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MCINTOSH FORMULA FOR THE GAP BETWEEN REGULAR OPERATORS

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Abstract. We derive an equivalent definition for the gap between two complemented submodules of a Hilbert C^*-module which is same as the one for closed subspaces of a Banach space. This gives an alternative way of defining gap between two regular operators. We give an alternative proof of the latter result. We also derive the McIntosh formula for computing the gap between two regular operators between Hilbert C^*-modules which is analogous to that of unbounded operators between Hilbert spaces.

1. Introduction

In this article we give an alternative definition for the gap between two complemented submodules of a Hilbert C^*-module using the distance concept as in the case of closed subspaces of a Banach space. We also derive the McIntosh formula for computing the gap of regular operators between Hilbert C^*-modules. The gap between two closed subspaces of a Hilbert space can be defined as the norm of the difference of the orthogonal projections onto these subspaces. The same notion can be applied to the graphs of operators to define the gap between two operators. This definition, as it involves projections is not applicable for closed subspaces of a Banach space. In this case it can be defined in terms of the distance between a point and a subspace [1, 9].

Restricted to the scalars, the gap between two complex numbers is the chordal distance between the corresponding images on the Riemann sphere centered at $(0, 0, \frac{1}{2})$ with radius $\frac{1}{2}$. This fact can be observed from the McIntosh formula (see

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[11] or [17]). Various gap concepts are useful in determining different properties of subspaces and operators (see [2, 4, 9, 17] for details). The McIntosh formula for the gap between two unbounded operators was discussed by Kulkarni and Ramesh in [11]. The Horn-Li-Merino formula for computing the gap between two unbounded operators between Hilbert spaces is discussed in [19].

Recently Sharifi [21], defined a metric equivalent to the gap metric for regular operators between Hilbert C^*-modules and discussed applications to Fredholm operators.

We organize the paper as follows: In section 2, we introduce notations and basic concepts about Hilbert C^*-modules and regular operators on Hilbert C^*-modules. In Section 3, we deduce the McIntosh formula for the gap between two regular operators and deduce an equivalent definition as in the case of subspaces of Hilbert spaces ([1, 9]).

In this article we extend the results of [11] to the case of unbounded regular operators between Hilbert C^*-modules.

2. Notations and Preliminaries

In this section we present definitions, notations and results that are frequently used in this article to prove main results. We assume that all C^*-algebras to be complex. For the theory of C^*-algebras we refer to [8, 16]. Here we present basics of Hilbert C^*-modules which can be found in [13, 14].

Definition 2.1. [13, page 2] Let $\mathcal{A}$ be a C^*-algebra. A pre-Hilbert $\mathcal{A}$-module E is a complex linear space, which is a right $\mathcal{A}$-module, compatible with that of the linear space structure (i.e., $\lambda(xa) = (\lambda x)a = x(\lambda a), \quad \text{for all } \lambda \in \mathbb{C},\ a \in \mathcal{A},\ x \in E$) equipped with an $\mathcal{A}$-valued inner product, that is the map $\langle \cdot, \cdot \rangle : E \times E \to \mathcal{A}$ satisfying:

(i) $\langle x, y + \lambda z\rangle = \langle x, y\rangle + \lambda\langle x, z\rangle \quad \text{for all } x, y, z \in E, \lambda \in \mathbb{C}$
(ii) $\langle x, ya\rangle = \langle x, y\rangle a, \quad \text{for all } x, y \in E,\ a \in \mathcal{A}$
(iii) $\langle x, y\rangle^* = \langle y, x\rangle, \quad \text{for all } x, y \in E$
(iv) $\langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0$ if and only if $x = 0$.

