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THE BEST LOWER BOUND FOR JENSEN'S INEQUALITY WITH THREE FIXED ORDERED VARIABLES

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Abstract. In this paper we establish the best lower bound for the weighted Jensen's discrete inequality with ordered variables applied to a convex function f, in the case when the bound depends on f, weights and three fixed variables. Some applications for particular cases of interest are provided.

1. Introduction

Let $\mathbb{I}$ be an interval in $\mathbb{R}$, let $\mathbf{x} = (x_1, x_2, \cdots, x_n) \in \mathbb{I}^n$, and let $\mathbf{p} = (p_1, p_2, \cdots, p_n)$ be a positive n-tuple such that $p_1 + p_2 + \cdots + p_n = 1$. If $f : \mathbb{I} \to \mathbb{R}$ is a convex function, then the well-known discrete Jensen's inequality [2] states that

$$\Delta_n(f, \mathbf{p}, \mathbf{x}) \ge 0,$$

where the functional

$$\Delta_n(f, \mathbf{p}, \mathbf{x}) = p_1 f(x_1) + p_2 f(x_2) + \cdots + p_n f(x_n) - f(p_1 x_1 + p_2 x_2 + \cdots + p_n x_n)$$

is so called Jensen's difference.

In [1], we presented the theorem below which establishes the best lower bound $L_{\mathbf{p},f}(x_i, x_k)$ of Jensen's difference $\Delta_n(f, \mathbf{p}, \mathbf{x})$ for

$$x_1 \le \cdots \le x_i \le \cdots \le x_k \le \cdots \le x_n$$

and fixed x_i and x_k.

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Theorem 1.1. *Let f be a convex function on $\mathbb{I}$, and let $x_1, x_2, \cdots, x_n \in \mathbb{I}$ $(n \geq 3)$ such that*

$$x_1 \leq x_2 \leq \cdots \leq x_n.$$

For fixed x_i and x_k $(i < k)$, Jensen's difference $\Delta_n(f, \boldsymbol{p}, \boldsymbol{x})$ is minimal when

$$x_1 = x_2 = \cdots = x_{i-1} = x_i, \qquad x_n = x_{n-1} = \cdots = x_{k+1} = x_k,$$

$$x_{i+1} = x_{i+2} = \cdots = x_{k-1} = \frac{P_{1,i}x_i + P_{k,n}x_k}{P_{1,i} + P_{k,n}},$$

where

$$P_{1,i} = p_1 + p_2 + \cdots + p_i, \qquad P_{k,n} = p_k + p_{k+1} + \cdots + p_n;$$

that is

$$\Delta_n(f, \boldsymbol{p}, \boldsymbol{x}) \geq L_{\boldsymbol{p},f}(x_i, x_k),$$

where

$$L_{\boldsymbol{p},f}(x_i, x_k) = P_{1,i}f(x_i) + P_{k,n}f(x_k) - (P_{1,i} + P_{k,n})f\left(\frac{P_{1,i}x_i + P_{k,n}x_k}{P_{1,i} + P_{k,n}}\right).$$

Towards proving Theorem 1.1, we have used the following three lemmas.

Lemma 1.2. *Let p, q be nonnegative real numbers, and let f be a convex function on $\mathbb{I}$. If $a, b, c, d \in \mathbb{I}$ are such that $c, d \in [a, b]$, and*

$$pa + qb = pc + qd,$$

then

$$pf(a) + qf(b) \geq pf(c) + qf(d).$$

Lemma 1.3. *Let f be a convex function on $\mathbb{I}$, and let $x_1, x_2, \cdots, x_n \in \mathbb{I}$ $(n \geq 3)$ be such that*

$$x_1 \leq x_2 \leq \cdots \leq x_n.$$

For fixed $x_i, x_{i+1}, \cdots, x_n$, where $i \in \{2, 3, \cdots, n\}$, Jensen's difference $\Delta_n(f, \boldsymbol{p}, \boldsymbol{x})$ is minimal when

$$x_1 = x_2 = \cdots = x_{i-1} = x_i.$$

Lemma 1.4. *Let f be a convex function on $\mathbb{I}$, and let $x_1, x_2, \cdots, x_n \in \mathbb{I}$ $(n \geq 3)$ be such that*

$$x_1 \leq x_2 \leq \cdots \leq x_n.$$

For fixed $x_1, x_2, \cdots, x_k$, where $k \in \{1, 2, \cdots, n-1\}$, Jensen's difference $\Delta_n(f, \boldsymbol{p}, \boldsymbol{x})$ is minimal when

$$x_k = x_{k+1} = \cdots = x_{n-1} = x_n.$$

In this paper, we will use these lemmas to establish the best lower bound of weighted Jensen's difference for three fixed variables. In addition, we will use the following lemma.

Lemma 1.5. *Let f be a convex function on $\mathbb{I}$, and let $x_1, x_2, \cdots, x_n \in \mathbb{I}$ $(n \geq 4)$ be such that*

$$x_1 \leq \cdots \leq x_i \leq \cdots \leq x_k \leq \cdots \leq x_n,$$

where $1 \leq i < i+1 < k \leq n$. We have

$$\Delta_n(f, \boldsymbol{p}, \boldsymbol{x}) \geq \Delta_n(f, \boldsymbol{p}, \boldsymbol{y}),$$

where

$$y_1 = x_1,\ y_2 = x_2,\ \cdots\ ,\ y_i = x_i,$$
$$y_{i+1} = \cdots = y_{k-1} = \frac{p_{i+1}x_{i+1} + \cdots + p_{k-1}x_{k-1}}{p_{i+1} + \cdots + p_{k-1}},$$
$$y_k = x_k,\ y_{k+1} = x_{k+1},\ \cdots\ ,\ y_n = x_n.$$

Note that the proof of this lemma follows immediately from Lemma 1.3, Lemma 1.4 and Jensen's inequality

$$p_{i+1}f(x_{i+1})+\cdots+p_{k-1}f(x_{k-1}) \geq (p_{i+1}+\cdots+p_{k-1})f\left(\frac{p_{i+1}x_{i+1} + \cdots + p_{k-1}x_{k-1}}{p_{i+1} + \cdots + p_{k-1}}\right).$$

2. Main result

We will establish the best lower bound $L_{\mathbf{p},f}(x_i, x_j, x_k)$ of Jensen's difference $\Delta_n(f, \mathbf{p}, \mathbf{x})$ for $x_1 \leq x_2 \leq \cdots \leq x_n$ and fixed x_i, x_j, x_k such that

$$1 \leq i < j < k \leq n.$$

To do this, we need Lemmas 1.2, 1.3, 1.4, 1.5 and Lemma 2.1 below.

