha,\beta) + \mathbb{B}^r_{k,2}(\alpha,\beta) + \mathbb{B}^r_{k,3}(\alpha,\beta) \right) \right)^{1/r}.$$

Moreover, $C \approx \mathbb{A} + \mathbb{B}$.
II) If $1 < \alpha + \beta < 2$ *then the inequality* (3.1) *holds if and only if* $\mathbb{A} + \mathbb{D} < \infty$, *where*

$$\begin{aligned} \mathbb{D} \;&:=\; \left(\sum_{k\in\mathbb{Z}} \int_{2^k}^{2^{k+1}} \frac{v(s)^q}{s^{(2-\alpha-\beta)q}} \left(\int_s^{2^{k+1}} \frac{v(t)^q dt}{t^{(2-\alpha-\beta)q}} \right)^{r/p} \left(\int_{2^{k-1}}^s \frac{u^{p'}(t)dt}{t^{(\beta-1)p'}} \right)^{r/p'} ds \right)^{1/r} \\ &=: \left(\sum_{k\in\mathbb{Z}} \mathbb{D}^r_k \right)^{\frac{1}{r}}. \end{aligned}$$

Moreover, $C \approx \mathbb{A} + \mathbb{D}$.

Proof. (I) ($\alpha + \beta > 2$) Arguing similarly to the proof of Theorem 3.1 part (I) and using Theorems 2.1, 2.2, we can see our aim in this part.
(II) ($1 < \alpha + \beta < 2$) Since L is a block-diagonal operator using ([27], Lemma 1), we have

$$\|L\| \approx \left(\sum_{k\in\mathbb{Z}} \|L_k\|^r \right)^{\frac{1}{r}}, \quad q < p, \tag{3.11}$$

and it is sufficient to show, that $\|L\| \ll \mathbb{D}$. Let

$$h(x) := \frac{\chi_{\Delta_k}(x)}{x^{(2-\alpha-\beta)q^2/r}} \left(\int_{2^{k-1}}^x \left[\int_{2^{k-1}}^s \frac{u^{p'}(t)dt}{t^{(\beta-1)p'}} \right]^{\frac{r}{q'}} \left[\int_s^{2^{k+1}} \frac{v_k(t)^q dt}{t^{(2-\alpha-\beta)q}} \right]^{\frac{r}{p}} \frac{u^{p'}(s)ds}{s^{(\beta-1)p'}} \right)^{\frac{q}{r}}.$$

Applying Hölder's inequality, we find

$$\begin{aligned} J_k :=\; & \left(\int_{\Delta_k} v(x)^q \left(\int_{2^{k-1}}^x (x-y)^{\alpha+\beta-2} f(y) \frac{u(y)}{y^{\beta-1}} dy \right)^q dx \right)^{\frac{1}{q}} \\ \le\; & \left(\int_{\Delta_k} v(x)^q h(x)^{\frac{r}{q}} dx \right)^{\frac{1}{r}} \\ \times\; & \left(\int_{\Delta_k} v(x)^q h(x)^{-\frac{p}{q}} \left(\int_{2^{k-1}}^x (x-y)^{\alpha+\beta-2} f(y) \frac{u(y)}{y^{\beta-1}} dy \right)^p dx \right)^{\frac{1}{p}}. \end{aligned}$$

Changing the order of integration and integrating by parts, we have

$$\int_{\Delta_k} v(x)^q h(x)^{\frac{r}{q}} dx = \int_{2^{k-1}}^{2^{k+1}} v_k(x)^q h(x)^{\frac{r}{q}} dx$$

$$= \int_{2^{k-1}}^{2^{k+1}} \frac{v_k(x)^q}{x^{(2-\alpha-\beta)q}} \int_{2^{k-1}}^{x} \left[\int_{2^{k-1}}^{s} \frac{u^{p'}(t)dt}{t^{(\beta-1)p'}}\right]^{\frac{r}{q'}}$$

$$\times \left[\int_{s}^{2^{k+1}} \frac{v_k(t)^q dt}{t^{(2-\alpha-\beta)q}}\right]^{\frac{r}{p}} \frac{u^{p'}(s)ds}{s^{(\beta-1)p'}} dx$$