For a pre-Hilbert $\mathcal{A}$-module, the Cauchy-Schwarz inequality

$$\langle y, x\rangle\langle x, y\rangle \leq \|\langle x, x\rangle\|\langle y, y\rangle, \text{ for all } x, y \in E,$$

holds and using this we can show that

$$\|x\| := \|\langle x, x\rangle\|^{\frac{1}{2}}, \quad \text{for all } x \in E, \tag{2.1}$$

defines a norm on E. A Hilbert $\mathcal{A}$-module is a pre-Hilbert $\mathcal{A}$-module E which is complete with respect to the norm given by (2.1). Through out we consider only Hilbert C^*-modules in our article.

Hilbert C^*-modules possess the properties of both Hilbert spaces as well as C^*-algebras. The failure of the projection theorem, the parallelogram law and the Riesz representation theorem makes these objects complicated compared to Hilbert spaces. Hilbert C^*-modules plays an important role in operator algebras, operator K-theory, and the theory of operator spaces (see for example [13, 15] for details). This subject is growing very rapidly and refer the reader to consult the website [5] for more details.

Let E and F be Hilbert $\mathcal{A}$-modules. A map $t : E \to F$ is said to be $\mathcal{A}$-*linear* if $t(xa) = t(x)a$ for all $x \in E$ and for all $a \in \mathcal{A}$ and is said to be *adjointable* if there exists an operator $t^* : F \to E$ with the property that

$$\langle x, ty\rangle = \langle t^*x, y\rangle, \quad \text{for all } \ x \in F, \ y \in E.$$

We denote the set of all adjointable bounded maps between E and F by $\mathcal{B}^a(E, F)$. In case, $E = F$, we denote $\mathcal{B}^a(E, F)$ by $\mathcal{B}^a(E)$. Note that $\mathcal{B}^a(E)$ is a C^*-algebra. If $t : E \to F$ is $\mathcal{A}$-linear, the range and the null space of t are denoted by $ran(t)$ and $ker(t)$ respectively.

We denote the identity operator on a Hilbert $\mathcal{A}$-module by 1 and the underlying Hilbert C^*-module can be understood without any confusion. Let $c \in \mathcal{A}$. Then $c = \dfrac{c + c^*}{2} - i(\dfrac{c - c^*}{2i})$. Here $Re(c) := \dfrac{c + c^*}{2}$ and $Im(c) := \dfrac{c - c^*}{2i}$ are self-adjoint elements of $\mathcal{A}$ and are called as the *real* and *imaginary* parts of c respectively.

Let E be a Hilbert $\mathcal{A}$-module and $x, y \in E$. We say x is orthogonal to y if $\langle x, y\rangle = 0$ and denote it by $x \perp y$. If F is a submodule of E, its orthogonal complement is $F^{\perp} := \{x \in E : x \perp y, \ \text{for all } y \in F\}$. If F_1 and F_2 are two submodules of E such that $F_1 \cap F_2 = \{0\}$, then $F_1 + F_2$ is called the *direct sum* of F_1 and F_2 and is denoted by $F_1 \oplus F_2$. The direct sum is said to be *orthogonal* if $F_1 \perp F_2$. A closed submodule F is said to be *topologically complemented* if there exists a submodule $G \subset E$ such that $E = F \oplus G$. A closed submodule F is said to be *complemented* or *orthogonally complemented* if $E = F \oplus F^{\perp}$. The orthogonal projection onto a complemented submodule N of E is denoted by p_N. Note that in this case $ran(p)^{\perp} = ran(1 - p) = ker(p)$ and $E = ran(p) \oplus ran(1 - p)$ ([13, Chapter 3]). For any complemented submodule M of E, we denote the distance between $x \in E$ and M by $d(x, M)$ and $S_M : = \{x \in M : \|x\| = 1\}$, the unit sphere of M.

