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WEAK ERGODICITY OF NONHOMOGENEOUS MARKOV CHAINS ON NONCOMMUTATIVE L^1-SPACES

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ABSTRACT. In this paper we study certain properties of Dobrushin's ergodicity coefficient for stochastic operators defined on noncommutative L^1-spaces associated with semi-finite von Neumann algebras. Such results extends the well-known classical ones to a noncommutative setting. This allows us to investigate the weak ergodicity of nonhomogeneous discrete Markov processes (NDMP) by means of the ergodicity coefficient. We provide a sufficient conditions for such processes to satisfy the weak ergodicity. Moreover, a necessary and sufficient condition is given for the satisfaction of the L^1-weak ergodicity of NDMP. It is also provided an example showing that L^1-weak ergodicity is weaker that weak ergodicity. We applied the main results to several concrete examples of noncommutative NDMP.

1. INTRODUCTION

It is known (see [19]) that the investigations of asymptotical behavior of iterations of Markov operators on commutative L^1-spaces are very important. On the other hand, these investigations are related with several notions of ergodicity of L^1-contractions of measure spaces. To the investigation of such ergodic properties of Markov operators were devoted lots of papers (see for example, [3, 19]). On the other hand, such kind of operators were studied in noncommutative settings. Since, the study of quantum dynamical systems has had an impetuous growth in the last years, in view of natural applications to various field of mathematics and

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physics. It is then of interest to understand among the various ergodic properties, which ones survive and are meaningful by passing from the classical to the quantum case. Due to noncommutativity, the latter situation is much more complicated than the former. The reader is referred e.g. to [2, 13, 14, 16, 26, 27, 33] for further details relative to some differences between the classical and the quantum situations. It is therefore natural to study the possible generalizations to quantum case of the various ergodic properties known for classical dynamical systems. Mostly, in those investigations homogeneous Markov processes were considered. Many ergodic type theorems have been proved for Markov operators acing in noncommutative L^p-spaces (see for example, [4, 5, 16, 20, 35])

On the other hand, nonhomogeneous Markov processes with general state space have become a subject of interest due to their applications in many branches of mathematics and natural sciences. In many papers (see for example, [21, 15, 28, 34]) the weak ergodicity of nonhomogeneous Markov process are given in terms of Dobrushin's ergodicity coefficient [9]. In [37] some sufficient conditions for weak and strong ergodicity of nonhomogeneous Markov processes are given and estimates of the rate of convergence are proved. Lots of papers were devoted to the investigation of ergodicity of nonhomogeneous Markov chains (see, for example [9]-[17],[32]).

Until now a limited number of investigations are devoted to the ergodic properties of nonhomogeneous Markov processes defined on noncommutative spaces (see [1, 7, 22, 28]). In this paper we are going to study ergodic properties of nonhomogeneous discrete Markov processes defined on noncommutative L^1-spaces. Note that in the context of inhomogeneous Markov chains, ergodicity refers to the asymptotic behavior of products of stochastic operators where the number of factors grows unbounded. In the simplest case, when all factors in the products are identical to the same stochastic operator T, ergodicity corresponds to the investigation of iterations of T. The Dobrushin's ergodicity coefficient is one of the effective tools to study a behavior of such products (see [15] for review) . Therefore, we will define such a ergodicity coefficient of a positive mapping defined on noncommutative L^1-space, and study its properties. In this direction we extend the results of [21] to a noncommutative setting. This allows us to investigate the weak ergodicity of nonhomogeneous discrete Markov processes by means of such ergodicity coefficient. We shall provide sufficient conditions for such processes to satisfy the weak ergodicity. Note that in [10] similar conditions were found for classical ones to satisfy weak ergodicity. Moreover, a necessary and sufficient condition is given for the satisfaction of the L^1-weak ergodicity of NDMP. Note that we also provided an example showing that L^1-weak ergodicity is weaker that weak ergodicity. We apply main results to certain concrete examples of noncommutative NDMP to show them weak ergodicity. It is worth to mention that in [30] a necessary and sufficient condition was found for noncommutative homogeneous Markov processes to satisfy the L^1-strong ergodicity (see also [31]).

2. Preliminaries

Throughout the paper M would be a von Neumann algebra with the unit $\mathbb{1}$ and let τ be a faithful normal semifinite trace on M. Recall that an element $x \in M$ is called *self-adjoint* if $x = x^*$. The set of all self-adjoint elements is denoted by M_{sa}. By M_* we denote a pre-dual space to M (see for more definitions [6]). Let $\mathfrak{N}_\tau = \{x \in M : \tau(|x|) < \infty\}$. Completion $\mathfrak{N}_\tau$ with respect to the norm $\|x\|_1 = \tau(|x|)$ is denoted by $L^1(M, \tau)$. It is known [25] that the spaces $L^1(M, \tau)$ and M_* are isometrically isomorphic, therefore they can be identified. Further we will use this fact without noting.

Theorem 2.1. [25] *The space $L^1(M, \tau)$ coincides with the set*

$$L^1 = \left\{ x = \int_{-\infty}^{\infty} \lambda de_\lambda : \int_{-\infty}^{\infty} |\lambda| d\tau(e_\lambda) < \infty \right\}.$$

Moreover,

$$\|x\|_1 = \int_{-\infty}^{\infty} |\lambda| d\tau(e_\lambda).$$

Besides, if $x, y \in L^1(M, \tau)$ such that $x \geq 0, y \geq 0$ and $x \cdot y = 0$ then $\|x + y\|_1 = \|x\|_1 + \|y\|_1$.

It is known [25] that the equality

$$L^1(M, \tau) = L^1(M_{sa}, \tau) + iL^1(M_{sa}, \tau) \tag{2.1}$$

is valid. Note that $L^1(M_{sa}, \tau)$ is a pre-dual to M_{sa}.

Let $T : L^1(M, \tau) \to L^1(M, \tau)$ be a linear bounded operator. We say that a linear operator T is *positive* is $Tx \geq 0$ whenever $x \geq 0$. A positive operator T is said to be a *contraction* if $\|T(x)\|_1 \leq \|x\|_1$ for all $x \in L^1(M_{sa}, \tau)$. A positive operator T is called *stochastic* if $\|Tx\|_1 = \|x\|_1$, $x \geq 0$. It is clear that any stochastic operator is a contraction. In what follows, by $\Sigma(M)$ we denote the set of all stochastic operators defined on $L^1(M, \tau)$. For a given $y \in L^1(M_{sa}, \tau)$ define a linear operator $T_y : L^1(M_{sa}, \tau) \to L^1(M_{sa}, \tau)$ as follows

$$T_y(x) = \tau(x)y$$

and extend it to $L^1(M, \tau)$ as $T_y x = T_y x_1 + iT_y x_2$, where $x = x_1 + ix_2$, $x_1, x_2 \in L^1(M_{sa}, \tau)$.

Recall that a family of contractions $\{T^{m,n} : L^1(M, \tau) \to L^1(M, \tau)\}$ ($m \leq n$, $m, n \in \mathbb{N}$) is called a *nonhomogeneous discrete Markov process (NDMP)* if one satisfies

$$T^{m,n} = T^{k,n} T^{m,k}$$

for every $m \leq k \leq n$. A NDMP $\{T^{m,n}\}$ is called *nonhomogeneous discrete Markov chain (NDMC)*, if each $T^{m,n}$ is a stochastic operator. A NDMP $\{T^{m,n}\}$ is called *uniformly asymptotically stable* or *uniformly ergodic* if there exist an element $y \in L^1(M_{sa}, \tau)$ such that

$$\lim_{n \to \infty} \|T^{m,n} - T_y\| = 0$$

for any $m \geq 0$.

Recall that if for a NDMP $\{T^{k,m}\}$ one has $T^{k,m} = (T^{0,1})^{m-k}$, then such a process becomes *homogeneous*. In what follows, by $\{T^n\}$ we denote homogeneous Markov process, where $T := T^{0,1}$.