$$= \int_{2^{k-1}}^{2^{k+1}} \left[\int_{2^{k-1}}^{s} \frac{u^{p'}(t)dt}{t^{(\beta-1)p'}}\right]^{\frac{r}{q'}} \left[\int_{s}^{2^{k+1}} \frac{v_k(t)^q dt}{t^{(2-\alpha-\beta)q}}\right]^{\frac{r}{q}} \frac{u^{p'}(s)ds}{s^{(\beta-1)p'}}$$

$$= \frac{p'}{r} \int_{2^{k-1}}^{2^{k+1}} \left[\int_{s}^{2^{k+1}} \frac{v_k(t)^q dt}{t^{(2-\alpha-\beta)q}}\right]^{\frac{r}{q}} d\left[\int_{2^{k-1}}^{s} \frac{u^{p'}(t)dt}{t^{(\beta-1)p'}}\right]^{\frac{r}{p'}}$$

$$= \frac{p'}{q} \int_{2^{k-1}}^{2^{k+1}} \left[\int_{s}^{2^{k+1}} \frac{v_k(t)^q dt}{t^{(2-\alpha-\beta)q}}\right]^{\frac{r}{p}} \left[\int_{2^{k-1}}^{s} \frac{u^{p'}(t)dt}{t^{(\beta-1)p'}}\right]^{\frac{r}{p'}} \frac{v_k(s)^q ds}{s^{(2-\alpha-\beta)q}}$$

$$= \frac{p'}{q} \int_{\Delta_k} \left[\int_{s}^{2^{k+1}} \frac{v_k(t)^q dt}{t^{(2-\alpha-\beta)q}}\right]^{\frac{r}{p}} \left[\int_{2^{k-1}}^{s} \frac{u^{p'}(t)dt}{t^{(\beta-1)p'}}\right]^{\frac{r}{p'}} \frac{v_k(s)^q ds}{s^{(2-\alpha-\beta)q}}$$

$$= \frac{p'}{q} \mathbb{D}_k^r.$$

Thus, from (3.12)

$$J_k \ll \mathbb{D}_k \left(\int_{\Delta_k} v(x)^q h(x)^{-\frac{p}{q}} \left(\int_{2^{k-1}}^{x} (x-y)^{\alpha+\beta-2} f(y) \frac{u(y)}{y^{\beta-1}} dy\right)^p dx\right)^{\frac{1}{p}}. \quad (3.12)$$

Now we show, that

$$\sup_{t\in\Delta_k} \left(\int_{t}^{2^{k+1}} \frac{v(x)^q h(x)^{-\frac{p}{q}} dx}{x^{(2-\alpha-\beta)p}}\right)^{\frac{1}{p}} \left(\int_{2^{k-1}}^{t} \frac{u^{p'}(s)ds}{s^{(\beta-1)p'}}\right)^{\frac{1}{p'}} \ll 1. \quad (3.13)$$

Let $t \in \Delta_k$. We write

$$\int_{t}^{2^{k+1}} \frac{v(x)^q h(x)^{-\frac{p}{q}} dx}{x^{(2-\alpha-\beta)p}} = \int_{t}^{2^{k+1}} \frac{v(x)^q dx}{x^{(2-\alpha-\beta)p}}$$

$$\times \left(\left[\int_{2^{k-1}}^{x} \left[\int_{2^{k-1}}^{s} \frac{u^{p'}(t)dt}{t^{(\beta-1)p'}}\right]^{\frac{r}{q'}} \left[\int_{s}^{2^{k+1}} \frac{v_k(x)^q dx}{x^{(2-\alpha-\beta)q}}\right]^{\frac{r}{p}} \frac{u^{p'}(s)ds}{s^{(\beta-1)p'}}\right]^{\frac{q}{r}} \frac{1}{x^{(2-\alpha-\beta)q^2/r}}\right)^{-\frac{p}{q}}$$