Lemma 2.1. *Let f be a convex function on $\mathbb{I}$, let $a_1, a_2, a_3, a_4, a_5 \in \mathbb{I}$ be such that*

$$a_1 \leq a_2 \leq a_3 \leq a_4 \leq a_5,$$

let r_1, r_2, r_3, r_4, r_5 be positive weights satisfying $r_1 + r_2 + r_3 + r_4 + r_5 = 1$, and let

$$A_1 = \frac{r_1a_1 + (r_2 + r_3)a_3 + r_5a_5}{r_1 + r_2 + r_3 + r_5}$$
$$A_2 = \frac{r_1a_1 + (r_3 + r_4)a_3 + r_5a_5}{r_1 + r_3 + r_4 + r_5}.$$

For fixed a_1, a_3, a_5, the best lower bound of Jensen's difference $\Delta_5(f, \boldsymbol{r}, \boldsymbol{a})$ is

$$\Lambda_{\boldsymbol{r},f}(a_1, a_3, a_5) = \begin{cases} \Lambda_1, & for \quad a_3 \leq \dfrac{r_1a_1 + r_5a_5}{r_1 + r_5} \\ \Lambda_2, & for \quad a_3 \geq \dfrac{r_1a_1 + r_5a_5}{r_1 + r_5} \end{cases},$$

where

$$\Lambda_1 = r_1f(a_1) + (r_2 + r_3)f(a_3) + r_5f(a_5) - (r_1 + r_2 + r_3 + r_5)f(A_1).$$
$$\Lambda_2 = r_1f(a_1) + (r_3 + r_4)f(a_3) + r_5f(a_5) - (r_1 + r_3 + r_4 + r_5)f(A_2).$$

In addition, we have

$$\Delta_5(f, \boldsymbol{r}, \boldsymbol{a}) = \Lambda_1$$

for $a_2 = a_3$ and $a_4 = A_1$, and

$$\Delta_5(f, \boldsymbol{r}, \boldsymbol{a}) = \Lambda_2$$

for $a_2 = A_2$ *and* $a_4 = a_3$.

For any $1 \le i < j \le n$, we introduce the notation

$$P_{i,j} = p_i + p_{i+1} + \cdots + p_j.$$

In addition, let us denote

$$X_1 = \frac{P_{1,i}x_i + P_{i+1,j}x_j + P_{k,n}x_k}{P_{1,i} + P_{i+1,j} + P_{k,n}},$$

$$X_2 = \frac{P_{1,i}x_i + P_{j,k-1}x_j + P_{k,n}x_k}{P_{1,i} + P_{j,k-1} + P_{k,n}},$$

and

$$L_1 = P_{1,i}f(x_i) + P_{i+1,j}f(x_j) + P_{k,n}f(x_k) - (P_{1,i} + P_{i+1,j} + P_{k,n})f(X_1),$$

$$L_2 = P_{1,i}f(x_i) + P_{j,k-1}f(x_j) + P_{k,n}f(x_k) - (P_{1,i} + P_{j,k-1} + P_{k,n})f(X_2).$$

Our main result is given by the following theorem.

Theorem 2.2. *Let* f *be a convex function on* $\mathbb{I}$*, let* $x_1, x_2, \cdots, x_n \in \mathbb{I}$ $(n \ge 4)$ *be such that*

$$x_1 \le x_2 \le \cdots \le x_n,$$

and let $p_1, p_2, \cdots, p_n$ *be positive weights satisfying* $p_1 + p_2 + \cdots + p_n = 1$. *For fixed* x_i, x_j, x_k $(1 \le i < j < k \le n)$*, the best lower bound of Jensen's difference* $\Delta_n(f, \boldsymbol{p}, \boldsymbol{x})$ *is*

$$L_{\boldsymbol{p},f}(x_i, x_j, x_k) = \begin{cases} L_1, \ for \quad x_j \le \dfrac{P_{1,i}x_i + P_{k,n}x_k}{P_{1,i} + P_{k,n}} \\ L_2, \ for \quad x_j \ge \dfrac{P_{1,i}x_i + P_{k,n}x_k}{P_{1,i} + P_{k,n}} \end{cases}.$$

In addition, we have $\Delta_n(f, \boldsymbol{p}, \boldsymbol{x}) = L_1$ *for*

$$\begin{aligned} x_1 = x_2 = \cdots = x_{i-1} = x_i, \\ x_{i+1} = x_{i+2} = \cdots = x_{j-1} = x_j, \\ x_{j+1} = x_{j+2} = \cdots = x_{k-1} = X_1, \\ x_k = x_{k+1} = \cdots = x_{n-1} = x_n, \end{aligned}$$

and $\Delta_n(f, \boldsymbol{p}, \boldsymbol{x}) = L_2$ *for*

$$\begin{aligned} x_1 = x_2 = \cdots = x_{i-1} = x_i, \\ x_{i+1} = x_{i+2} = \cdots = x_{j-1} = X_2, \\ x_j = x_{j+1} = \cdots = x_{k-2} = x_{k-1}, \\ x_k = x_{k+1} = \cdots = x_{n-1} = x_n. \end{aligned}$$

From Theorem 2.2, for the particular case

$$x_j = \frac{P_{1,i}x_i + P_{k,n}x_k}{P_{1,i} + P_{k,n}},$$

which implies

$$X_1 = X_2 = x_j$$

and

$$L_1 = L_2 = P_{1,i}f(x_i) + P_{k,n}f(x_k) - (P_{1,i} + P_{k,n})f\left(\frac{P_{1,i}x_i + P_{k,n}x_k}{P_{1,i} + P_{k,n}}\right),$$

we get Theorem 1.1.

On the other hand, according to Jensen's inequality, we have

$$\begin{aligned} L_2 - L_1 = (P_{j,k-1} - P_{i+1,j})f(x_j) + (P_{1,i} + P_{i+1,j} + P_{k,n})f(X_1) \\ - (P_{1,i} + P_{j,k-1} + P_{k,n})f(X_2) \geq 0 \end{aligned}$$

for $P_{i+1,j} \leq P_{j,k-1}$, and

$$\begin{aligned} L_1 - L_2 = (P_{i+1,j} - P_{j,k-1})f(x_j) + (P_{1,i} + P_{j,k-1} + P_{k,n})f(X_2) \\ - (P_{1,i} + P_{i+1,j} + P_{k,n})f(X_1) \geq 0 \end{aligned}$$

for $P_{i+1,j} \geq P_{j,k-1}$. Thus, from Theorem 2.2, we obtain the following proposition.