For most of the material in this section we refer to [12, 13, 18]. Throughout we denote the Hilbert $\mathcal{A}$-modules by E and F. Let $t : D(t) \subseteq E \to F$ be $\mathcal{A}$-linear, where $D(t) \subseteq E$ is the domain of t. If $D(t)$ is a dense submodule of E, then t is called *densely defined*. For such an operator we define a submodule

$$D(t^*) := \{y \in F : \exists \ z \ \in E \text{ with } \langle tx, y\rangle = \langle x, z\rangle \text{ for all } x \in D(t)\}. \tag{2.2}$$

For $y \in D(t^*)$, the element z in (2.2) is unique and we define $z = t^*y$. This defines an $\mathcal{A}$-linear map $t^* : D(t^*) \to E$ satisfying

$$\langle tx, y\rangle = \langle x, t^*y\rangle \quad \text{for all } \ x \in D(t), y \in D(t^*).$$

The graph of t is defined by $G(t) := \{(x, tx) : x \in D(t)\} \subseteq E \oplus F$. The graph of the zero operator is $G(0) = \{(x, 0) : x \in D(0)\}$. Note that $\{(0, 0)\}$ is a graph an operator s if and only if $s = 0$ and $D(s) = \{0\}$.

If $G(t)$ is a closed submodule, then t is called a *closed operator*[13]. The closed graph theorem asserts that everywhere defined closed operator is bounded [20]. The map t^* if exists, is always closed whether t is closed or not.

If s and t are $\mathcal{A}$-linear maps such that $D(s) \subseteq D(t)$ and $tx = sx$ for all $x \in D(s)$, then s is called the *restriction* of t and t is called an *extension* of s. If

s is a restriction of t, then we denote this by $s \subseteq t$. If $t \subseteq t^*$, then t is said to be *symmetric* and *self-adjoint* if $t = t^*$. We say t to be *positive* if $t = t^*$ and its spectrum $\sigma(t)$ is a subset of $[0, \infty)$. If t_1, t_2 are self-adjoint $\mathcal{A}$-linear maps such that $t_1 - t_2 \geq 0$, then we write this by $t_1 \geq t_2$.

If E and F are Hilbert $\mathcal{A}$-modules, then $E \oplus F$ is a Hilbert $\mathcal{A}$-module with respect to the inner product given by

$$\langle (x_1, y_1), (x_2, y_2) \rangle = \langle x_1, x_2 \rangle + \langle y_1, y_2 \rangle, \quad \text{for all } x_i \in E,\ y_i \in F,\ i = 1, 2.$$

The induced norm is given by $\|(x, y)\| = \|\langle x, x \rangle + \langle y, y \rangle\|^{\frac{1}{2}}$ for all $(x, y) \in E \oplus F$.

Definition 2.2. [13, Chapter 9] Let $t : D(t)(\subseteq E) \to F$ be a $\mathcal{A}$-linear map. Then t is said to be *regular* if

(1) t is densely defined and closed
(2) t^* is densely defined
(3) $ran(1 + t^*t)$ is dense in E.

We denote the set of all regular operators between E and F by $\mathcal{R}(E, F)$. In case $E = F$, $\mathcal{R}(E, F) = \mathcal{R}(E)$. The operator $v : E \oplus F \to F \oplus E$ given by $v(x, y) = (-y, x)$ for all $x \in E,\ y \in F$ is a unitary operator and if $t \in \mathcal{R}(E, F)$, then $E \oplus F = G(t) \oplus v(G(t^*))$ [13, Theorem 9.3].

Proposition 2.3. [21, 13] *For $t \in \mathcal{R}(E, F)$, let $Q_t := (1 + t^*t)^{\frac{-1}{2}}$ and $F_t := tQ_t$. Then*

(1) $Q_t \in \mathcal{B}^a(E)$, $0 \leq Q_t \leq 1$ *and* $ran(Q_t) = D(t)$
(2) $F_t \in \mathcal{B}^a(E, F)$, $\ F_t^* = F_{t^*}$ *and* $\|F_t\| \leq 1$
(3) $\|F_t\| < 1$ *if and only if* $t \in \mathcal{B}^a(E, F)$.

The operator F_t is called the bounded transform or z-transform of t.