$$= \int_{t}^{2^{k+1}} \frac{v(x)^q dx}{x^{(2-\alpha-\beta)q}} \left(\int_{2^{k-1}}^{x} \left(\int_{2^{k-1}}^{s} \frac{u^{p'}(t)dt}{t^{(\beta-1)p'}}\right)^{\frac{r}{q'}} \left[\int_{s}^{2^{k+1}} \frac{v_k(z)^q dz}{z^{(2-\alpha-\beta)q}}\right]^{\frac{r}{p}} \frac{u^{p'}(s)ds}{s^{(\beta-1)p'}}\right)^{-\frac{p}{r}}$$

$$\leq \left(\int_{2^{k-1}}^{t}\left[\int_{2^{k-1}}^{s}\frac{u^{p'}(t)dt}{t^{(\beta-1)p'}}\right]^{\frac{r}{q'}}\frac{u^{p'}(s)ds}{s^{(\beta-1)p'}}\right)^{-\frac{p}{r}}=\left(\frac{r}{p'}\right)^{\frac{p}{r}}\left(\int_{2^{k-1}}^{t}\frac{u^{p'}(s)ds}{s^{(\beta-1)p'}}\right)^{-\frac{p}{p'}},$$

and (3.13) follows. Applying the arguments from the proof of Theorem 3.1 with $p = q$ we see, that

$$\left(\int_{\Delta_k} v(x)^q h(x)^{-\frac{p}{q}}\left(\int_{2^{k-1}}^{x}(x-y)^{\alpha+\beta-2}f(y)\frac{u(y)}{y^{\beta-1}}dy\right)^p dx\right)^{\frac{1}{p}} \ll \|f\chi_{[2^{k-1},2^{k+1}]}\|_p.$$

Thus, (3.12) brings

$$\|L_k f\|_q \ll \mathbb{D}_k\|f\chi_{[2^{k-1},2^{k+1}]}\|_p.$$

Consequently, $\|L_k\| \ll \mathbb{D}_k$ and by (3.11) $\|L\| \ll \mathbb{D}$. □

4. Compactness

Theorem 4.1. *Let* $\max(\frac{1}{\alpha},1) < p \leq q < \infty$. *Let* $v \in \mathfrak{M}^+$ *and* $u \in \mathfrak{M}^+$ *is monotone decreasing on* $[0,\infty)$.
I) If $\alpha+\beta > 2$ *the operator* $L_{\alpha,\beta}$ *from* L^p *to* L^q *is compact iff,* $A + B < \infty$ *and*

$$\lim_{t\to 0} A_i(t) = \lim_{t\to\infty} A_i(t) = 0, \quad i = 0,1,$$

$$\lim_{t\to 0} B_i(t) = \lim_{t\to\infty} B_i(t) = 0, \quad i = 0,1.$$

II) If $1 < \alpha+\beta < 2$ *the operator* $L_{\alpha,\beta}$ *from* L^p *to* L^q *is compact iff,* $A + D < \infty$ *and*

$$\lim_{t\to 0} A_i(t) = \lim_{t\to\infty} A_i(t) = 0, \quad i = 0,1, \tag{4.1}$$

$$\lim_{k\to-\infty} D_k = \lim_{k\to+\infty} D_k = 0. \tag{4.2}$$

Proof. (I) $(\alpha+\beta > 2)$ Since in this case, we have *Oinarov–kernel*, therefore the proof of compactness follows from representation of the operator by sum of a compact operator and an operator with a small norm and using Theorems 2.1, 2.2.
(II) $(1 < \alpha+\beta < 2)$ *Necessity.* Since the operator $L_{\alpha,\beta}$ is compact, then $L_{\alpha,\beta}$ is bounded from L^p to L^q and it follows from Theorem 3.1 that $A+B < \infty$. We use the well-known fact that a compact operator maps a weakly convergent sequence into a strongly convergent one. Put

$$f_t(x) = \frac{\chi_{[0,\frac{t}{2}]}(x)(\ln(\frac{t}{x}))^{(\beta-1)(p'-1)}u^{p'-1}(x)}{\left(\int_0^{\frac{t}{2}}(\ln\frac{t}{y})^{(\beta-1)p'}u^{p'}(y)dy\right)^{1/p}}, \ \ t > 0.$$