Proposition 2.3. *Let f be a convex function on $\mathbb{I}$, let $x_1, x_2, \cdots, x_n \in \mathbb{I}$ $(n \geq 4)$ such that*

$$x_1 \leq \cdots \leq x_i \leq \cdots \leq x_j \leq \cdots \leq x_k \leq \cdots \leq x_n,$$

and let $p_1, p_2, \cdots, p_n$ be positive weights satisfying $p_1 + p_2 + \cdots + p_n = 1$.

(a) *If $P_{i+1,j} \leq P_{j,k-1}$, then*

$$\Delta_n(f, \boldsymbol{p}, \boldsymbol{x}) \geq L_1;$$

(b) *If $P_{i+1,j} \geq P_{j,k-1}$, then*

$$\Delta_n(f, \boldsymbol{p}, \boldsymbol{x}) \geq L_2.$$

Applying Theorem 2.2 and Proposition 2.3 for $f(x) = e^x$ and using the substitutions $a_1 = e^{x_1}$, $a_2 = e^{x_2}$,$\cdots$,$a_n = e^{x_n}$, we obtain

Corollary 2.4. *Let $p_1, p_2, \cdots, p_n$ $(n \geq 4)$ be positive real numbers such that $p_1 + p_2 + \cdots + p_n = 1$, and let*

$$0 < a_1 \leq \cdots \leq a_i \leq \cdots \leq a_j \leq \cdots \leq a_k \leq \cdots \leq a_n.$$

(a) *If $a_j^{P_{1,i}+P_{k,n}} \leq a_i^{P_{1,i}} a_k^{P_{k,n}}$ or $P_{i+1,j} \leq P_{j,k-1}$, then*

$$\begin{aligned} p_1a_1 + p_2a_2 + \cdots + p_na_n - a_1^{p_1}a_2^{p_2}\cdots a_n^{p_n} \geq P_{1,i}a_i + P_{i+1,j}a_j + P_{k,n}a_k \\ - (P_{1,i} + P_{i+1,j} + P_{k,n})\left(a_i^{P_{1,i}}a_j^{P_{i+1,j}}a_k^{P_{k,n}}\right)^{\frac{1}{P_{1,i}+P_{i+1,j}+P_{k,n}}}, \end{aligned}$$

with equality for

$$\begin{aligned} a_1 &= a_2 = \cdots = a_{i-1} = a_i, \\ a_{i+1} &= a_{i+2} = \cdots = a_{j-1} = a_j, \\ a_{j+1} &= a_{j+2} = \cdots = a_{k-1} = \left(a_i^{P_{1,i}} a_j^{P_{i+1,j}} a_k^{P_{k,n}}\right)^{\frac{1}{P_{1,i}+P_{i+1,j}+P_{k,n}}}, \\ a_n &= a_{n-1} = \cdots = a_{k+1} = a_k; \end{aligned}$$

(b) *If* $a_j^{P_{1,i}+P_{k,n}} \geq a_i^{P_{1,i}} a_k^{P_{k,n}}$ *or* $P_{i+1,j} \geq P_{j,k-1}$, *then*

$$\begin{aligned} p_1a_1+p_2a_2 + \cdots + p_na_n - a_1^{p_1}a_2^{p_2}\cdots a_n^{p_n} &\geq P_{1,i}a_i + P_{j,k-1}a_j + P_{k,n}a_k \\ &- (P_{1,i} + P_{j,k-1} + P_{k,n}) \left(a_i^{P_{1,i}} a_j^{P_{j,k-1}} a_k^{P_{k,n}}\right)^{\frac{1}{P_{1,i}+P_{j,k-1}+P_{k,n}}}, \end{aligned}$$

with equality for

$$\begin{aligned} a_1 &= a_2 = \cdots = a_{i-1} = a_i, \\ a_{i+1} &= a_{i+2} = \cdots = a_{j-1} = \left(a_i^{P_{1,i}} a_j^{P_{j,k-1}} a_k^{P_{k,n}}\right)^{\frac{1}{P_{1,i}+P_{j,k-1}+P_{k,n}}}, \\ a_{k-1} &= a_{k-2} = \cdots = a_{j+1} = a_j, \\ a_n &= a_{n-1} = \cdots = a_{k+1} = a_k. \end{aligned}$$

Applying Theorem 2.2 and Proposition 2.3 for $f(x) = -\ln x$, we obtain

Corollary 2.5. *Let* $p_1, p_2, \cdots, p_n$ $(n \geq 4)$ *be positive real numbers such that* $p_1 + p_2 + \cdots + p_n = 1$, *let*

$$0 < x_1 \leq \cdots \leq x_i \leq \cdots \leq x_j \leq \cdots \leq x_k \leq \cdots \leq x_n,$$

and let

$$X_1 = \frac{P_{1,i}x_i + P_{i+1,j}x_j + P_{k,n}x_k}{P_{1,i} + P_{i+1,j} + P_{k,n}}, \qquad X_2 = \frac{P_{1,i}x_i + P_{j,k-1}x_j + P_{k,n}x_k}{P_{1,i} + P_{j,k-1} + P_{k,n}}.$$

(a) *If* $x_j \leq \dfrac{P_{1,i}x_i + P_{k,n}x_k}{P_{1,i} + P_{k,n}}$ *or* $P_{i+1,j} \leq P_{j,k-1}$, *then*

$$\frac{p_1x_1 + p_2x_2 + \cdots + p_nx_n}{x_1^{p_1}x_2^{p_2}\cdots x_n^{p_n}} \geq \frac{X_1^{P_{1,i}+P_{i+1,j}+P_{k,n}}}{x_i^{P_{1,i}} x_j^{P_{i+1,j}} x_k^{P_{k,n}}},$$

with equality for

$$\begin{aligned} x_1 &= x_2 = \cdots = x_{i-1} = x_i, \\ x_{i+1} &= x_{i+2} = \cdots = x_{j-1} = x_j, \\ x_{j+1} &= x_{j+2} = \cdots = x_{k-1} = X_1, \\ x_n &= x_{n-1} = \cdots = x_{k+1} = x_k; \end{aligned}$$

(b) *If* $x_j \geq \dfrac{P_{1,i}x_i + P_{k,n}x_k}{P_{1,i} + P_{k,n}}$ *or* $P_{i+1,j} \geq P_{j,k-1}$, *then*

$$\frac{p_1x_1 + p_2x_2 + \cdots + p_nx_n}{x_1^{p_1}x_2^{p_2}\cdots x_n^{p_n}} \geq \frac{X_2^{P_{1,i}+P_{j,k-1}+P_{k,n}}}{x_i^{P_{1,i}} x_j^{P_{j,k-1}} x_k^{P_{k,n}}},$$

with equality for

$$x_1 = x_2 = \cdots = x_{i-1} = x_i,$$
$$x_{i+1} = x_{i+2} = \cdots = x_{j-1} = X_2,$$
$$x_{k-1} = x_{k-2} = \cdots = x_{j+1} = x_j,$$
$$x_n = x_{n-1} = \cdots = x_{k+1} = x_k.$$

3. Proof of Lemma 2.1

Let us denote

$$R_1 = r_1 + r_2 + r_3 + r_5, \quad B_1 = \frac{r_1a_1 + r_2a_2 + r_3a_3 + r_5a_5}{r_1 + r_2 + r_3 + r_5},$$
$$R_2 = r_1 + r_3 + r_4 + r_5, \quad B_2 = \frac{r_1a_1 + r_3a_3 + r_4a_4 + r_5a_5}{r_1 + r_3 + r_4 + r_5}.$$

We have two cases to consider.