3. McIntosh formula

Recall that if M and N are closed subspaces of a Hilbert space H and $p, q : H \to H$ are orthogonal projections with $R(p) = M$ and $R(q) = N$, then the gap between M and N is defined to be $\theta(M, N) = \|p - q\|$. It can be shown that [1, page 70],

$$\theta(M, N) = \max\{\theta_0(M, N), \theta_0(N, M)\},$$

$$\text{where} \quad \theta_0(M, N) = \begin{cases} \sup\{d(x, N) : x \in S_M\} & \text{if } M \neq \{0\}, \\ 0, & \text{if } M = \{0\}. \end{cases}$$

The latter definition is useful to define the gap between two closed subspaces of a Banach space where as the former one is not as it involves orthogonal projections. We prove that these two definitions are equivalent in the setting of Hilbert C^*-modules.

If $t \in \mathcal{R}(E, F)$, then $G(t)$ is complemented in $E \oplus F$ [13, Chapter 9]. Let $s \in \mathcal{R}(E, F)$, $p = p_{G(t)}$ and $q = p_{G(s)}$. Then $\theta(t, s) := \|p - q\|$ is called the *gap* between t and s. The gap between t and 0 is called the gap of t and is denoted by $\theta(t)$. In this section we prove a formula for computing the gap between two regular operators, which is due to McIntosh in the case of $m \times n$ matrices. This formula for the case of bounded operators was proved by Nakamoto in [17] and this is further generalized to the case of unbounded closed operators in Hilbert spaces by Kulkarni and the author in [11].

Proposition 3.1. *Let E be a Hilbert $\mathcal{A}$-module and $p_1, p_2 : E \to E$ be orthogonal projections. Then*

$$\|p_1p_2\| = \sup\left\{\frac{\|\langle x, y\rangle\|}{\|x\|\,\|y\|} : 0 \neq p_1x = x,\ 0 \neq y = p_2y\right\}.$$

Proof. First note that if $t \in \mathcal{B}^a(E, F)$, then

$$\|t\| = \sup\left\{\frac{\|\langle y, tx\rangle\|}{\|x\|\,\|y\|} : 0 \neq x \in E,\ 0 \neq y \in F\right\}.$$

This result follows from the observation: If $x \in E$, then

$$\|x\| = \sup\{\|\langle x, y\rangle\| : y \in S_E\}.$$

By this observation, we have

$$\begin{aligned}\|p_1p_2\| &= \sup\left\{\frac{\|\langle y, p_1p_2x\rangle\|}{\|x\|\,\|y\|} : 0 \neq x \in E,\ 0 \neq y \in E\right\}\\ &= \sup\left\{\frac{\|\langle p_1y, p_2x\rangle\|}{\|x\|\,\|y\|} : 0 \neq x \in E,\ 0 \neq y \in E\right\}\\ &\leq \sup\left\{\frac{\|\langle p_1y, p_2x\rangle\|}{\|p_2x\|\,\|p_1y\|} : 0 \neq x \in E,\ 0 \neq y \in E\right\}\\ &= \sup\left\{\frac{\|\langle z, w\rangle\|}{\|z\|\,\|w\|} : 0 \neq z = p_1z,\ 0 \neq w = p_2w\right\}.\end{aligned}$$

On other hand,

$$\begin{aligned}\|\langle p_1y, p_2x\rangle\| &= \|\langle p_1^2y, p_2^2x\rangle\|\\ &= \|\langle p_1y, p_1p_2p_2x\rangle\|\\ &\leq \|p_1p_2\|\|p_2x\|\|p_1y\|.\end{aligned}$$

This shows that $\sup\left\{\frac{\|\langle x, y\rangle\|}{\|x\|\,\|y\|} : 0 \neq p_1x = x,\ 0 \neq y = p_2y\right\} \leq \|p_1p_2\|$. ☐

Lemma 3.2. [3, lemma 1.1] *Let $p, p_2 \in \mathcal{B}^a(E)$ be orthogonal projections. Then*

$$\|p_1 - p_2\| = \max\{\|p_1(1 - p_2)\|, \|p_2(1 - p_1)\|\}.$$

Theorem 3.3. *Let M, N be complemented submodules of E. Then*

$$\theta(M, N) = \max\left\{\sup_{x \in S_M} d(x, N), \sup_{y \in S_N} d(y, M)\right\}.$$

Here we assume that $S_L = \{0\}$ if $L = \{0\}$.