3. Dobrushin ergodicity coefficient

Let M be a von Neumann algebra with faithful normal finite trace τ. Let $L^1(M,\tau)$ be a L^1-space associated with M.

Let $T : L^1(M,\tau) \to L^1(M,\tau)$ be a linear bounded operator. Define

$$X = \{x \in L^1(M_{sa},\tau) : \ \tau(x) = 0\},$$

$$\delta(T) = \sup_{x\in X, x\neq 0} \frac{\|Tx\|_1}{\|x\|_1}, \qquad \alpha(T) = \|T\| - \delta(T). \tag{3.1}$$

The magnitude $\delta(T)$ is called the *Dobrushin ergodicity coefficient* of T.

Remark 3.1. We note that in a commutative case, the notion of the Dobrushin ergodicity coefficient was studied in [8],[9],[36].

We have the following theorem which extends the results of [8],[36].

Theorem 3.2. *Let $T : L^1(M,\tau) \to L^1(M,\tau)$ be a linear bounded operator. Then the following inequality holds*

$$\|Tx\|_1 \le \delta(T)\|x\|_1 + \alpha(T)|\tau(x)| \tag{3.2}$$

for every $x \in L^1(M_{sa},\tau)$.

Proof. Let assume that x is positive. Then $\|x\|_1 = \tau(x)$ and we have

$$\delta(T)\|x\|_1 + \alpha(T)|\tau(x)| = \delta(T)\tau(x) + (\|T\| - \delta(T))\tau(x) = \|T\|\|x\|_1 \ge \|Tx\|_1.$$

So (3.2) is valid. If $x \le 0$ the same argument is used to prove (3.2). Now let $x \in X$ then (3.2) easily follows from (3.1).

Suppose that x is not in one of the above three cases. Then $x = x^+ - x^-$, $\|x^+\|_1 \neq 0$, $\|x^-\|_1 \neq 0$,$\|x^+\|_1 \neq \|x^-\|_1$ (see [6]). Let $\|x^+\|_1 > \|x^-\|_1$. Put

$$y = \frac{\|x^-\|_1}{\|x^+\|_1}x^+ - x^-, \qquad z = \frac{\|x^+\|_1 - \|x^-\|_1}{\|x^+\|_1}x^+.$$

Then $x = y + z$ and $\|x\|_1 = \|y\|_1 + \|z\|_1$, here it has been used Theorem 2.1. It is clear that $y \in X$ and $z \ge 0$, therefore the inequality (3.2) is valid for y and z. Hence, one gets

$$\begin{aligned} \|Tx\|_1 &\le \|Ty\|_1 + \|Tz\|_1 \\ &\le \delta(T)\|y\|_1 + \delta(T)\|z\|_1 + \alpha(T)\tau(z) \\ &= \delta(T)\|x\|_1 + \alpha(T)|\tau(x)|. \end{aligned}$$

This completes the proof. □

Note that the proved theorem extends the results of [8],[36],[24]. Now before formulating a main result of this section we need an auxiliary result. Next lemma has been proved in [24], but for the sake of completeness we provide its proof.

First denote

$$D = \{x \in L^1(M,\tau) : \ x \ge 0, \|x\|_1 = 1\}.$$

Lemma 3.3. *For every $x, y \in L^1(M_{sa}, \tau)$ such that $x-y \in X$ there exist $u, v \in D$, such that*

$$x - y = \frac{\|x-y\|_1}{2}(u - v).$$

Proof. We have $x - y = (x-y)^+ - (x-y)^-$. Define

$$u = \frac{(x-y)^+}{\|(x-y)^+\|_1}, \quad v = \frac{(x-y)^-}{\|(x-y)^-\|_1}.$$

It is clear that $u, v \in D$. Since $x - y \in X$ implies that

$$\begin{aligned} \tau(x-y) &= \tau((x-y)^+) - \tau((x-y)^-) \\ &= \|(x-y)^+\|_1 - \|(x-y)^-\|_1 = 0 \end{aligned}$$

therefore $\|(x-y)^+\|_1 = \|(x-y)^-\|_1$. Using this and the fact $\|x-y\|_1 = \|(x-y)^+\|_1 + \|(x-y)^-\|_1$ we get $\|(x-y)^+\|_1 = \|x-y\|_1/2$. Consequently, we obtain

$$\begin{aligned} u - v &= \frac{(x-y)^+}{\|x-y\|_1/2} - \frac{(x-y)^-}{\|x-y\|_1/2} \\ &= \frac{2}{\|x-y\|_1}(x-y). \end{aligned}$$

□

The next result establishes several properties of the Dobrushin ergodicity coefficient in a noncommutative setting. Note that when M is commutative and τ is finite, similar properties were studied in [21, 15].

Theorem 3.4. *Let $T, S : L^1(M, \tau) \to L^1(M, \tau)$ be stochastic operators. Then the following assertions hold true:*

(i) $0 \le \delta(T) \le 1$;
(ii) $|\delta(T) - \delta(S)| \le \delta(T - S) \le \|T - S\|$;
(iii) $\delta(TS) \le \delta(T)\delta(S)$;
(iv) *if $K : L^1(M_{sa}, \tau) \to L^1(M_{sa}, \tau)$ is a linear bounded operator with $K^*\mathbb{1} = 0$, then $\|TK\| \le \|K\|\delta(T)$;*
(v) *one has*

$$\delta(T) = \sup\left\{\frac{\|Tu - Tv\|_1}{2} : \ u, v \in D\right\}. \tag{3.3}$$

(vi) *if $\delta(T) = 0$, then there is $y \in L^1(M, \tau)$, $y \ge 0$ such that $T = T_y$.*

Proof. (i) is obvious. Let us prove (ii). From (3.1) we immediately find that $\delta(T - S) \le \|T - S\|$. Now let us establish the first inequality. Without loss of generality, we may assume that $\delta(T) \ge \delta(S)$. For an arbitrary $\varepsilon > 0$ from (3.1) one can find $x_\varepsilon \in X$ with $\|x_\varepsilon\|_1 = 1$ such that

$$\delta(T) \le \|Tx_\varepsilon\|_1 + \varepsilon.$$

Then we have

$$\begin{aligned}
\delta(T)-\delta(S) &\leq \|Tx_\varepsilon\|_1+\varepsilon-\sup_{x\in X,\|x\|_1=1}\|Sx\|_1\\
&\leq \|Tx_\varepsilon\|_1-\|Sx_\varepsilon\|_1+\varepsilon\\
&\leq \|(T-S)x_\varepsilon\|_1+\varepsilon\\
&\leq \sup_{x\in X,\|x\|_1=1}\|(T-S)x\|_1+\varepsilon\\
&= \delta(T-S)+\varepsilon,
\end{aligned}$$

and the arbitrariness of ε implies the assertion.

(iii). Let $x\in X$, then the stochasticity of S implies $\tau(Sx)=0$, hence due to (3.2) one finds

$$\begin{aligned}
\|TSx\|_1 &\leq \delta(T)\|Sx\|_1+\alpha(T)|\tau(Sx)|\\
&\leq \delta(T)\delta(S)\|x\|_1
\end{aligned}$$

which yields $\delta(TS)\leq\delta(T)\delta(S)$.

(iv). Let K be as above. Then according to (3.2) for every $x\in L^1(M_{sa},\tau)$ we have

$$\begin{aligned}
\|TKx\|_1 &\leq \delta(T)\|Kx\|_1+\alpha(T)|\tau(Kx)|\\
&\leq \delta(T)\|Kx\|_1+\alpha(T)|\tau(K^*(\mathbb{1})x)|\\
&\leq \|K\|\delta(T)\|\varphi\|_1
\end{aligned}$$

which yields the assertion.

(v). For $x\in X$, $x\neq 0$ using Lemma 3.3 we have

$$\begin{aligned}
\frac{\|Tx\|_1}{\|x\|_1} &= \frac{\|T(x^+-x^-)\|_1}{\|x^+-x^-\|_1}\\
&= \frac{\frac{\|x^+-x^-\|_1}{2}\|T(u-v)\|_1}{\|x^+-x^-\|_1}\\
&= \frac{\|Tu-Tv\|_1}{2}.
\end{aligned}$$

The equality (3.1) with the last one implies (3.3).