Then $\|f_t\|_p = 1$ and for any fixed $g \in L^{p'}$ we have by Hölder's inequality that

$$\left|\int_0^\infty f_t(x)g(x)dx\right| \leq \left(\int_0^{\frac{t}{2}}|g(x)|^{p'}dx\right)^{1/p'} \to 0, \ \ t \to 0.$$

Therefore, $f_t \to 0$ is a weakly convergent sequence, and by the hypotheses, we have

$$\lim_{t\to 0}\|L_{\alpha,\beta}f_t\|_q = 0.$$

However, using *Oinarov–kernel condition*

$$\|L_{\alpha,\beta}f_t\|_q = \bigg(\int_0^\infty v^q(x)\bigg(\int_0^x \frac{(\ln\frac{x}{y})^{\beta-1}f_t(y)u(y)dy}{(x-y)^{1-\alpha}}\bigg)^q dx\bigg)^{1/q} \gg A_1(t).$$

Consequently, $\lim_{t\to 0} A_1(t) = 0$. With the same argument with the sequence

$$f_s(x) = \frac{\chi_{[0,\frac{s}{2}]}(x)u^{p'-1}(x)}{\Big(\int_0^{\frac{s}{2}} u^{p'}(y)dy\Big)^{1/p}}, \quad s>0,$$

we obtain $\lim_{t\to 0} A_0(t) = 0$. The second condition in (4.1) follows from the compactness of the dual operator $L^*_{\alpha,\beta}$ on applying similar observations. Let

$$f_{k,t}(x) = \frac{\chi_{[2^{k-1},t]}(x)(\frac{u(x)}{x^{(\beta-1)}})^{p'-1}}{\Big(\int_{2^{k-1}}^t (\frac{u(y)}{y^{(\beta-1)}})^{p'-1}dy\Big)^{1/p}}, \quad t\in[2^k,2^{k+1}], \ k\in\mathbb{Z}.$$

Hence, $\|f_{k,t}\|_p = 1$ and for any fixed $g\in L^{p'}$ we have by Hölder's inequality that

$$\bigg|\int_0^\infty f_{k,t}(x)g(x)dx\bigg| = \bigg|\int_{2^k}^{2^{k+1}} f_{k,t}(x)g(x)dx\bigg| \le \bigg(\int_{2^k}^{2^{k+1}} |g(x)|^{p'}dx\bigg)^{1/p'} \to 0,$$

when $k\to\pm\infty$. Therefore, $f_{k,t}\to 0$ weakly, and we have

$$\lim_{k\to\pm\infty}\ \sup_{t\in[2^k,2^{k+1}]} \|L_{\alpha,\beta}f_{k,t}\|_q = 0.$$

If $x<2^{k-1}$, then $L_{\alpha,\beta}f_{k,t}(x) = 0$, so for all $x>t$ we write,

$$\|L_{\alpha,\beta}f_{k,t}\|_q \ge \bigg(\int_t^\infty v^q(x)\bigg(\int_{2^{k-1}}^t \frac{f_{k,t}(y)u(y)dy}{(x-y)^{2-\alpha-\beta}y^{\beta-1}}\bigg)^q dx\bigg)^{1/q}$$

$$\ge \bigg(\int_t^{2^{k+1}} v^q(x)\bigg(\int_{2^{k-1}}^t \frac{u^{p'}(y)dy}{(x-y)^{2-\alpha-\beta}y^{(\beta-1)p'}}\bigg)^q dx\bigg)^{1/q}$$

$$\gg \bigg(\int_t^{2^{k+1}} \frac{v(x)^q dx}{x^{(2-\alpha-\beta)q}}\bigg)^{1/q}\bigg(\int_{2^{k-1}}^t \frac{u^{p'}(s)ds}{s^{(\beta-1)p'}}\bigg)^{1/p'}, \quad t\in[2^k,2^{k+1}].$$