Case 1: $(r_1 + r_5)a_3 \le r_1a_1 + r_5a_5$. We need to show that $\Delta_5(f, \mathbf{r}, \mathbf{a}) \ge \Lambda_1$, that is,

$$r_2f(a_2) + r_4f(a_4) + R_1f(A_1) \ge r_2f(a_3) + f(r_1a_1 + r_2a_2 + r_3a_3 + r_4a_4 + r_5a_5).$$

By Jensen's inequality, we have

$$r_4f(a_4) + R_1f(B_1) \ge f(r_1a_1 + r_2a_2 + r_3a_3 + r_4a_4 + r_5a_5).$$

Thus, it suffices to show that

$$r_2f(a_2) + R_1f(A_1) \ge r_2f(a_3) + R_1f(B_1). \tag{3.1}$$

From $(r_1 + r_5)a_3 \le r_1a_1 + r_5a_5$, it is easy to prove that

$$a_3, B_1 \in [a_2, A_1].$$

In addition, we have

$$r_2a_2 + R_1A_1 = r_2a_3 + R_1B_1.$$

Therefore, (3.1) is true according to Lemma 1.2.

Case 2: $(r_1 + r_5)a_3 \ge r_1a_1 + r_5a_5$. We can write the desired inequality as

$$r_2f(a_2) + r_4f(a_4) + R_2f(A_2) \ge r_4f(a_3) + f(r_1a_1 + r_2a_2 + r_3a_3 + r_4a_4 + r_5a_5).$$

Using Jensen's inequality

$$r_2f(a_2) + R_2f(B_2) \ge f(r_1a_1 + r_2a_2 + r_3a_3 + r_4a_4 + r_5a_5),$$

it suffices to show that

$$r_4f(a_4) + R_2f(A_2) \ge r_4f(a_3) + R_2f(B_2). \tag{3.2}$$

From $(r_1 + r_5)a_3 \ge r_1a_1 + r_5a_5$, we get

$$a_3, B_2 \in [A_2, a_4].$$

Since

$$r_4a_4 + R_2A_2 = r_4a_3 + R_2B_2,$$

(3.2) follows by Lemma 1.2. Thus, the proof of Lemma 1.5 is completed.

4. Proof of Theorem 2.2

Let us denote

$$X_{ij} = \frac{p_{i+1}x_{i+1} + \cdots + p_{j-1}x_{j-1}}{P_{i+1,j-1}},$$
$$Y_{jk} = \frac{p_{j+1}x_{j+1} + \cdots + p_{k-1}x_{k-1}}{P_{j+1,k-1}}.$$

According to Lemmas 1.3, 1.4 and 1.5, we have

$$\Delta_n(f,\mathbf{p},\mathbf{x}) \ge \Delta_n(f,\mathbf{p},\mathbf{y}), \tag{4.1}$$

where

$$y_1 = y_2 = \cdots = y_i = x_i,$$
$$y_{i+1} = \cdots = y_{j-1} = X_{ij},$$
$$y_j = x_j,$$
$$y_{j+1} = \cdots = y_{k-1} = Y_{jk},$$
$$y_k = y_{k+1} = \cdots = y_n = x_k,$$

and hence

$$\begin{aligned}\Delta_n(f,\mathbf{p},\mathbf{y}) =& P_{1,i}f(x_i) + P_{i+1,j-1}f(X_{ij}) + p_jf(x_j) + P_{j+1,k-1}f(Y_{jk}) + P_{k,n}f(x_k)\\ &- f(P_{1,i}x_i + P_{i+1,j-1}X_{ij} + p_jx_j + P_{j+1,k-1}Y_{jk} + P_{k,n}x_k).\end{aligned}$$

Case 1: $(P_{1,i} + P_{k,n})x_j \le P_{1,i}x_i + P_{k,n}x_k$. According to (4.1), it suffices to prove that $\Delta_n(f,\mathbf{p},\mathbf{y}) \ge L_1$, which is equivalent to

$$\begin{aligned}&P_{i+1,j-1}f(X_{ij}) + P_{j+1,k-1}f(Y_{jk}) + (P_{1,i} + P_{i+1,j-1} + p_j + P_{k,n})f(X_1) \ge\\ \ge& P_{i+1,j-1}f(x_j) + f(P_{1,i}x_i + P_{i+1,j-1}X_{ij} + p_jx_j + P_{j+1,k-1}Y_{jk} + P_{k,n}x_k).\end{aligned} \tag{4.2}$$

Using the substitutions

$$r_1 = P_{1,i},\quad r_2 = P_{i+1,j-1},\quad r_3 = p_j,\quad r_4 = P_{j+1,k-1},\quad r_5 = P_{k,n},$$
$$a_1 = x_i,\quad a_2 = X_{ij},\quad a_3 = x_j,\quad a_4 = Y_{jk},\quad a_5 = x_k,$$

the condition $(P_{1,i} + P_{k,n})x_j \le P_{1,i}x_i + P_{k,n}x_k$ becomes $(r_1 + r_5)a_3 \le r_1a_1 + r_5a_5$, while the inequality (4.2) turns into

$$\begin{aligned}&r_2f(a_2) + r_4f(a_4) + (r_1 + r_2 + r_3 + r_5)f(A_1) \ge\\ &\ge r_2f(a_3) + f(r_1a_1 + r_2a_2 + r_3a_3 + r_4a_4 + r_5a_5),\end{aligned} \tag{4.3}$$

where

$$A_1 = \frac{r_1a_1 + (r_2 + r_3)a_3 + r_5a_5}{r_1 + r_2 + r_3 + r_5}.$$

The inequality (4.3) is equivalent to $\Delta_5(f,\mathbf{r},\mathbf{a}) \ge \Lambda_1$ in Lemma 2.1.