(vi). Let $\delta(T)=0$, then from (3.3) one gets $Tu=Tv$ for all $u,v\in D$. Therefore, denote $y:=Tu$. It is clear that $y\in D$. Moreover, $Ty=y$. Let $x\in L^1(M,\tau)$, $x\geq 0$, then noting $\|x\|_1=\tau(x)$ we find

$$Tx=\|x\|_1T\left(\frac{x}{\|x\|_1}\right)=\tau(x)y.$$

If $z\in L^1(M_{sa},\tau)$, then $z=z_+-z_-$, where $z_+,z_-\geq 0$. Therefore

$$T(z)=T(z_+)-T(z_-)=\tau(z_+)y-\tau(z_-)y=\tau(z)y.$$

In general, if $z\in L^1(M,\tau)$, then $z=z_1+iz_2$, where $z_1,z_2\in L^1(M_{sa},\tau)$, hence

$$Tz=Tz_1+iTz_2=\tau(z_1)y+i\tau(z_2)y=\tau(z)y.$$

$\square$

4. Uniform ergodicity

In this section, as an application of Theorem 3.4 we are going to prove the uniform mixing of stochastic operator.

First we recall that a NDMP $\{T^{k,n}\}$ defined on $L^1(M,\tau)$ is *weakly ergodic* if for each $k \in \mathbb{N} \cup \{0\}$ one has

$$\lim_{n\to\infty} \sup_{x,y\in D} \|T^{k,n}x - T^{k,n}y\|_1 = 0.$$

Note that taking into account Theorem 3.4 (v) we obtain that the weak ergodicity is equivalent to the condition $\delta(T^{k,n}) \to 0$ as $n \to \infty$.

Theorem 4.1. *Let $\{T^n\}$ be a discrete homogeneous Markov chain on $L^1(M,\tau)$. The following assertions are equivalent:*

(i) *the chain $\{T^n\}$ is weakly ergodic;*
(ii) *there exists $\rho \in [0,1)$ and $n_0 \in \mathbb{N}$ such that $\delta(T^{n_0}) \le \rho$;*
(iii) *T is uniformly ergodic.*

Proof. The implications (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (i) are obvious. Therefore, to complete the proof, it is enough to show the implication (ii) $\Rightarrow$ (iii). Let $\rho \in [0,1)$ and $n_0 \in \mathbb{N}$ such that $\delta(T^{n_0}) \le \rho$. Now from (iii) and (i) of Theorem 3.4 one gets

$$\delta(T^n) \le \rho^{[n/n_0]} \to 0 \quad \text{as} \quad n \to \infty, \tag{4.1}$$

where $[a]$ stands for the integer part of a.

Let us show that $\{T^n\}$ is a Cauchy sequence w.r.t. to the norm. Indeed, using (iv) of Theorem 3.4 and (4.1) we have

$$\|T^n - T^{n+m}\| = \|T^{n-1}(T - T^{m+1})\| \le \delta(T^{n-1})\|T - T^{m+1}\| \to 0 \quad \text{as} \quad n \to \infty.$$

Hence, there is a stochastic operator Q such that $\|T^n - Q\| \to 0$. Let us show that $Q = T_y$, for some $y \in L^1(M,\tau)$. To do so, due to (vi) of Theorem 3.4 it is enough to establish $\delta(Q) = 0$.

So, using (ii) of Theorem 3.4 we have

$$|\delta(T^n) - \delta(Q)| \le \|T^n - Q\|.$$

Now passing to the limit $n \to \infty$ at the last inequality and taking into account (4.1), we obtain $\delta(Q) = 0$, this is the desired assertion. □

Remark 4.2. Note that the proved theorem is a non-commutative version Bartoszek's result [3]. A similar result has been obtained in [4, 5, 24] without using Dobrushin ergodicity coefficient, when M is a von Neumman algebra with a finite trace.

Remark 4.3. In the proved theorem the condition $\rho < 1$ is crucial, otherwise the statement of the theorem fails. For instance, let us consider the following example. Let $M = \ell_\infty$. Then the corresponding L^1-space coincides with ℓ_1. Define $T : \ell_1 \to \ell_1$ by

$$T(x_1, x_2, x_3 \dots) = (x_1 + x_2, x_3, \dots). \tag{4.2}$$

It is clear that T is a stochastic operator. One can see that $\delta(T^n) = 1$ for all $n \in \mathbb{N}$. On the other hand, $\mathbf{y} = (1, 0, 0 \dots)$ is an invariant vector for T, and one has

$$\|T^n - T_{\mathbf{y}}\| = \sup_{\|\mathbf{x}\|_1=1} \|T^n\mathbf{x} - T_{\mathbf{y}}\mathbf{x}\|_1 \geq \|T^n(\mathbf{e}_{n+1}) - T_{\mathbf{y}}(\mathbf{e}_{n+1})\|_1 = 1$$

where $\mathbf{e}_{n+1} = (\underbrace{0, \dots, 0}_{n}, 1, 0, \dots)$. Hence, T is not uniform ergodic. Note that it satisfies the weaker condition, i.e. for every $\mathbf{x} \in \ell_1$ one has

$$\|T^n\mathbf{x} - T_{\mathbf{y}}\mathbf{x}\|_1 = \left|\sum_{i=1}^{n} x_i - \|\mathbf{x}\|_1\right| + \sum_{i \geq n+1} |x_i| \to 0 \quad \text{as} \quad n \to \infty. \tag{4.3}$$

By $\Sigma(M)_{ue}$ we denote the set of all stochastic operators for which the corresponding homogeneous Markov chain is uniformly ergodic.

Theorem 4.4. *The set $\Sigma(M)_{ue}$ is a norm dense and open subset of $\Sigma(M)$.*

Proof. Take an arbitrary $T \in \Sigma(M)$ and $0 < \varepsilon < 2$. Given $y \in S$ let us denote

$$T_\varepsilon = \left(1 - \frac{\varepsilon}{2}\right)T + \frac{\varepsilon}{2}T_y.$$

It is clear that $T_\varepsilon \in \Sigma(M)$ and $\|T - T_\varepsilon\| < \varepsilon$. Now we show that $T_\varepsilon \in \Sigma(M)_{ue}$. Indeed, by using Lemma 3.3 we have

$$\begin{aligned}
\|T_\varepsilon(x - y)\|_1 &= \frac{\|x - y\|_1}{2}\|T_\varepsilon(u - v)\|_1 \\
&= \frac{\|x - y\|_1}{2}\left\|\left(1 - \frac{\varepsilon}{2}\right)T(u - v) + \frac{\varepsilon}{2}T_y(u - v)\right\|_1 \\
&= \frac{\|x - y\|_1}{2}\left\|\left(1 - \frac{\varepsilon}{2}\right)T(u - v) + \frac{\varepsilon}{2}y - \frac{\varepsilon}{2}y\right\|_1 \\
&= \frac{\|x - y\|_1}{2}\left\|\left(1 - \frac{\varepsilon}{2}\right)T(u - v)\right\|_1 \\
&\leq \left(1 - \frac{\varepsilon}{2}\right)\|x - y\|_1
\end{aligned}$$

which implies $\delta(T_\varepsilon) \leq 1 - \frac{\varepsilon}{2}$. Here $u, v \in D$. Hence, due to Theorem 4.1 we infer that $T_\varepsilon \in \Sigma(M)_{ue}$.