Therefore,

$$\sup_{t\in[2^k,2^{k+1}]} \|L_{\alpha,\beta}f_{k,t}\| \gg \sup_{t\in[2^k,2^{k+1}]} D_k(t).$$

Consequently, $\lim_{k\to\pm\infty} D_k = 0$. *Sufficiency.* We follow on applying similar arguments from ([19], Theorem 3). Let $0<a<b<\infty$ and

$$P_a f = \chi_{[0,a]}f, \quad Q_b f = \chi_{[b,\infty)}f, \quad P_{ab}f = \chi_{[a,b]}f.$$

Then

$$L_{\alpha,\beta}f = (P_a + P_{ab} + Q_b)L_{\alpha,\beta}(P_a + P_{ab} + Q_b)f$$

$$= P_aL_{\alpha,\beta}P_af + Q_bL_{\alpha,\beta}Q_bf + Q_bL_{\alpha,\beta}P_{ab}f + Q_aL_{\alpha,\beta}P_af + P_{ab}L_{\alpha,\beta}P_{ab}f.$$

We consider each operator from the sum separately and prove, that $L_{\alpha,\beta}$ is compact as a limit of compact operators. For instance, let $v_a := v\chi_{[0,a]}$ and $u_a := u\chi_{[0,a]}$. Then

$$P_a L_{\alpha,\beta} P_a f(x) = v_a(x) \int_0^x \frac{\ln^{\beta-1}(\frac{x}{y}) f(y) u_a(y) dy}{(x-y)^{1-\alpha}},$$

and, applying Theorem 3.1, we see, that

$$\|P_a L_{\alpha,\beta} P_a\|_{L^p \to L^q} \ll \big(\sup_{0<t<a} A_0(t) + \sup_{0<t<a} A_1(t) + \sup_{\{k:2^k<a\}} D_k \big).$$

Hence, by (4.1) and (4.2), we have,

$$\lim_{a\to 0} \|P_a L_{\alpha,\beta} P_a\|_{L^p\to L^q} = 0.$$

Similarly, we find that

$$\lim_{b\to\infty} \|Q_b L_{\alpha,\beta} Q_b\|_{L^p\to L^q} = 0,$$

$$\lim_{b\to\infty} \|Q_b L_{\alpha,\beta} P_{ab}\|_{L^p\to L^q} = 0,$$

$$\lim_{a\to 0} \|Q_a L_{\alpha,\beta} P_a\|_{L^p\to L^q} = 0.$$

To prove that $P_{ab} L_{\alpha,\beta} P_{ab}$ is compact we suppose without a loss of generality, that both factors on the right hand side of (3.2), (3.3) are finite, that is

$$\int_t^\infty \frac{v^q(x)dx}{x^{(1-\alpha)q}} \in (0,\infty),$$

and

$$\int_0^{\frac{t}{2}} \ln^{(\beta-1)p'}(\frac{t}{y}) u^{p'}(y) dy \in (0,\infty),$$

also

$$\int_t^{2^{k+1}} \frac{v(s)^q ds}{s^{(2-\alpha-\beta)q}} \in (0,\infty),$$

$$\int_{2^{k-1}}^t \frac{u^{p'}(s)ds}{s^{(\beta-1)p'}} \in (0,\infty),$$

for all $t \in (0,\infty)$, $k \in \mathbb{Z}$. The kernel of the integral operator $P_{ab} L_{\alpha,\beta} P_{ab}$ is

$$\varphi_{a,b}(x,y) := v(x)\chi_{[a,b]}(x) \frac{\ln^{\beta-1}(\frac{x}{y})}{(x-y)^{1-\alpha}} u(y) \chi_{[a,x]}(y) \chi_{[a,b]}(y).$$