Case 2: $(P_{1,i} + P_{k,n})x_j \ge P_{1,i}x_i + P_{k,n}x_k$. According to (4.1), it suffices to prove that $\Delta_n(f,\mathbf{p},\mathbf{y}) \ge L_2$, which is equivalent to

$$\begin{aligned}&P_{i+1,j-1}f(X_{ij}) + P_{j+1,k-1}f(Y_{jk}) + (P_{1,i} + p_j + P_{j+1,k-1} + P_{k,n})f(X_2) \ge\\ \ge& P_{j+1,k-1}f(x_j) + f(P_{1,i}x_i + P_{i+1,j-1}X_{ij} + p_jx_j + P_{j+1,k-1}Y_{jk} + P_{k,n}x_k).\end{aligned} \tag{4.4}$$

Using the same substitutions as the ones from the case 1, the condition $(P_{1,i} + P_{k,n})x_j \ge P_{1,i}x_i + P_{k,n}x_k$ becomes $(r_1 + r_5)a_3 \ge r_1a_1 + r_5a_5$, while the inequality (4.4) turns into

$$r_2f(a_2) + r_4f(a_4) + (r_1 + r_3 + r_4 + r_5)f(A_2) \ge$$
$$\ge r_4f(a_3) + f(r_1a_1 + r_2a_2 + r_3a_3 + r_4a_4 + r_5a_5), \tag{4.5}$$

where

$$A_2 = \frac{r_1a_1 + (r_3 + r_4)a_3 + r_5a_5}{r_1 + r_3 + r_4 + r_5}.$$

Since (4.5) is equivalent to the inequality $\Delta_5(f, \mathbf{r}, \mathbf{a}) \ge \Lambda_2$ in Lemma 2.1, the proof is completed.

5. Applications

Proposition 5.1. *If $a_1, a_2, \cdots, a_n$ $(n \ge 3)$ are positive real numbers such that*

$$a_1 \le a_2 \le \cdots \le a_n,$$

then (see [3])

$$(a) \qquad a_1 + a_2 + \cdots + a_n - n\sqrt[n]{a_1a_2\cdots a_n} \ge \frac{1}{3}(2\sqrt{a_1} - \sqrt{a_{n-1}} - \sqrt{a_n})^2;$$

$$(b) \qquad a_1 + a_2 + \cdots + a_n - n\sqrt[n]{a_1a_2\cdots a_n} \ge \frac{1}{2}(2\sqrt{a_2} - \sqrt{a_{n-1}} - \sqrt{a_n})^2.$$

Proof. (a) In the case $n \ge 4$, we apply Corollary 2.4 for $p_1 = p_2 = \cdots = p_n = \dfrac{1}{n}$, $i = 1$, $j = n - 1$ and $k = n$. We have

$$P_{1,i} = \frac{1}{n}, \quad P_{i+1,j} = \frac{n-2}{n}, \quad P_{j,k-1} = \frac{1}{n}, \quad P_{k,n} = \frac{1}{n}.$$

Since $P_{i+1,j} > P_{j,k-1}$, by Corollary 2.4 we have

$$a_1 + a_2 + \cdots + a_n - n\sqrt[n]{a_1a_2\cdots a_n} \ge a_1 + a_{n-1} + a_n - 3\sqrt[3]{a_1a_{n-1}a_n}.$$

Notice that this inequality is also true (as identity) for $n = 3$. Therefore, it suffices to prove that

$$a_1 + a_{n-1} + a_n - 3\sqrt[3]{a_1a_{n-1}a_n} \ge \frac{1}{3}(2\sqrt{a_1} - \sqrt{a_{n-1}} - \sqrt{a_n})^2,$$

which is equivalent to

$$2(a_{n-1} + a_n) + 4\sqrt{a_1}(\sqrt{a_{n-1}} + \sqrt{a_n}) - 2\sqrt{a_{n-1}a_n} - 9\sqrt[3]{a_1a_{n-1}a_n} - a_1 \ge 0.$$

Taking into account that $a_{n-1} + a_n \ge 2\sqrt{a_{n-1}a_n}$ and $\sqrt{a_{n-1}} + \sqrt{a_n} \ge 2\sqrt[4]{a_{n-1}a_n}$, it is enough to show that

$$2\sqrt{a_{n-1}a_n} + 8\sqrt[4]{a_1^2a_{n-1}a_n} - 9\sqrt[3]{a_1a_{n-1}a_n} - a_1 \ge 0.$$

Since this inequality is homogeneous in a_1, a_{n-1} and a_n, without loss of generality, assume that $a_1 = 1$, $a_n \ge a_{n-1} \ge 1$. In addition, using the notation $x = \sqrt[12]{a_{n-1}a_n}$, $x \ge 1$, we can write the inequality as

$$2x^6 - 9x^4 + 8x^3 - 1 \ge 0.$$

This is true since

$$2x^6 - 9x^4 + 8x^3 - 1 = (x-1)^3(2x^3 + 6x^2 + 3x + 1) \geq 0.$$

(b) If $n = 3$, then the inequality is equivalent to

$$2a_1 + a_2 + a_3 + 2\sqrt{a_2a_3} \geq 6\sqrt[3]{a_1a_2a_3},$$

which is a consequence of the AM-GM inequality.

Consider now that $n \geq 4$, and apply Corollary 2.4 for $p_1 = p_2 = \cdots = p_n = \frac{1}{n}$, $i = 2$, $j = n-1$ and $k = n$. We have

$$P_{1,i} = \frac{2}{n}, \quad P_{i+1,j} = \frac{n-3}{n}, \quad P_{j,k-1} = \frac{1}{n}, \quad P_{k,n} = \frac{1}{n}.$$

By Corollary 2.4, since $P_{i+1,j} \geq P_{j,k-1}$, we have

$$a_1 + a_2 + \cdots + a_n - n\sqrt[n]{a_1a_2\cdots a_n} \geq 2a_2 + a_{n-1} + a_n - 4\sqrt[4]{a_2^2a_{n-1}a_n}.$$

Therefore, it suffices to prove that

$$2a_2 + a_{n-1} + a_n - 4\sqrt[4]{a_2^2a_{n-1}a_n} \geq \frac{1}{2}(2\sqrt{a_2} - \sqrt{a_{n-1}} - \sqrt{a_n})^2,$$

which is equivalent to the obvious inequality

$$(\sqrt{a_{n-1}} - \sqrt{a_n})^2 + 4\sqrt{a_2}(\sqrt[4]{a_{n-1}} - \sqrt[4]{a_n})^2 \geq 0.$$

Both inequalities in (a) and (b) become equalities if and only if $a_1 = a_2 = \cdots = a_n$.

