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ON LINEAR FUNCTIONAL EQUATIONS AND COMPLETENESS OF NORMED SPACES

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Abstract. The aim of this note is to give a type of characterization of Banach spaces in terms of the stability of functional equations. More precisely, we prove that a normed space X is complete if there exists a functional equation of the type

$$\sum_{i=1}^{n} a_i f(\varphi_i(x_1, \dots, x_k)) = 0 \qquad (x_1, \dots, x_k \in D)$$

with given real numbers $a_1, \dots, a_n$, given mappings $\varphi_1 \dots, \varphi_n \colon D^k \to D$ and unknown function $f \colon D \to X$, which has a Hyers–Ulam stability property on an infinite subset D of the integers.

1. Introduction

The basic problem of the stability of functional equations was formulated by Ulam in 1940 in the following form. Suppose that a function f satisfies the so called Cauchy (or additive) functional equation $f(x + y) = f(x) + f(y)$ only approximately. Then does there exist an additive function which approximates f? (cf. also [18].) In 1941, in the paper [6], Hyers gave the following answer to this question. If B_1 and B_2 are Banach spaces and for a nonnegative real number ε and a function $f : B_1 \to B_2$ we have $\|f(x + y) - f(x) - f(y)\| \leq \varepsilon$ $(x, y \in B_1)$ then there exists a unique function $a : B_1 \to B_2$ satisfying $a(x+y)-a(x)-a(y) = 0$ $(x, y \in B_1)$ and $\|f(x)-a(x)\| \leq \varepsilon$ $(x \in B_1)$. We note that the same problem was

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posed (and solved) in 1925 in Pólya and Szegő's book [13] (Teil I, Aufgabe 99) in the context of natural numbers. Stability problems of the type above have been investigated by several authors during the last decades. Surveys of the results can be found, among others, in the papers of Forti [2] Ger [4], Moszner [11] and in the books [7] by Hyers, Isac and Rassias and [8] by Jung .

In most stability theorems for functional equations, the completeness of the target space of the unknown functions contained in the equation is assumed. In the present paper, we investigate the question, whether the stability of a functional equation implies this completeness.

During the 25$^{\text{th}}$ International Symposium on Functional Equations in 1988, this problem was considered by Schwaiger, who proved that if X is a normed space then the stability of Cauchy's functional equation

$$f(x+y) = f(x) + f(y) \qquad (x, y \in \mathbb{Z}) \tag{1}$$

for functions $f\colon \mathbb{Z} \to X$ implies the completeness of X (cf. [14]). Forti and Schwaiger proved in [3] that an analogous statement is valid if the domain of f is an abelian group containing an element of infinite order. In [10], Moszner shoved that the assumptions of this theorem are essential and presented some applications of the result above.

At the Conference on Inequalities and Applications '07 in 2007, Moslehian formulated the problem whether a similar property is valid for the square norm equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \qquad (x, y \in \mathbb{Z}) \tag{2}$$

(cf. [9]). This question was answered in the affirmative by Ger and, independently, by Volkmann during the same meeting as well as, in a more general setting, by Najati in [12]. In [5], an analogous theorem was presented for monomial functional equations on the set of positive integers. In this paper, we prove a generalization of these results for a class of linear functional equations.

2. Stability and completeness

Let D be an infinite subset of $\mathbb{Z}$ and let $X \neq \{0\}$ be a normed space. Let us consider the functional equation

$$\sum_{i=1}^{n} a_i f(\varphi_i(x_1, \ldots, x_k)) = 0 \qquad (x_1, \ldots, x_k \in D) \tag{3}$$

where n is a positive integer, $a_1, \ldots, a_n$ are given real numbers, $\varphi_1 \ldots, \varphi_n \colon D^k \to D$ are given functions and $f\colon D \to X$ is an unknown function.

We say that this functional equation satisfies a (strong) Hyers–Ulam stability property on D, if there exists a polynomial p of degree of at least 1 on D such that pb solves (3) with some $b \in X$, $b \neq 0$; furthermore, if, for a function $f\colon D \to X$, the left hand side of equation (3) is bounded, then there exists a function $g\colon D \to X$, which has the form $g(x) = p(x)c$, $(x \in D)$, with a $c \in X$, satisfies equation (3) and the difference $f - g$ is uniformly bounded on D.