Then

$$J := \left(\int_0^\infty \left(\int_0^\infty |\varphi_{a,b}(x,y)|^{p'} dy \right)^{q/p'} dx \right)^{1/q} =$$

$$\left(\int_a^b v^q(x) \left(\int_a^x \frac{\ln^{(\beta-1)p'}(\frac{x}{y}) u^{p'}(y) dy}{(x-y)^{(1-\alpha)p'}} \right)^{q/p'} dx \right)^{1/q}$$

$$\le \Big(\int_a^b v^q(x)\Big(\int_0^x \frac{\ln^{(\beta-1)p'}(\frac{x}{y})u^{p'}(y)dy}{(x-y)^{(1-\alpha)p'}}\Big)^{q/p'}dx\Big)^{1/q}$$

$$= \Big(\int_a^b v^q(x)\Big(\int_0^{x/2} \frac{\ln^{(\beta-1)p'}(\frac{x}{y})u^{p'}(y)dy}{(x-y)^{(1-\alpha)p'}}\Big)^{q/p'}dx\Big)^{1/q}$$

$$+\Big(\int_a^b v^q(x)\Big(\int_{x/2}^x \frac{\ln^{(\beta-1)p'}(\frac{x}{y})u^{p'}(y)dy}{(x-y)^{(1-\alpha)p'}}\Big)^{q/p'}dx\Big)^{1/q}.$$

Hence, using instruction of the proof of Theorem 3.1,

$$J \ll \Big(\int_a^b \frac{v^q(x)}{x^{(1-\alpha)q}}\Big(\int_0^{x/2} \ln^{(\beta-1)p'}(\frac{x}{y})u^{p'}(y)dy\Big)^{q/p'}dx\Big)^{1/q}$$

$$+\Big(\int_a^b \frac{v^q(x)}{x^{(2-\alpha-\beta)q}}\Big(\int_{x/2}^x \frac{u^{p'}(y)dy}{y^{(\beta-1)p'}}\Big)^{q/p'}dx\Big)^{1/q} < \infty.$$

By well-known result ([6], Chapter XI, Sec 3.2) it implies, that $P_{ab}L_{\alpha,\beta}P_{ab}$ is compact. Therefore, $L_{\alpha,\beta}$ is compact as a limit of compact operators. □

Theorem 4.2. *Let $p > \frac{1}{\alpha}, 0 < q < p < \infty$ and $\frac{1}{r} := \frac{1}{q} - \frac{1}{p}$. Let $v \in \mathfrak{M}^+$ and $u \in \mathfrak{M}^+$ is monotone decreasing on $[0,\infty)$. Then*
I) If $\alpha+\beta > 2$, the operator $L_{\alpha,\beta} : L^p \to L^q$, is compact if and only if $\mathbb{A}+\mathbb{B} < \infty$.
II) If $1 < \alpha+\beta < 2$, the operator $L_{\alpha,\beta} : L^p \to L^q$, is compact if and only if $\mathbb{A}+\mathbb{D} < \infty$.

Proof. (I) $(\alpha+\beta > 2)$ It follows from Theorems 2.1, 2.2, and applying Ando's theorem (see [11, 17] and [24]).
(II) $(1 < \alpha+\beta < 2)$ Necessity follows immediately from Theorem 3.2 and to prove sufficiency we apply Ando's theorem, its extension ([8], Theorem 5.5) and Theorem 2.3 (see also [11, 17]). □

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References

1. K F. Andersen and E.T. Sawyer, *Weighted norm inequalities for the Riemann–Liouville and Weyl fractional integral operators*, Trans. Amer. Math. Soc. **308** (1988), 547–558.
2. T. Chen and G. Sinnamon, *Generalized Hardy operators and normalizing measures*, J. Inequal. Appl. **7** (2002), 829–866.
3. A. Gogatishvili and J. Lang, *The generalized Hardy operators with kernel and variable integral limits in Banach function spaces*, J. Inequal. Appl. **4** (1999), 1–16.
4. G.G. Hardy, J.E. Littlewood and G. Polya, *Inequalities*, Cambridge University Press, 1934.
5. H.P. Heinig and G. Sinnamon, *Mapping properties of integral averaging operators*, Studia Math. **129** (1998), 157–177.
6. L.V. Kantorovich and G.P. Akilov, *Functional analysis*, Nauka, Moscow, 1977 (Russian); Pergamon, Oxford, 1982 (English).
7. V. Kokilashvili and A. Meskhi, *Criteria for the boundedness and compactness of operators with power-logarithmic kernels*, Anal. Math. **27** (2001), 173–185.