Now let us show that $\Sigma(M)_{ue}$ is a norm open set. First we establish that for each $n \in \mathbb{N}$ the set

$$\Sigma(M)_{ue,n} = \left\{T \in \Sigma(M) : \quad \delta(T^n) < 1\right\}$$

is open. Indeed, take any $T \in \Sigma(M)_{ue,n}$, then $\alpha := \delta(T^n) < 1$. Choose $0 < \beta < 1$ such that $\alpha + \beta < 1$. Then for any $S \in \Sigma(M)$ with $\|S - T\| < \beta/n$ by using (ii)

Theorem 3.4 we find

$$\begin{aligned} |\delta(S^n) - \delta(T^n)| &\leq \|S^n - T^n\| \\ &\leq \|S^{n-1}(S-T)\| + \|(S^{n-1} - T^{n-1})T\| \\ &\leq \|S-T\| + \|S^{n-1} - T^{n-1}T\| \\ &\dots \\ &\leq n\|S-T\| < \beta. \end{aligned}$$

Hence, the last inequality yields that $\delta(S^n) < \delta(T^n) + \beta < 1$, i.e. $S \in \Sigma(M)_{ue,n}$.

Now from the equality

$$\Sigma(M)_{ue} = \bigcup_{n\in\mathbb{N}} \Sigma(M)_{ue,n}$$

we obtain that $\Sigma(M)_{ue}$ is open. The completes the proof. □

Remark 4.5. Note that a similar result has been probed in [5] when $M = B(H)$. So, the proved theorem extends Theorem 2.4 for general von Neumann algebras.

5. Weak ergodicity of nonhomogeneous Markov chains

In this section we study weak ergodicity of nonhomogeneous discrete Markov chains defined on $L^1(M,\tau)$.

Theorem 5.1. *Let $\{T^{k,n}\}$ be a NDMC defined on $L^1(M,\tau)$. If for each $k \in \mathbb{N}\cup\{0\}$ there exist $\lambda_k \in [0,1]$, a number $n_k \in \mathbb{N}$ such that $\delta(T^{k,k+n_k}) \leq \lambda_k$ with*

$$\sum_{j'\geq 0}(1-\lambda_{j'}) = \infty \tag{5.1}$$

for every subsequence $\{j'\}$ of $\{j\}_{j\in\mathbb{N}}$. Then the process $\{T^{k,n}\}$ is weak ergodic.

Proof. Take any $k \in \mathbb{N}\cup\{0\}$. Then due to the condition of Theorem there exist $\lambda_k \in [0,1]$, a number $n_1 \in \mathbb{N}$ such that $\delta(T^{k,k+n_k}) \leq \lambda_k$. For $\ell_1 := k+n_k$ we again apply the given condition, then one can find λ_{ℓ_1}, n_{ℓ_1} such that $\delta(T^{\ell_1,\ell_1+n_{\ell_1}}) \leq \lambda_{\ell_1}$. Now continuing this procedure one finds sequences $\{\ell_j\}$ and $\{\lambda_{\ell_j}\}$ such that

$$\ell_0 = k,\ \ \ell_1 = \ell_0 + n_k,\ \ \ell_2 = \ell_1 + n_{\ell_1}, \dots, \ell_m = \ell_{m-1} + n_{\ell_{m-1}}, \dots$$

and $\delta(T^{\ell_j,\ell_{j+1}}) \leq \lambda_{\ell_j}$.

Now for large enough n one can find M such that

$$M = \max\{j:\ \ell_j + n_j \leq n\}.$$

Then due to (iii) of Theorem 3.4 we get

$$\begin{aligned} \delta(T^{k,n}) &= \delta\big(T^{n,\ell_M} T^{\ell_{M-1},\ell_M} \cdots T^{\ell_0,\ell_1}\big) \\ &\leq \prod_{j=0}^{M-1} \delta(T^{\ell_{M-j},\ell_{M-j+1}}) \\ &\leq \prod_{j=0}^{M-1} \lambda_{\ell_j}. \end{aligned}$$

Now taking into account (5.1), the last inequality implies the weak ergodicity of $\{T^{k,n}\}$. □

It is well-known [32] that one of the most significant conditions for weak ergodicity is the Doeblin's Condition. Now we are going to define some noncommutative analogous of such a condition.

We say that a NDMP $\{T^{k,n}\}$ defied on $L^1(M,\tau)$ satisfies *condition* $\mathfrak{D}$ if there exists $\mu \in D$ and for each k there exist a constant $\lambda_k \in [0,1]$, an integer $n_k \in \mathbb{N}$, and for every $\varphi \in D$, one can find $\sigma_{k,\varphi} \in L^1(M_+,\tau)$ with $\sup\limits_{\varphi} \|\sigma_{k,\varphi}\|_1 \leq \frac{\lambda_k}{4}$ such that

$$T^{k,n_k}\varphi + \sigma_{k,\varphi} \geq \lambda_k \mu, \tag{5.2}$$

and

$$\sum_{j'\geq 0} \lambda_{j'} = \infty \tag{5.3}$$

for every subsequence $\{j'\}$ of $\{j\}_{j\in\mathbb{N}}$.

Theorem 5.2. *Assume that a NDMC $\{T^{k,n}\}$ defined on $L^1(M,\tau)$ satisfies condition $\mathfrak{D}$. Then the process $\{T^{k,n}\}$ is weak ergodic.*

Proof. Fix $k \in \mathbb{N}\cup\{0\}$, and take any two elements $u, v \in D$. According to condition $\mathfrak{D}$, there exist $\lambda_k \in [0,1]$, $n_k \in \mathbb{N}$ such that for those u and v one can find $\sigma_{k,u}, \sigma_{k,v} \in L^1(M_+,\tau)$ with $\|\sigma_{k,u}\|_1 \leq \frac{\lambda_k}{4}$, $\|\sigma_{k,v}\|_1 \leq \frac{\lambda_k}{4}$ such that

$$T^{k,n_k}u + \sigma_{k,u} \geq \lambda_k\mu, \quad T^{k,n_k}v + \sigma_{k,v} \geq \lambda_k\mu. \tag{5.4}$$

Now denote $\sigma_k = \sigma_{k,u} + \sigma_{k,v}$, then we have

$$\|\sigma_k\|_1 \leq \frac{\lambda_k}{2}. \tag{5.5}$$

From (5.4) one finds

$$T^{k,n_k}u + \sigma_k \geq T^{k,n_k}u + \sigma_{k,u} \geq \lambda_k\mu. \tag{5.6}$$

Similarly,

$$T^{k,n_k}v + \sigma_k \geq \lambda_k\mu. \tag{5.7}$$

Therefore, using stochasticity of $T^{k,n}$, and inequality (5.6) with (5.5) implies

$$\begin{aligned}\|T^{k,n_k}u + \sigma_k - \lambda_k\mu\|_1 &= \tau(T^{k,n_k}u) - \underbrace{(\lambda_k\tau(\mu) - \tau(\sigma_k))}_{c_1} \\ &= 1 - c_1 \leq 1 - \frac{\lambda_k}{2}.\end{aligned}$$

By the same argument and using (5.7), we find

$$\|T^{k,n_k}v + \sigma_k - \lambda_k\mu\|_1 = 1 - c_1 \leq 1 - \frac{\lambda_k}{2}.$$

Let us denote

$$u_1 = \frac{1}{1-c_1}(T^{k,n_k}u + \sigma_k - \lambda_k\mu),$$

$$v_1 = \frac{1}{1-c_1}(T^{k,n_k}v + \sigma_k - \lambda_k\mu).$$

It is clear that $u_1, v_1 \in D$.

So, one has

$$\|T^{k,n_k}u - T^{k,n_k}v\|_1 = (1-c_1)\|u_1 - v_1\|_1 \leq 2\left(1 - \frac{\lambda_k}{2}\right). \tag{5.8}$$

Hence, from (3.3) and (5.8) we obtain

$$\delta(T^{k,n_k}) \leq \left(1 - \frac{\lambda_k}{2}\right).$$

Consequently, from (5.3) one gets

$$\sum_{j'\geq 0}(1 - \left(1 - \frac{\lambda_{j'}}{2}\right)) = \sum_{j'\geq 0}\frac{\lambda_{j'}}{2} = \infty$$

which implies that the condition of Theorem 5.1 is satisfied, and this completes the proof. □

6. L^1-WEAK ERGODICITY

Let $\{T^{k,n}\}$ be a NDMP defined on $L^1(M,\tau)$.