Obviously, the class of equations given in (3) contains the Cauchy equation (1), the square norm equation (2) and several other well-known functional equations as a special case. It is also easy to see that monomial functional equations, polynomial functional equations and, using the notation above, even linear functional equations of the form

$$\sum_{i=1}^{n} a_i f(p_i x + q_i y) = 0 \qquad (x, y \in \mathbb{Z}), \tag{4}$$

where $p_1, \ldots, p_n$ and $q_1, \ldots, q_n$ are given integers and $f : \mathbb{Z} \to X$ is an unknown function, are contained in the class (3). Concerning the investigations of functional equations of this type, we refer to Székelyhidi's works [16], [17], Wilson's article [19] and the references therein. The Hyers–Ulam stability of the class of equations (3), in this general form, has not been investigated yet. The stability of (4), under certain circumstances, was proved in Székelyhidi's paper [15].

In the following, we consider a reverse statement. We note that, obviously, there are several equations belonging to classes (3) and (4), which do not satisfy the Hyers–Ulam stability property above (although they are stable in the sense of Hyers–Ulam).

Theorem. *Let X be a normed space. If there exists a functional equation of type (3) which has the Hyers–Ulam stability property above on an infinite subset D of the integers then X is complete.*

Proof. Let $\bar{X}$ be the completion of X and let $\bar{c} \in \bar{X}$. Let, furthermore, $g\colon D \to \bar{X}$ be given by the formula $g(x) = p(x)\bar{c}$, $(x \in D)$, where p is the polynomial considered in the definition of the stability property. Since X is dense in $\bar{X}$, there exists a function $h\colon D \to X$, such that

$$\|g(x) - h(x)\| \le 1 \qquad (x \in D). \tag{5}$$

Obviously, g is a solution of (3); therefore,

$$\begin{aligned}
&\left\|\sum_{i=1}^{n} a_i h(\varphi_i(x_1, \ldots, x_k))\right\| \\
&\le \left\|\sum_{i=1}^{n} a_i h(\varphi_i(x_1, \ldots, x_k)) - \sum_{i=1}^{n} a_i g(\varphi_i(x_1, \ldots, x_k))\right\| + \left\|\sum_{i=1}^{n} a_i g(\varphi_i(x_1, \ldots, x_k))\right\| \\
&\le \sum_{i=1}^{n} |a_i| =: K \qquad (x_1, \ldots, x_k \in D)
\end{aligned}$$

The stability property implies the existence of a $c \in X \setminus \{0\}$ such that

$$\|h(x) - p(x)c\| \le L \qquad (x \in D), \tag{6}$$

with some $L \in \mathbb{R}$. Using inequalities (5) and (6), we obtain

$$\|p(x)\bar{c} - p(x)c\| \le \|p(x)\bar{c} - h(x)\| + \|h(x) - p(x)c\| \le 1 + L,$$

thus,

$$\|\bar{c} - c\| \le \frac{1 + L}{|p(x)|} \qquad (x \in D,\ p(x) \ne 0).$$

Letting $|x| \longrightarrow \infty$, we get that $\bar{c} = c$ which yields our statement. $\square$

Remark. The result above leads to the question whether there exists a functional equation which is stable in the sense of Hyers–Ulam but the target space of the function contained in the equation is a not necessarily complete linear normed space. In [1], Badora, Przebieracz and Volkmann presented a functional equation which has that property. They proved that if G is a group, X is a linear normed space and there exists a nonnegative real number ε such that a function $f\colon G \to X$ satisfies the inequality

$$\|f(xy) - f(yx)\| \leq \varepsilon \qquad (x, y \in G),$$

then there exists a function $g\colon G \to X$ which solves the equation

$$g(xy) - g(yx) = 0 \qquad (x, y \in G)$$

and fulfils

$$\|f(x) - g(x)\| \leq \varepsilon \qquad (x \in G).$$

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