□

Proposition 5.2. *If $a_1, a_2, \cdots, a_n$ $(n \geq 3)$ are positive real numbers such that*

$$a_1 \leq a_2 \leq \cdots \leq a_n,$$

then

$$(a) \qquad a_1 + a_2 + \cdots + a_n - n\sqrt[n]{a_1a_2\cdots a_n} \geq \frac{1}{4}(\sqrt{a_1} + \sqrt{a_2} - 2\sqrt{a_n})^2;$$

$$(b) \qquad a_1 + a_2 + \cdots + a_n - n\sqrt[n]{a_1a_2\cdots a_n} \geq \frac{1}{2}(\sqrt{a_1} + \sqrt{a_2} - 2\sqrt{a_{n-1}})^2.$$

Proof. (a) In the case $n \geq 4$, we apply Corollary 2.4 for $p_1 = p_2 = \cdots = p_n = \frac{1}{n}$, $i = 1$, $j = 2$ and $k = n$. We have

$$P_{1,i} = \frac{1}{n}, \quad P_{i+1,j} = \frac{1}{n}, \quad P_{j,k-1} = \frac{n-2}{n}, \quad P_{k,n} = \frac{1}{n}.$$

Since $P_{i+1,j} < P_{j,k-1}$, by Corollary 2.4 we have

$$a_1 + a_2 + \cdots + a_n - n\sqrt[n]{a_1a_2\cdots a_n} \geq a_1 + a_2 + a_n - 3\sqrt[3]{a_1a_2a_n}.$$

Clearly, this inequality is also true (as identity) for $n = 3$. Then, it suffices to prove that

$$a_1 + a_2 + a_n - 3\sqrt[3]{a_1a_2a_n} \geq \frac{1}{4}(\sqrt{a_1} + \sqrt{a_2} - 2\sqrt{a_n})^2,$$

which is equivalent to

$$3(a_1+a_2)+4\sqrt{a_n}(\sqrt{a_1}+\sqrt{a_2})-2\sqrt{a_1a_2}-12\sqrt[3]{a_1a_2a_n}\ge 0.$$

Taking into account that $a_1+a_2\ge 2\sqrt{a_1a_2}$ and $\sqrt{a_1}+\sqrt{a_2}\ge 2\sqrt[4]{a_1a_2}$, it is enough to show that

$$\sqrt{a_1a_2}+2\sqrt[4]{a_1a_2a_n^2}-3\sqrt[3]{a_1a_2a_n}\ge 0,$$

which follows by the AM-GM inequality.

(b) For $n=3$, the inequality is equivalent to

$$a_1+a_2+2a_3+2\sqrt{a_1a_2}\ge 6\sqrt[3]{a_1a_2a_3},$$

which is a consequence of the AM-GM inequality.

For $n\ge 4$, we apply Corollary 2.4 for $p_1=p_2=\cdots=p_n=\dfrac{1}{n}$, $i=1$, $j=2$ and $k=n-1$. We have

$$P_{1,i}=\frac{1}{n},\quad P_{i+1,j}=\frac{1}{n},\quad P_{j,k-1}=\frac{n-3}{n},\quad P_{k,n}=\frac{2}{n}.$$

By Corollary 2.4, since $P_{i+1,j}\le P_{j,k-1}$, we have

$$a_1+a_2+\cdots+a_n-n\sqrt[n]{a_1a_2\cdots a_n}\ge a_1+a_2+2a_{n-1}-4\sqrt[4]{a_1a_2a_{n-1}^2}.$$

Therefore, it suffices to prove that

$$a_1+a_2+2a_{n-1}-4\sqrt[4]{a_1a_2a_{n-1}^2}\ge\frac{1}{2}(\sqrt{a_1}+\sqrt{a_2}-2\sqrt{a_{n-1}})^2,$$

which is equivalent to the obvious inequality

$$(\sqrt{a_1}-\sqrt{a_2})^2+4\sqrt{a_{n-1}}(\sqrt[4]{a_1}-\sqrt[4]{a_2})^2\ge 0.$$

Both inequalities in (a) and (b) become equalities if and only if $a_1=a_2=\cdots=a_n$.

□

Proposition 5.3. *If $a_1,a_2,\cdots,a_n$ $(n\ge 4)$ are positive real numbers such that*

$$a_1\le a_2\le\cdots\le a_n,$$

then (see [3])

$$a_1+a_2+\cdots+a_n-n\sqrt[n]{a_1a_2\cdots a_n}\ge 2\left(1-\frac{1}{n}\right)(\sqrt{a_1}-2\sqrt{a_2}+\sqrt{a_3})^2.$$

Proof. Apply Corollary 2.4 for $p_1=p_2=\cdots=p_n=\dfrac{1}{n}$, $i=1$, $j=2$ and $k=3$. We have

$$P_{1,i}=\frac{1}{n},\quad P_{i+1,j}=\frac{1}{n},\quad P_{j,k-1}=\frac{1}{n},\quad P_{k,n}=\frac{n-2}{n}.$$

By Corollary 2.4, we have

$$a_1+a_2+\cdots+a_n-n\sqrt[n]{a_1a_2\cdots a_n}\ge a_1+a_2+(n-2)a_3-n\sqrt[n]{a_1a_2a_3^{n-2}}. \quad (5.1)$$

Therefore, it suffices to prove that

$$a_1+a_2+(n-2)a_3-n\sqrt[n]{a_1a_2a_3^{n-2}}\ge 2\left(1-\frac{1}{n}\right)(\sqrt{a_1}-2\sqrt{a_2}+\sqrt{a_3})^2.$$

Write this inequality as $f(x) \geq 0$, where $0 < x \leq a \leq b$ and

$$f(x) = x + a + (n-2)b - n\sqrt[n]{ab^{n-2}x} - 2\left(1 - \frac{1}{n}\right)(\sqrt{x} - 2\sqrt{a} + \sqrt{b})^2.$$

We have

$$f'(x) = 1 - \sqrt[n]{\frac{ab^{n-2}}{x^{n-1}}} + 2\left(1 - \frac{1}{n}\right)\left(\frac{2\sqrt{a} - \sqrt{b}}{\sqrt{x}} - 1\right).$$

Clearly, $f'(x)$ increases when b decreases. Therefore, replacing b with a, we have

$$f'(x) \leq 1 - \sqrt[n]{\frac{a^{n-1}}{x^{n-1}}} + 2\left(1 - \frac{1}{n}\right)\left(\sqrt{\frac{a}{x}} - 1\right).$$

Substituting

$$\frac{a}{x} = t^{2n}, \quad t \geq 1,$$

we get $f'(x) \leq g(t)$, where

$$g(t) = 1 - t^{2n-2} + 2\left(1 - \frac{1}{n}\right)(t^n - 1).$$

Since

$$g'(t) = 2(n-1)t^{n-1}(1 - t^{n-2}) \leq 0,$$

$g(t)$ is decreasing, $g(t) \leq g(1) = 0$, $f'(x) \leq g(t) \leq 0$, $f(x)$ is decreasing, $f(x) \geq f(a)$. Thus, to show that $f(x) \geq 0$ for $0 < x \leq a \leq b$, we only need to show that $f(a) \geq 0$; that is,

$$2a + (n-2)b - n\sqrt[n]{a^2 b^{n-2}} - 2\left(1 - \frac{1}{n}\right)(\sqrt{b} - \sqrt{a})^2 \geq 0.$$