Definition 6.1. We say that $\{T^{k,n}\}$ satisfies

(i) the L^1-*weak ergodicity* if for any $u, v \in S$ and $k \in \mathbb{N}\cup\{0\}$ one has

$$\lim_{n\to\infty}\|T^{k,n}u - T^{k,n}v\|_1 = 0. \tag{6.1}$$

(ii) the L^1-*strong ergodicity* if there exists $y \in S$ such that for every $k \in \mathbb{N}\cup\{0\}$ and $u \in S$ one has

$$\lim_{n\to\infty}\|T^{k,n}u - y\|_1 = 0. \tag{6.2}$$

Remark 6.2. It is clear that the weak ergodicity implies the L^1-weak ergodicity. But, the reverse is not true. Indeed, let $M = \ell_\infty$, then the corresponding L^1-space coincides with ℓ_1. Consider an operator $T : \ell_1 \to \ell_1$ given by (4.2). To define a NDMC $\{T^{k,m}\}$ is enough to provide a sequence of stochastic operators $\{T_k\}_{k=1}^\infty$, and in this case one has

$$T^{k,m} = T_m \cdots T_k.$$

Let us define a sequence $\{T_k\}$ by

$$T_k = \begin{cases} T, & \text{if } \sqrt{k} \in \mathbb{N} \\ I, & \text{otherwise} \end{cases} \tag{6.3}$$

where I is the identity mapping.

Denote

$$L_{k,m} = \#\{n \in \mathbb{N} : \ \sqrt{n} \in \mathbb{N}, \ k \leq n \leq m\}.$$

It is clear that for $\mathbf{y} = (1, 0, 0 \dots)$ is an invariant vector for $T^{k,m}$ for every k, m. Moreover, one has $T^{k,m} = T^{L_{k,m}}$, so using (4.3) for each $k \geq 0$ we get

$$\|T^{k,n}\mathbf{x} - T_{\mathbf{y}}\mathbf{x}\|_1 = \|T^{L_{k,n}}\mathbf{x} - T_{\mathbf{y}}\mathbf{x}\|_1 \to 0 \quad \text{as} \quad n \to \infty.$$

for every $\mathbf{x} \in \ell_1$. This means that the defined NDMC is L^1-strong ergodic. But it is not weak ergodic, since $\delta(T^{k,m}) = 1$ (see Remark 4.2).

Remark 6.3. Note that if for each $k \geq 0$ there exists $y_k \in S$ such that for every $u \in S$ one has

$$\lim_{n\to\infty} \|T^{k,n}u - y_k\|_1 = 0, \tag{6.4}$$

then the process is the L^1-strong ergodic. Indeed, it is enough to show that $y_0 = y_k$ for all $k \geq 1$. For any $u, v \in S$, one has $T^{0,n}u \to y_0$, $T^{k,n}u \to y_k$ as $n \to \infty$. From this we conclude that $T^{k,n}(T^{0,k}u) \to y_k$ as $n \to \infty$. Now the equality $T^{0,n}u = T^{k,n}T^{0,k}u$ implies that $y_0 = y_k$.

We say that a NDMP $\{T^{k,n}\}$ defined on $L^1(M, \tau)$ satisfies *condition E* if there exists a dense set $\mathfrak{N}$ in D and for each k there exists $\gamma_k \in [0, 1)$, and every $u, v \in \mathfrak{N}$, one can find $n_0 = n_0(u, v, k) \in \mathbb{N}$ such that

$$\|T^{k,k+n_0}u - T^{k,k+n_0}v\|_1 \leq \gamma_k \|u - v\|_1 \tag{6.5}$$

with

$$\sum_{n=1}^{\infty} (1 - \gamma_{k_n}) = \infty \tag{6.6}$$

for any increasing subsequence $\{k_n\}$ of $\mathbb{N}$.

Theorem 6.4. *Let $\{T^{k,n}\}$ be a NDMP defined on $L^1(M, \tau)$. The following conditions are equivalent:*

(i) *$\{T^{k,n}\}$ satisfies the condition E;*
(ii) *$T^{k,n}$ is L^1-weak ergodic.*

Proof. The implication (ii) $\Rightarrow$ (i) is obvious. Therefore, let us consider (i) $\Rightarrow$ (ii). Assume that $u, v \in D$ and $k \in \mathbb{N} \cup \{0\}$ are fixed.

Then for an arbitrary $\varepsilon > 0$, one can find $\varphi, \psi \in \mathfrak{N}$ such that

$$\|u - \varphi\|_1 < \varepsilon 2^{-4}, \quad \|v - \psi\|_1 < \varepsilon 2^{-4}. \tag{6.7}$$

Due to condition E one can find $\lambda_k \in [0, 1)$ and n_0 such that

$$\|T^{k,k+n_0}\varphi - T^{k,k+n_0}\psi\|_1 \leq \gamma_k \|\varphi - \psi\|_1. \tag{6.8}$$

From (6.8) and (6.7) we obtain

$$\begin{aligned}
\|T^{k,k+n_0}u - T^{k,k+n_0}v\|_1 \quad &\leq \quad \|T^{k,k+n_0}u - T^{k,k+n_0}\varphi\|_1 + \|T^{k,k+n_0}v - T^{k,k+n_0}\psi\|_1 \\
&\quad\quad + \|T^{k,k+n_0}\varphi - T^{k,k+n_0}\psi\|_1 \\
&\leq \quad \varepsilon 2^{-3} + \gamma_k \|\varphi - \psi\|_1 \\
&\leq \quad \varepsilon 2^{-3} + \gamma_k \big(\|u - \varphi\|_1 + \|v - \psi\|_p + \|u - v\|_1\big) \\
&\leq \quad \varepsilon 2^{-2} + \gamma_k \|u - v\|_1.
\end{aligned}$$

Now we claim that there are numbers $\{n_i\}_{i=0}^m \subset \mathbb{N}$ and

$$\|T^{k,K_m}u - T^{k,K_m}v\|_1 \leq \varepsilon 2^{-2}\big(1 + 2 + \cdots + 2^{(m-1)}\big) + \left(\prod_{j=0}^{m-1} \gamma_{K_j}\right)\|u - v\|_1, \tag{6.9}$$

where $K_0 = k, K_{j+1} = k + \sum\limits_{i=0}^{j} n_i$, $j = 0, \dots, m-2$.

Let us prove the inequality (6.9) by induction.

When $m = 1$ we have already proved it. Assume that (6.9) holds at m.

Denote $u_m := T^{k,K_m}u, v_m := T^{k,K_m}v$. It is clear that $u_m, v_m \in D$. Then one can find $\varphi_m, \psi_m \in \mathfrak{N}$ such that

$$\|u_m - \varphi_m\|_1 < \varepsilon 2^{-(m+4)}, \quad \|v_m - \psi_m\|_1 < \varepsilon 2^{-(m+4)}. \tag{6.10}$$

By condition E one can find $n_{m+1} \in \mathbb{N}$ and $\gamma_{K_m} \in [0, 1)$ such that

$$\|T^{K_m,K_m+n_{m+1}}\varphi_m - T^{K_m,K_m+n_{m+1}}\psi_m\|_1 \leq \gamma_{K_m}\|\varphi_m - \psi_m\|_1. \tag{6.11}$$

Now using (6.11),(6.10) and our assumption one gets

$$\begin{aligned}
\|T^{k,K_{m+1}}u - T^{k,K_{m+1}}v\|_1 \;&\leq\; \big\|T^{K_m,K_{m+1}}(u_m - \varphi_m)\big\| + \big\|T^{K_m,K_{m+1}}(v_m - \psi_m)\big\| \\
&\quad + \big\|T^{K_m,K_{m+1}}(\varphi_m - \psi_m)\big\| \\
&\leq\; \varepsilon 2^{-(m+3)} + \gamma_{K_m}\|\varphi_m - \psi_m\|_1 \\
&\leq\; \varepsilon 2^{-(m+3)} + \gamma_{K_m}\big(\|u_m - \varphi_m\|_1 + \|v_m - \psi_m\|_p \\
&\quad + \|u_m - v_m\|_1\big) \\
&\leq\; \varepsilon 2^{-(m+2)} + \gamma_{K_m}\Big(\varepsilon 2^{-2}\big(1 + 2 + \cdots + 2^{(m-1)}\big) \\
&\quad + \left(\prod_{j=0}^{m-1} \gamma_{K_j}\right)\|u - v\|_1\Big) \\
&\leq\; \varepsilon 2^{-2}\big(1 + 2 + \cdots + 2^{m}\big) + \left(\prod_{j=0}^{m} \gamma_{K_j}\right)\|u - v\|_1
\end{aligned}$$

Hence, (6.9) is valid for all $m \in \mathbb{N}$.