Proof. Let $p_1 = p_M, p_2 = p_N$ and $x \in M$. First we show that

$$d(x, N) = \|(1 - p_2)p_1x\|.$$

Let $y \in E$. Then we have

$$\begin{aligned}\langle x - p_2 y, x - p_2 y\rangle &= \langle x - p_2 x + p_2(x-y), x - p_2 x + p_2(x-y)\rangle \\ &= \langle x - p_2 x, x - p_2 x\rangle + \langle p_2(x-y), p_2(x-y) \\ &\geq \langle x - p_2 x, x - p_2 x\rangle.\end{aligned}$$

Hence $\|(1-p_2)p_1 x\| = \|x - p_2 x\| \leq \|x - p_2 y\|$ for each $y \in E$. So $\|(1-p_2)p_1 x\| \leq d(x, N)$.

On the other hand, $d(x,N) \leq \|x - p_2 x\| \leq \|(1-p_2)p_1 x\|$. Thus $d(x,N) = \|(1-p_2)p_1 x\|$.

Now consider

$$\begin{aligned}\sup_{x \in S_M} d(x,N) = \sup_{x \in S_M} \|(1-p_2)p_1 x\| &\leq \sup_{w \in S_E} \|(1-p_2)p_1 w\| \\ &= \|(1-p_2)p_1\| \\ &= \|p_1(1-p_2)\|.\end{aligned}$$

On the other hand let $z \in S_M$. Then $z = p_1 x$ for some $0 \neq x \in E$. So $d(z,N) = \|(1-p_2)z\|$. Thus

$$\begin{aligned}\sup_{z \in S_M} d(z,N) &= \sup_{x \in E} \frac{\|(1-p_2)p_1 x\|}{\|p_1 x\|} \\ &\geq \sup_{x \in E} \frac{\|(1-p_2)p_1 x\|}{\|x\|} \\ &= \|(1-p_2)p_1\|.\end{aligned}$$

Similarly, we can show that $\sup\limits_{y \in S_N} d(y, M) = \|(1-p_1)p_2\|$. □

Theorem 3.4. *Let $s, t \in \mathcal{R}(E,F)$. Then*

$$\theta(s,t) = \max\big\{\theta_0(G(t), G(s)), \theta_0(G(s), G(t))\big\},$$

where $\theta_0(M,N) = \begin{cases} \sup\{d(x,N) : x \in S_M\} & \text{if } M \neq \{0\}, \\ 0, & \text{if } M = \{0\}.\end{cases}$

Proof. We know that $G(t)$ and $G(s)$ are complemented submodules of $E \oplus F$. Now applying Theorem 3.3, we get the conclusion. □

Theorem 3.5 (McIntosh formula)**.** *Let $s, t \in \mathcal{R}(E,F)$. Then*

$$\theta(s,t) = \max\big\{\|F_t Q_s - Q_{t^*} F_s\|, \|F_s Q_t - Q_{s^*} F_t\|\big\}.$$

Proof. Let $p = p_{G(t)}$ and $q = p_{G(s)}$. First, we calculate $\|p(1-q)\|$ with the help of Proposition 3.1. Note that $1-q$ is an orthogonal projection onto the submodule $\{(-t^* y, y) : y \in D(t^*)\}$. This can be seen from the facts that $G(t)$ is complemented in $E \oplus F$ and $G(t)^{\perp} = v(G(t^*))$, where $v : E \oplus F \to F \oplus E$ given by $v(x,y) = (-y, x)$ for all $x \in E,\ y \in F$ is a unitary map. Hence by Proposition