Due to homogeneity, we may set $a = 1$. In addition, substituting $b = t^{2n}$, $t \geq 1$, we need to prove that $h(t) \geq 0$, where

$$h(t) = 2 + (n-2)t^{2n} - nt^{2n-4} - 2\left(1 - \frac{1}{n}\right)(t^n - 1)^2.$$

We have

$$h'(t) = 2t^{n-1}h_1(t), \quad h_1(t) = (n^2 - 4n + 2)t^n - n(n-2)t^{n-4} + 2(n-1).$$

For $n = 4$, we have $h_1(t) = 2(t^4 - 1) \geq 0$, and for $n > 4$, we have

$$h_1'(t) = nt^{n-5}[(n^2 - 4n + 2)t^4 - (n-2)(n-4)]$$

$$\geq nt^{n-5}[(n^2 - 4n + 2) - (n-2)(n-4)] = 2n(n-3)t^{n-5} > 0,$$

$h_1(t)$ is increasing, $h_1(t) \geq h_1(1) = 0$. Thus, $h'(t) \geq 0$ for $n \geq 4$, $h(t)$ is increasing, $h(t) \geq h(1) = 0$. This completes the proof. Equality occurs if and only if $a_1 = a_2 = \cdots = a_n$.

□

Proposition 5.4. *If $a_1, a_2, \cdots, a_n$ $(n \ge 4)$ are positive real numbers such that*

$$a_1 \le a_2 \le \cdots \le a_n,$$

then (see [3])

$$a_1 + a_2 + \cdots + a_n - n\sqrt[n]{a_1 a_2 \cdots a_n} \ge \frac{2}{9}\left(1 - \frac{1}{n}\right)(3\sqrt{a_1} - \sqrt{a_2} - 2\sqrt{a_3})^2.$$

Proof. According to (5.1), it suffices to prove that

$$a_1 + a_2 + (n-2)a_3 - n\sqrt[n]{a_1 a_2 a_3^{n-2}} \ge \frac{2}{9}\left(1 - \frac{1}{n}\right)(3\sqrt{a_1} - \sqrt{a_2} - 2\sqrt{a_3})^2.$$

Write this inequality as $f(x) \ge 0$, where $0 < a \le b \le x$ and

$$f(x) = a + b + (n-2)x - n\sqrt[n]{abx^{n-2}} - \frac{2}{9}\left(1 - \frac{1}{n}\right)(3\sqrt{a} - \sqrt{b} - 2\sqrt{x})^2.$$

We have

$$f'(x) = (n-2)\left(1 - \sqrt[n]{\frac{ab}{x^2}}\right) + \frac{4}{9}\left(1 - \frac{1}{n}\right)\left(\frac{3\sqrt{a} - \sqrt{b}}{\sqrt{x}} - 2\right).$$

Clearly, $f'(x)$ decreases when b increases. Therefore, replacing b with x, we have

$$f'(x) \ge (n-2)\left(1 - \sqrt[n]{\frac{a}{x}}\right) + \frac{4}{9}\left(1 - \frac{1}{n}\right)\left(\frac{3\sqrt{a} - \sqrt{x}}{\sqrt{x}} - 2\right)$$
$$= (n-2)\left(1 - \sqrt[n]{\frac{a}{x}}\right) - \frac{4}{3}\left(1 - \frac{1}{n}\right)\left(1 - \sqrt{\frac{a}{x}}\right).$$

Substituting

$$\frac{a}{x} = t^{2n}, \quad 0 < t \le 1,$$

we get $f'(x) \ge g(t)$, where

$$g(t) = (n-2)(1 - t^2) - \frac{4}{3}\left(1 - \frac{1}{n}\right)(1 - t^n).$$

Since

$$g'(t) = 2t\left[2 - n + \frac{2}{3}(n-1)t^{n-2}\right] \le 2t\left[2 - n + \frac{2}{3}(n-1)\right] = \frac{2(4-n)t}{3} \le 0,$$

g(t) is decreasing, $g(t) \ge g(1) = 0$, $f'(x) \ge g(t) \ge 0$, $f(x)$ is increasing, $f(x) \ge f(b)$. Thus, to show that $f(x) \ge 0$ for $x \ge b$, we only need to show that $f(b) \ge 0$; that is,

$$a + (n-1)b - n\sqrt[n]{ab^{n-1}} - 2\left(1 - \frac{1}{n}\right)(\sqrt{a} - \sqrt{b})^2 \ge 0.$$

Due to homogeneity, we may set $a = 1$. In addition, substituting $b = t^{2n}$, $t \ge 1$, we need to prove that $h(t) \ge 0$, where

$$h(t) = 1 + (n-1)t^{2n} - nt^{2n-2} - 2\left(1 - \frac{1}{n}\right)(1 - t^n)^2.$$

We have

$$h'(t) = 2(n-1)t^{n-1}h_1(t), \quad h_1(t) = (n-2)t^n - nt^{n-2} + 2.$$

Since

$$h_1'(t) = n(n-2)t^{n-3}(t^2-1) \geq 0,$$

$h_1(t)$ is increasing, $h_1(t) \geq h_1(1) = 0$. Thus, $h'(t) \geq 0$, $h(t)$ is increasing, $h(t) \geq h(1) = 0$. This completes the proof. Equality occurs if and only if $a_1 = a_2 = \cdots = a_n$.