Due to (6.6) one can find $m \in \mathbb{N}$ such that $\prod_{j=0}^m \gamma_{K_j} < \varepsilon/4$. Take any $n \geq K_m$, then we have

$$n = K_m + r, \quad 0 \leq r < n_{m+1}$$

hence from (6.9) one finds

$$\begin{aligned}
\|T^{k,n}u - T^{k,n}v\|_1 \;&=\; \big\|T^{K_m,n}\big(T^{k,K_m}u - T^{k,K_m}v\big)\big\|_1 \\
&\leq\; \|T^{k,K_m}u - T^{k,K_m}v\|_1 \\
&\leq\; \varepsilon 2^{-2}\big(1 + 2 + \cdots + 2^{(m-1)}\big) + \varepsilon/2 < \varepsilon
\end{aligned}$$

which implies the L^1-weak ergodicity.

This completes the proof. □

Now we consider two conditions for NDMP which are analogous of Deoblin's condition.

Definition 6.5. Let $\{T^{k,n}\}$ be a NDMP on $L^1(M,\tau)$ and $\mathfrak{N} \subset D$. We say that $\{T^{k,n}\}$ satisfies

(a) *condition* $\mathfrak{D}_1$ on $\mathfrak{N}$ if for each k there exist $y_k \in D$ and a constant $\lambda_k \in [0,1]$, and for every $u, v \in \mathfrak{N}$, one can find an integer $n_k \in \mathbb{N}$ and $\sigma_{k,u}, \sigma_{k,v} \in L^1(M_+,\tau)$ with $\|\sigma_{k,u}\|_1 \le \lambda_k/4$, $\|\sigma_{k,v}\|_1 \le \lambda_k/4$ such that
$$T^{k,n_k}u + \sigma_{k,u} \ge \lambda_k y_k, \quad T^{k,n_k}v + \sigma_{k,v} \ge \lambda_k y_k, \tag{6.12}$$
with
$$\sum_{n=1}^{\infty} \lambda_{k_n} = \infty \tag{6.13}$$
for any increasing subsequence $\{k_n\}$ of $\mathbb{N}$.

(b) *condition* $\mathfrak{D}_2$ on $\mathfrak{N}$ if for each k there exist $y_k \in D$ and a constant $\lambda_k \in [0,1]$, and for every $u \in \mathfrak{N}$, one can find a sequence $\{\sigma_{k,u}^{(n)}\} \subset L^1(M_+,\tau)$ with $\|\sigma_{k,u}^{(n)}\|_1 \to 0$ as $n \to \infty$ such that
$$T^{k,n}u + \sigma_{k,u}^{(n)} \ge \lambda_k y_k, \quad \text{for all} \quad n \ge k \tag{6.14}$$
where $\{\lambda_k\}$ satisfies (6.13).

Next theorem shows that condition $\mathfrak{D}_2$ is stronger than $\mathfrak{D}_1$.

Theorem 6.6. *Assume that a NDMC $\{T^{k,n}\}$ defined on $L^1(M,\tau)$. Then for the following statements:*

(i) *$\{T^{k,n}\}$ satisfies condition $\mathfrak{D}_2$ on D;*
(ii) *$\{T^{k,n}\}$ satisfies condition $\mathfrak{D}_2$ on a dense set $\mathfrak{N}$ in D;*
(iii) *$\{T^{k,n}\}$ satisfies condition $\mathfrak{D}_1$ on a dense set $\mathfrak{N}$ in D.*
(iv) *$\{T^{k,n}\}$ is the L^1-weak ergodic;*
(v) *$\{T^{k,n}\}$ satisfies condition $\mathfrak{D}_1$ on D;*

the implications hold true: (i)⇒(ii)⇒(iii)⇔(iv)⇔(v).

Proof. The implication (i)⇒ (ii) is obvious. Consider (ii)⇒ (iii). For a fixed $k \ge 0$, take arbitrary $u, v \in \mathfrak{N}$. Due to condition $\mathfrak{D}_2$ one can find $\lambda_k \in [0,1]$, $y_k \in D$ and two sequences $\{\sigma_{k,u}^{(n)}\}$, $\{\sigma_{k,v}^{(n)}\}$ with
$$\|\sigma_{k,u}^{(n)}\|_1 \to 0, \quad \|\sigma_{k,u}^{(n)}\|_1 \to 0 \quad \text{as} \quad n \to \infty \tag{6.15}$$
such that
$$T^{k,n}u + \sigma_{k,u}^{(n)} \ge \lambda_k y_k, \ T^{k,n}v + \sigma_{k,v}^{(n)} \ge \lambda_k y_k, \ \text{for all} \ \ n \ge k. \tag{6.16}$$
Due to (6.15) we choose n_k such that
$$\|\sigma_{k,u}^{(n_k)}\|_1 \le \frac{\lambda_k}{4}, \quad \|\sigma_{k,u}^{(n_k)}\|_1 \le \frac{\lambda_k}{4}.$$
Therefore, by denoting $\sigma_{k,u} = \sigma_{k,u}^{(n_k)}$, $\sigma_{k,v} = \sigma_{k,u}^{(n_k)}$ from (6.16) one finds
$$T^{k,n_k}u + \sigma_{k,u} \ge \lambda_k y_k, \ T^{k,n_k}v + \sigma_{k,v} \ge \lambda_k y_k,$$

which yields condition $\mathfrak{D}_1$ on $\mathfrak{N}$.

(iii)$\Rightarrow$ (iv). Fix $k \in \mathbb{N} \cup \{0\}$, and take any two elements $u, v \in D$. Then due Lemma 3.3 one finds $\varphi, \psi \in D$ such that

$$u - v = \frac{\|u - v\|_1}{2}(\varphi - \psi). \tag{6.17}$$

Since $\mathfrak{N}$ is dense, for any $\varepsilon > 0$ one can find $u_1, v_1 \in \mathfrak{N}$ such that

$$\|\varphi - u_1\|_1 < \varepsilon, \quad \|\psi - v_1\|_1 < \varepsilon. \tag{6.18}$$

According to condition $\mathfrak{D}_1$, there exist $y_k \in D$ and $\lambda_k \in [0, 1]$ such that for those u_1 and v_1 one can find $n_k \in \mathbb{N}$ and $\sigma_{k,u_1}, \sigma_{k,v_1} \in L^1(M_+, \tau)$ with $\|\sigma_{k,u_1}\|_1 \leq \frac{\lambda_k}{4}$, $\|\sigma_{k,v_1}\|_1 \leq \frac{\lambda_k}{4}$ one has

$$T^{k,n_k} u_1 + \sigma_{k,u_1} \geq \lambda_k y_k, \quad T^{k,n_k} v_1 + \sigma_{k,v_1} \geq \lambda_k y_k. \tag{6.19}$$

Now denote $\sigma_k = \sigma_{k,u_1} + \sigma_{k,v_1}$, then we have

$$\|\sigma_k\|_1 \leq \frac{\lambda_k}{2}. \tag{6.20}$$

From (6.19) one finds

$$T^{k,n_k} u_1 + \sigma_k \geq T^{k,n_k} u_1 + \sigma_{k,u_1} \geq \lambda_k y_k. \tag{6.21}$$