3.1,

$$\begin{aligned}\|p(1-q)\| &= \sup\left\{\frac{\|\langle z,w\rangle\|}{\|z\|\,\|w\|} : 0\neq z=pz,\ 0\neq w=(1-q)w\right\}\\ &= \sup\left\{\frac{\|\langle (x,sx),(-t^*y,y)\rangle\|}{\|(x,sx)\|\,\|(-t^*y,y)\|} : 0\neq x\in D(s),\ 0\neq y\in D(t^*)\right\}\\ &= \sup\left\{\frac{\|\langle x,-t^*y\rangle+\langle sx,y\rangle\|}{\|(x,sx)\|\,\|(-t^*y,y)\|} : 0\neq x\in D(s),\ 0\neq y\in D(t^*)\right\}.\end{aligned}$$

The operators $Q_s : E\to D(s)$ and $Q_{t^*} : F\to D(t^*)$ are bijective. Hence there exists unique $0\neq u\in E$ and unique $0\neq v\in F$ such that $x=Q_su$ and $y=Q_{t^*}v$. It can be verified easily that $\|(x,sx)\|^2=\|u\|^2$ and $\|(-t^*y,y)\|^2=\|v\|^2$. Hence

$$\begin{aligned}\|p(1-q)\| &= \sup_{0\neq u\in E,\ 0\neq v\in F}\left\{\frac{\|\langle (Q_su,-t^*Q_{t^*}v)\rangle+\langle F_su,Q_{t^*}v\rangle\|}{\|u\|\,\|v\|}\right\}\\ &= \sup_{0\neq u\in E,\ 0\neq v\in F}\left\{\frac{\|\langle (Q_su,-F_t^*v)\rangle+\langle F_su,Q_{t^*}v\rangle\|}{\|u\|\,\|v\|}\right\}\\ &= \sup_{0\neq u\in E,\ 0\neq v\in F}\left\{\frac{\|\langle (F_tQ_su,-v)\rangle+\langle Q_{t^*}F_su,v\rangle\|}{\|u\|\,\|v\|}\right\}\\ &= \sup_{0\neq u\in E,\ 0\neq v\in F}\left\{\frac{\|\langle \big(Q_{t^*}F_s-F_tQ_s\big)u,v\rangle\|}{\|u\|\,\|v\|}\right\}\\ &= \|Q_{t^*}F_s-F_tQ_s\|\\ &= \|F_tQ_s-Q_{t^*}F_s\|.\end{aligned}$$

With a similar computation, we can conclude that $\|q(1-p)\|=\|Q_{s^*}F_t-F_sQ_t\|$. Now the theorem follows from Lemma 3.2. □

Corollary 3.6. *Let $t\in\mathcal{R}(E,F)$. Then $t\in\mathcal{B}^a(E,F)$ if and only if $\theta(t)<1$.*

Proof. By Theorem 3.5, $\theta(t)=\|F_t\|$. By Proposition 2.3, $t\in\mathcal{B}^a(E,F)$ if and only if $\|F_t\|<1$. □

Remark 3.7. If $s,t\in\mathcal{R}(E,F)$ are such that $D(t)=D(s)$, then

$$\theta(s,t)=\max\left\{\|Q_{s^*}(s-t)Q_t\|,\|Q_{t^*}(s-t)Q_s\|\right\},$$

which is a formula obtained by Nakamoto in [17] for bounded operators between Hilbert spaces. Using the result $tQ_t=Q_{t^*}t$ on $D(t)$ [7, remark 2.2], we have

$$\begin{aligned}F_tQ_s-Q_{t^*}F_s &= tQ_tQ_s-Q_{t^*}sQ_s\\ &= Q_{t^*}tQ_s-Q_{t^*}sQ_s\\ &= Q_{t^*}(t-s)Q_s.\end{aligned}$$

A similar argument holds in the case of $F_sQ_t-Q_{s^*}F_t$.