□

Proposition 5.5. *If $a_1, a_2, \cdots, a_n$ $(n \geq 4)$ are positive real numbers such that*

$$a_1 \leq a_2 \leq \cdots \leq a_n,$$

then (see [4])

$$a_1 + a_2 + \cdots + a_n - n\sqrt[n]{a_1 a_2 \cdots a_n} \geq \frac{4}{9}\left(1 - \frac{2}{n}\right)(\sqrt{a_1} + 2\sqrt{a_2} - 3\sqrt{a_3})^2.$$

Proof. According to (5.1), it suffices to prove that

$$a_1 + a_2 + (n-2)a_3 - n\sqrt[n]{a_1 a_2 a_3^{n-2}} \geq \frac{4}{9}\left(1 - \frac{2}{n}\right)(\sqrt{a_1} + 2\sqrt{a_2} - 3\sqrt{a_3})^2.$$

Write this inequality as $f(x) \geq 0$, where $0 < x \leq a \leq b$ and

$$f(x) = x + a + (n-2)b - n\sqrt[n]{ab^{n-2}x} - \frac{4}{9}\left(1 - \frac{2}{n}\right)(\sqrt{x} + 2\sqrt{a} - 3\sqrt{b})^2.$$

We have

$$f'(x) = 1 - \sqrt[n]{\frac{ab^{n-2}}{x^{n-1}}} - \frac{4}{9}\left(1 - \frac{2}{n}\right)\left(1 + \frac{2\sqrt{a} - 3\sqrt{b}}{\sqrt{x}}\right).$$

Clearly, $f'(x)$ increases when b decreases. Therefore, replacing b with a, we have

$$f'(x) \leq 1 - \sqrt[n]{\frac{a^{n-1}}{x^{n-1}}} - \frac{4}{9}\left(1 - \frac{2}{n}\right)\left(1 - \sqrt{\frac{a}{x}}\right).$$

Substituting

$$\frac{a}{x} = t^{2n}, \quad t \geq 1,$$

we get

$$f'(x) \leq 1 - t^{2n-2} + \frac{4}{9}\left(1 - \frac{2}{n}\right)(t^n - 1) \leq \frac{2n-2}{n}(t^n - 1) - (t^{2n-2} - 1)$$

$$= 2(n-1)(t-1)\left(\frac{t^{n-1} + t^{n-2} + \cdots + 1}{n} - \frac{t^{2n-3} + t^{2n-4} + \cdots + 1}{2n-2}\right) \leq 0.$$

Therefore, $f(x)$ is decreasing, and hence $f(x) \geq f(a)$. To show that $f(x) \geq 0$ for $0 < x \leq a \leq b$, we only need to show that $f(a) \geq 0$; that is,

$$2a + (n-2)b - n\sqrt[n]{a^2 b^{n-2}} - 4\left(1 - \frac{2}{n}\right)(\sqrt{a} - \sqrt{b})^2 \geq 0.$$

Due to homogeneity, we may set $a = 1$. In addition, substituting $b = t^{2n}$, $t \geq 1$, we need to prove that $h(t) \geq 0$, where

$$h(t) = 2 + (n-2)t^{2n} - nt^{2n-4} - 4\left(1 - \frac{2}{n}\right)(t^n - 1)^2.$$

We have

$$h'(t) = 2(n-2)t^{n-1}h_1(t), \quad h_1(t) = (n-4)t^n - nt^{n-4} + 4.$$

Since

$$h_1'(t) = n(n-4)t^{n-5}(t^4 - 1) \geq 0,$$

$h_1(t)$ is increasing, $h_1(t) \geq h_1(1) = 0$. Thus, $h'(t) \geq 0$, $h(t)$ is increasing, $h(t) \geq h(1) = 0$. This completes the proof. For $n > 4$, equality occurs if and only if $a_1 = a_2 = \cdots = a_n$. If $n = 4$, then equality holds for $a_1 = a_2$ and $a_3 = a_4$.

□

Proposition 5.6. *Let $a_1, a_2, \cdots, a_n$ $(n \geq 3)$ and m be positive real numbers such that*

$$a_1 \leq a_2 \leq \cdots \leq a_n,$$

and

$$a_1 + a_2 + \cdots + a_n = m\sqrt[n]{a_1 a_2 \cdots a_n}.$$

(*a*) *If*

$$n \leq m \leq (n - \frac{i}{2})\sqrt[n]{2^i}, \quad i \in \{2, 3, \cdots, n-1\},$$

then a_{i-1}, a_i, a_{i+1} are the side-lengths of a degenerate or non-degenerate triangle (see [5]);

(*b*) *If*

$$n \leq m \leq \max_{i \in \{2,3,\cdots,n-1\}} (n - \frac{i}{2})\sqrt[n]{2^i},$$

then among the numbers $a_1, a_2, \cdots, a_n$ there exist three which are the side-lengths of a degenerate or non-degenerate triangle.

Proof. (a) The condition $m \geq n$ follows by the AM-GM Inequality

$$a_1 + a_2 + \cdots + a_n \geq n\sqrt[n]{a_1 a_2 \cdots a_n}.$$

For the sake of contradiction, assume that a_{i-1}, a_i, a_{i+1} are not the side-lengths of a triangle; that is, $a_{i-1} + a_i < a_{i+1}$. Setting $p_1 = p_2 = \cdots = p_n = \dfrac{1}{n}$ in Corollary 2.5 and replacing then i, j, k by $i-1, i, i+1$, respectively, we have

$$\frac{a_1 + a_2 + \cdots + a_n}{\sqrt[n]{a_1 a_2 \cdots a_n}} \geq \frac{(i-1)a_{i-1} + a_i + (n-i)a_{i+1}}{\sqrt[n]{a_{i-1}^{i-1} a_i a_{i+1}^{n-i}}},$$

and hence

$$m \geq g(a_{i-1}, a_i, a_{i+1}),$$

where

$$g(a_{i-1}, a_i, a_{i+1}) = \frac{(i-1)a_{i-1} + a_i + (n-i)a_{i+1}}{\sqrt[n]{a_{i-1}^{i-1} a_i a_{i+1}^{n-i}}}$$

Since $a_{i+1} > a_{i-1} + a_i$ and

$$\frac{\partial g}{\partial a_{i+1}} = \frac{(n-i)[(i-1)(a_{i+1}-a_{i-1}) + a_{i+1} - a_i]}{n a_{i+1} \sqrt[n]{a_{i-1}^{i-1} a_i a_{i+1}^{n-i}}} > 0,$$

we get

$$m > g(a_{i-1}, a_i, a_{i-1} + a_i) = \frac{(n-1)a_{i-1} + (n-i+1)a_i}{\sqrt[n]{a_{i-1}^{i-1} a_i (a_{i-1} + a_i)^{n-i}}}.$$

Due to homogeneity, we consider $a_{i-1} = 1$. Denoting $a_i = x$, $x \geq 1$, we have $m > h(x)$, where

$$h(x) = \frac{n-1+(n-i+1)x}{\sqrt[n]{x(1+x)^{n-i}}}.$$

Since

$$\begin{aligned} n\sqrt[n]{x^{n-1}(1+x)^i}\,h'(x) &= (i-1)(n-i+1)x^2 - n + 1 \\ &\geq (i-1)(n-i+1) - n + 1 = (i-2)(n-i) \geq 0, \end{aligned}$$

$h(x)$ is increasing, $h(x) \geq h(1) = (n - \frac{i}{2})\sqrt[n]{2^i}$, and hence $m > (n - \frac{i}{2})\sqrt[n]{2^i}$, which is false.

(b) The conclusion follows immediately from (a).

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