Similarly,

$$T^{k,n_k} v_1 + \sigma_k \geq \lambda_k y_k. \tag{6.22}$$

Therefore, using stochasticity of $T^{k,n}$, and inequality (6.21) with (6.20) implies

$$\begin{aligned} \|T^{k,n_k} u_1 + \sigma_k - \lambda_k y_k\|_1 &= \tau(T^{k,n_k} u_1) - (\underbrace{\lambda_k \tau(y_k) - \tau(\sigma_k)}_{c_1}) \\ &= 1 - c_1 \leq 1 - \frac{\lambda_k}{2}. \end{aligned}$$

Similarly, using (6.22) one gets

$$\|T^{k,n_k} v_1 + \sigma_k - \lambda_k y\|_1 = 1 - c_1 \leq 1 - \frac{\lambda_k}{2}.$$

Let us denote

$$u_2 = \frac{1}{1 - c_1}(T^{k,n_k} u_1 + \sigma_k - \lambda_k y_k),$$

$$v_2 = \frac{1}{1 - c_1}(T^{k,n_k} v_1 + \sigma_k - \lambda_k y_k).$$

It is clear that $u_2, v_2 \in D$.

So, one has

$$T^{k,n_k} u_1 - T^{k,n_k} v_1 = (1 - c_1)(u_2 - v_2). \tag{6.23}$$

Now from (6.17) and (6.23) we obtain

$$\begin{aligned}
\|T^{k,n_k}u - T^{k,n_k}v\|_1 &= \frac{\|u-v\|_1}{2}\|T^{k,n_k}\varphi - T^{k,n_k}\psi\|_1 \\
&\leq \frac{\|u-v\|_1}{2}\big(\|T^{k,n_k}(\varphi-u_1)\|_1 + \|T^{k,n_k}(\psi-v_1)\|_1 \\
&\quad + \|T^{k,n_k}u_1 - T^{k,n_k}v_1\|_1\big) \\
&\leq \frac{\|u-v\|_1}{2}\big(2\varepsilon + 2(1-c_1)\big) \\
&\leq (\varepsilon + 1 - c_1)\|u-v\|_1 \\
&\leq \left(\varepsilon + 1 - \frac{\lambda_k}{2}\right)\|u-v\|_1.
\end{aligned}$$

Due to the arbitrariness of ε and taking into account (6.13) with Theorem 6.4 we get the desired assertion.

(iv)⇒(v). Let $\{T^{k,n}\}$ be the L^1-weak ergodic. Take any $k \in \mathbb{N}\cup\{0\}$, and fix some element $v_0 \in D$. Then for any $u, v \in D$ from (6.1) one gets

$$\|T^{k,n}u - T^{k,n}v_0\|_1 \to 0, \quad \|T^{k,n}v - T^{k,n}v_0\|_1 \to 0 \quad \text{as} \quad n \to \infty. \tag{6.24}$$

Therefore, one can find $n_k \in \mathbb{N}$ such that

$$\|T^{k,n_k}u - T^{k,n_k}v_0\|_1 \leq \frac{1}{4}, \quad \|T^{k,n_k}v - T^{k,n_k}v_0\|_1 \leq \frac{1}{4}. \tag{6.25}$$

Let us denote

$$\sigma_{k,u} = (T^{k,n_k}u - T^{k,n_k}v_0)_-, \quad \sigma_{k,v} = (T^{k,n_k}v - T^{k,n_k}v_0)_-,$$

where $T^{k,n_k}u - T^{k,n_k}v_0 = (T^{k,n_k}u - T^{k,n_k}v_0)_+ - (T^{k,n_k}u - T^{k,n_k}v_0)_-$ is the Jordan decomposition (see [6]). From (6.25) we obtain

$$\|\sigma_{k,u}\|_1 = \|(T^{k,n_k}u - T^{k,n_k}v_0)_-\|_1 \leq \|T^{k,n_k}u - T^{k,n_k}v_0\|_1 \leq \frac{1}{4}.$$

Similarly, one finds

$$\|\sigma_{k,v}\|_1 \leq \frac{1}{4}.$$

It is clear that

$$\begin{aligned}
T^{k,n_k}u + \sigma_{k,u} &= T^{k,n_k}v_0 + T^{k,n_k}u - T^{k,n_k}v_0 + \sigma_{k,u} \\
&= T^{k,n_k}v_0 + (T^{k,n_k}u - T^{k,n_k}v_0)_+ \\
&\geq T^{k,n_k}v_0.
\end{aligned}$$

Using the same argument, we have

$$T^{k,n_k}v + \sigma_{k,v} \geq T^{k,n_k}v_0.$$

By denoting $\lambda_k = 1$ and $y_k = T^{k,n_k}v_0$, we conclude that the process $\{T^{k,m}\}$ satisfies condition $\mathfrak{D}_1$ on D.

The implication (v)⇒(iii) is obvious. This completes the proof. $\square$

Note that if M is a commutative von Nuemann algebra with finite trace, then analogous theorem to the previous one has been proved in [23].

Corollary 6.7. *Let $\{T^{k,n}\}$ be a NDMC on $L^1(M,\tau)$. If for each k there exist $y_k \in D$ and a constant $\lambda_k \in [0,1]$, and for every $u \in D$, one can find an $\sigma_{k,u} \in L^1(M_+,\tau)$ with $\|\sigma_{k,u}\|_1 \leq \lambda_k/4$ such that*

$$T^{k,k+1}u + \sigma_{k,u} \geq \lambda y_k \tag{6.26}$$

with (6.13). Then $\{T^{k,n}\}$ is the L^1-weak ergodic.

It turns out that the L^1-strong ergodicity implies condition $\mathfrak{D}_2$. Namely, one has

Theorem 6.8. *Let $\{T^{k,n}\}$ be a NDMC on $L^1(M,\tau)$. If $\{T^{k,m}\}$ is the L^1-strong ergodic, then it satisfies condition $\mathfrak{D}_2$ on D.*

Proof. Take any $k \geq 0$ and fix arbitrary $u \in D$. Then from the L^1-strong ergodicity one gets

$$\lim_{n\to\infty} \|T^{k,n}u - y\|_1 = 0, \tag{6.27}$$

since $T_y u = y$. Denote

$$\sigma^{(n)}_{k,u} = (T^{k,n}u - y)_-.$$

From (6.27) we obtain

$$\|\sigma^{(n)}_{k,u}\|_1 = \|(T^{k,n_k}u - y_k)_-\|_1 \leq \|T^{k,n_k}u - y_k\|_1 \to 0 \quad \text{as} \quad n \to \infty.$$

It is clear that

$$T^{k,n}u + \sigma^{(n)}_{k,u} = y + T^{k,n}u - y + \sigma^{(n)}_{k,u} = y + (T^{k,n}u - y)_+ \geq y_k.$$

This implies that condition $\mathfrak{D}_2$ is satisfied on D. □

Remark 6.9. Note that Theorem 6.6 and 6.8 are still valid if one replaces (M,τ) with an arbitrary von Neumann algebra. In this setting, the proofs remain the same as provided ones.

Remark 6.10. Note that in [30] it was proved that if the process $\{T^{k,m}\}$ is homogeneous, then condition $\mathfrak{D}_2$ implies the L^1-strong ergodicity, i.e. these two notions are equivalent.

Problem 6.11. Let $\{T^{k,m}\}$ be a homogeneous Markov chain, then does condition $\mathfrak{D}_1$ implies the L^1-strong ergodicity?

7. Examples

In this section we shall provide certain examples of NDMC which satisfy conditions $\mathfrak{D}$ and $\mathfrak{D}_i$, $i = 1, 2$.

First recall some notions which are needed for our construction. Let $M = M_2(\mathbb{C})$ be the algebra of 2×2 matrices. By $\sigma_1, \sigma_2, \sigma_3$ we denote the Pauli matrices, i.e.