8. M.A. Krasnoselskii et al., *Integral operators in spaces of summable functions* Nauka, Moscow, 1966 (Russian); Noordhoff International Publishing, Leiden, 1976 (English).
9. A. Kufner and L.-E. Persson, *Weighted inequalities of Hardy type*, World Scientific, New Jersey. 2003.
10. A. Kufner, L. Maligranda and L.-E. Persson, *The Hardy inequality. About its history and some related results*, Vydavatelsky Servis, Pilsen. 2007.
11. E.N. Lomakina and V.D. Stepanov, *On the Hardy-type integral operators in Banach function spaces*, Publ. Mat. **1**, no. 42 (1998), 165–194.
12. M. Lorente, *A characterization of two weighted norm inequalities for one-sided operators of fractional type*, Canad. J. Math. **49** (1997), 1010–1033.
13. F. Martin-Reyes and E. Sawyer, *Weighted inequalities for Riemann–Liouville fractional integrals of order one and greater*, Proc. Amer. Math. Soc. **106** (1989), 727–733.
14. V.G. Mazya, *Sobolev spaces,* Springer, 1985.
15. A. Meskhi, *Solution of some weight problems for the Riemann–Liouville and Weyl operators*, Georgian Math J. **5** (1998), 565–574.
16. A. Meskhi, *Criteria for the boundedness and compactness of integral transforms with positive kernels*, Proc. Edinburgh Math. Soc. **44** (2001), 267–284.
17. R. Oinarov, *Two-sided estimates of the norm of some classes of integral operators*, Proc. Steklov Inst. Math. **204**, no. 3 (1994), 205–214.
18. B. Opic and A. Kufner, *Hardy-type inequalities*, Pitman Research Notes in Mathematics, Series 219, Longman Science and Technology, Harlow. 1990.
19. D.V. Prokhorov, *On the boundedness and compactness of a class of integral operators*, J. London Math. Soc. **61** (2000), no. 2, 617–628.
20. D.V. Prokhorov and V.D. Stepanov, *Weighted estimates of Riemann–Liouville operators and applications*, Proc. Steklov Inst. Math. **243** (2003), 278–301.
21. G. Sinnamon and V.D. Stepanov, *The weighted Hardy inequality: new proofs and the case* $p = 1$, J. London Math. Soc. **54** (1996), 89–101.
22. V.D. Stepanov, *Two-weighted estimates for Riemann–Liouville integrals*, Math. USSR-Izv. **36**, (1991), 669–681.
23. V.D. Stepanov, *Weighted inequalities for a class of Volterra convolution operators*, J. London Math. Soc. (2) **45** (1992), 232–242.
24. V.D. Stepanov, *Weighted norm inequalities for integral operators and related topics*,Nonlinear analysis, function spaces and applications, Vol. 5 (Prague, 1994), 139–175, Prometheus, Prague, 1994.
25. V.D. Stepanov, *Weighted norm inequalities of Hardy type for a class of integral operators*, J. London Math. Soc. (2) **50** (1994), 105–120.
26. V.D. Stepanov and E.P. Ushakova, *On integral operators with variable limits of integration*, Proc. Steklov Inst. Math. **232** (2001), 290–309.
27. V.D. Stepanov and E.P.Ushakova, *Hardy operator with variable limits on monotone functions*, J. Funct. Spaces Appl. **1**, no. 1 (2003), 1–15.
28. V.D. Stepanov and E.P. Ushakova, *On the geometric mean operator with variable limits of integration*, Proc. Steklov Inst. Math. **260** (2008), 254–278.
29. V.D. Stepanov and E.P. Ushakova, *On boundedness of a certain class of Hardy–Steklov type operators in Lebesgue spaces*, Banach J. Math. Anal. **4** (2010), 28–52.
30. V.D. Stepanov and E.P.Ushakova, *Kernel operators with variable intervals of integration in Lebesgue spaces and applications*, Math. Inequal. Appl. **13**, no. 3 (2010), 449–510.

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