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

It is known (see [6]) that the set $\{\mathbb{1}, \sigma_1, \sigma_2, \sigma_3\}$ forms a basis for $M_2(\mathbb{C})$. Every matrix $x \in M_2(\mathbb{C})$ can be written in this basis as $x = w_0\mathbb{1} + \mathbf{w}\cdot\sigma$ with $w_0 \in \mathbb{C}, \mathbf{w} = (w_1, w_2, w_3) \in \mathbb{C}^3$, here by $\mathbf{w}\cdot\sigma$ we mean the following

$$\mathbf{w}\cdot\sigma = w_1\sigma_1 + w_2\sigma_2 + w_3\sigma_3.$$

The following facts hold (see [29]):

(a) a matrix $x \in M_2(\mathbb{C})$ is positive if and only if $\|\mathbf{w}\| \leq w_0$, where

$$\|\mathbf{w}\| = \sqrt{|w_1|^2 + |w_2|^2 + |w_3|^2}\ ;$$

(b) a linear functional φ on $M_2(\mathbb{C})$ is a state if and only if

$$\varphi(w_0\mathbb{1} + \mathbf{w}\cdot\sigma) = w_0 + \langle \mathbf{w}, \mathbf{f}\rangle,$$

where $\mathbf{f} = (f_1, f_2, f_3) \in \mathbb{R}^3$, $\|\mathbf{f}\| \leq 1$. Here $\langle\cdot,\cdot\rangle$ stands for the standard scalar product on $\mathbb{C}^3$.

(c) A mapping $\Phi : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ is unital, positive and preserves the trace if and only if

$$\Phi(w_0\mathbb{1} + \mathbf{w}\cdot\sigma) = w_0\mathbb{1} + (T\mathbf{w})\cdot\sigma, \tag{7.1}$$

and T is 3×3 real matrix with $\|T(\mathbf{w})\| \leq \|\mathbf{w}\|$ for all $\mathbf{w} \in \mathbb{C}^3$.

As we mentioned above, to define a NDMC $\{T^{k,m}\}$ is enough to provide a sequence of stochastic operators $\{T_k\}_{k=1}^{\infty}$ and in this case one has

$$T^{k,m} = T_m \cdots T_k.$$

1. Now we want to construct NDMC which satisfies condition $\mathfrak{D}_2$. Now let us consider a sequence of unital, positive and trace preserving mappings $\{\Phi_k\}$ of $M_2(\mathbb{C})$. According to (c) to each mapping Φ_k corresponds a real matrix $T^{(k)}$, which will assumed to be diagonal, i.e.

$$T^{(k)} = \begin{pmatrix} \lambda_1^{(k)} & 0 & 0 \\ 0 & \lambda_2^{(k)} & 0 \\ 0 & 0 & \lambda_3^{(k)} \end{pmatrix}, \tag{7.2}$$

where $|\lambda_i^{(k)}| \leq 1$, $i = 1, 2, 3$. Denote $\nu_k = \max\{|\lambda_1^{(k)}|, |\lambda_2^{(k)}||\lambda_3^{(k)}|\}$

Now define $T_k = \Phi_k^*$, $k \in \mathbb{N}$.

Take any $\lambda \in (0, 1)$. Assume that

$$\nu_k \leq 1 - \lambda \ \text{ for all } \ k \in \mathbb{N}.$$

Then for any φ state on $M_2(\mathbb{C})$ one has

$$T_k\varphi \geq \lambda\tau, \quad \forall k \in \mathbb{N}. \tag{7.3}$$

Indeed, to establish (7.3) it is enough to show that

$$\varphi(\Phi_k(x)) \geq \lambda\tau(x) \tag{7.4}$$

for all $x \in M_2(\mathbb{C})$, $x \geq 0$. Now taking into account (b), (c) one can rewrite (7.4) as follows

$$1 + \langle T^{(k)}\mathbf{w}, \mathbf{f}\rangle \geq \lambda, \quad \text{for all } \ \|\mathbf{w}\| \leq 1, \|\mathbf{f}\| \leq 1, \tag{7.5}$$

where $x = \mathbb{1} + \mathbf{w}\cdot\sigma$. Here we should note that without loss of generality one may assume that $w_0 = 1$.

Due to (7.2) the last inequality can be written as

$$(1-\lambda)+\sum_{i=1}^{3}\lambda_i^{(k)}w_if_i\geq 0,$$

where $\mathbf{w}=(w_1,w_2,w_3)$, $\mathbf{f}=(f_1,f_2,f_3)$.

The last inequality is satisfied since

$$\begin{aligned}\left|\sum_{i=1}^{3}\lambda_i^{(k)}w_if_i\right| &\leq \sum_{i=1}^{3}|\lambda_i^{(k)}||w_i||f_i|\\ &\leq \nu_k\|\mathbf{w}\|\|\mathbf{f}\|\\ &\leq 1-\lambda\end{aligned}$$

Due to equality $T^{k,k+1}=T_k$ with the inequality (7.3) implies the satisfaction of condition $\mathfrak{D}$ for NDMP $\{T^{k,m}\}$. Hence, by Theorem 5.2 the process $T^{k,m}$ is weak ergodic.

2. Now let us consider an other example. Take two unital, positive and trace preserving mappings Φ_1 and Φ_2 of $M_2(\mathbb{C})$. The corresponding real matrices we denote by T and S, which will be assumed to be diagonal, i.e.

$$T=\begin{pmatrix}\mu & 0 & 0\\ 0 & \mu & 0\\ 0 & 0 & \mu\end{pmatrix},\quad S=\begin{pmatrix}\mu_1 & 0 & 0\\ 0 & \mu_1 & 0\\ 0 & 0 & \mu_1\end{pmatrix}\tag{7.6}$$

where $|\mu|<1$, and $|\mu_1|=1$.

Now define

$$T_k=\begin{cases}\Phi_1^*, & \text{if } \sqrt{k}\in\mathbb{N}\\ \Phi_2^*, & \text{otherwise}\end{cases}\tag{7.7}$$

Denote

$$L_{k,m}=\#\{n\in\mathbb{N}:\ \sqrt{n}\in\mathbb{N},\ k\leq n\leq m\}.$$

Let $\lambda\in(0,1)$ be a given number. Then for each $k\geq 0$ and any φ state on $M_2(\mathbb{C})$ one can find n_k such that

$$T^{k,n_k}\varphi\geq\lambda\tau.\tag{7.8}$$

Indeed, taking into account (b), (c) with (7.6), (7.7) the last inequality can be rewritten as

$$(1-\lambda)+\sum_{i=1}^{3}\mu^{L_{k,n_k}}\mu_1^{M_k}w_if_i\geq 0,\quad \text{for all }\ \|\mathbf{w}\|\leq 1,\|\mathbf{f}\|\leq 1,\tag{7.9}$$

where $\mathbf{w}=(w_1,w_2,w_3)$, $\mathbf{f}=(f_1,f_2,f_3)$ and $M_k=(n_k^2-k^2)/2$.

Now we choose n_k such that $|\mu|^{L_{k,n_k}} \leq 1-\lambda$, therefore, (7.9) is satisfied since

$$\begin{aligned} \left|\sum_{i=1}^{3} \mu^{L_{k,n_k}} \mu_1^{M_k} w_i f_i\right| &\leq |\mu|^{L_{k,n_k}} \sum_{i=1}^{3} |w_i||f_i| \\ &\leq |\mu|^{L_{k,n_k}} \\ &\leq 1-\lambda \end{aligned}$$

Hence, (7.8) is satisfied. This implies the satisfaction of condition $\mathfrak{D}_1$ for NDMP $\{T^{k,m}\}$. Hence, by Theorem 6.6 the process $T^{k,m}$ is L^1-weak ergodic. Due to equality $T_k\mathbb{1} = \mathbb{1}$ for all $k \in \mathbb{N}$, we conclude this process is the L^1-strong ergodic. So, due to Theorem 6.8 the process satisfies condition $\mathfrak{D}_2$.

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