Remark 3.8. If $s,t\in\mathcal{R}(E,F)$ are both self-adjoint, then

$$\theta(s,t)=\|F_tQ_s-Q_tF_s\|.$$

To see this, suppose that $s = s^*$ and $t = t^*$. Now by Theorem 3.5, it follows that $\theta(s,t) = \max\{\|F_tQ_s - Q_tF_s\|, \|F_sQ_t - Q_sF_t\|\}$. But $F_sQ_t - Q_sF_t = -(F_tQ_s - Q_tF_s)^*$ and hence $\|F_tQ_s - Q_tF_s\| = \|F_sQ_t - Q_sF_t\|$.

Corollary 3.9. *Let $s \in \mathcal{R}(E,F)$ and $t \in \mathcal{B}^a(E,F)$. Then*

$$\theta(s+t, s) \leq \|t\|.$$

Proof. Follows from Theorem 3.5 and from Nakamoto's formula since $\|Q_x\| \leq 1$ for any regular operator x. □

Theorem 3.10. *Let $s, t \in \mathcal{B}^a(E,F)$. Then*

$$\theta(s,t) \leq \|s-t\| \leq (1+\|t\|^2)^{\frac{1}{2}}\ (1+\|s\|^2)^{\frac{1}{2}}\ \theta(s,t).$$

Proof. The first inequality follows from the fact that $\|Q_r\| \leq 1$ for any $r \in \mathcal{B}^a(E,F)$.

Let $p = p_{G(t)}$ and $q = p_{G(s)}$. Note that

$$\begin{aligned}\|s-t\| &= \left\|(1+ss^*)^{\frac{1}{2}} Q_{s^*}(s-t) Q_t (1+t^*t)^{\frac{1}{2}}\right\| \\ &\leq (1+\|t\|^2)^{\frac{1}{2}}\ (1+\|s\|^2)^{\frac{1}{2}}\ \|Q_{s^*}(s-t)Q_t\|.\end{aligned}$$

Similarly, we can show that $\|s-t\| \leq (1+\|t\|^2)^{\frac{1}{2}}\ (1+\|s\|^2)^{\frac{1}{2}}\ \|Q_{t^*}(s-t)Q_s\|$. Now the result follows from Remark 3.7. □

We give an alternative proof of Theorem 3.4 using the McIntosh formula. We need the following lemma in our proof.

Proposition 3.11. *Let $s, t \in \mathcal{R}(E,F)$. Let $x \in D(t)$ and $w = (x, tx)$. Then we have $d(w, G(s)) = \|Q_{s^*}tx - F_sx\|$.*

Proof. By definition,

$$\begin{aligned}&d(w, G(s))^2 \\ &= \inf_{y \in D(s)} \Big\{\|\langle x-y, x-y\rangle + \langle tx - sy, tx - sy\rangle\|\Big\} \\ &= \inf_{y \in D(s)} \Big\{\|\langle x, x\rangle + \langle tx, tx\rangle + \langle y, y\rangle + \langle sy, sy\rangle - 2\mathrm{Re}\big(\langle x, y\rangle + \langle tx, sy\rangle\big)\|\Big\} \\ &= \inf_{y \in D(s)} \Big\{\|\langle x, x\rangle + \langle tx, tx\rangle + \langle (1+s^*s)^{\frac{1}{2}}y, (1+s^*s)^{\frac{1}{2}}y\rangle - 2\mathrm{Re}\big(\langle x, y\rangle + \langle tx, sy\rangle\big)\|\Big\}.\end{aligned}$$

Let $y = Q_sz$. Then

$$\begin{aligned}d(w, G(s))^2 &= \inf_{z \in E} \Big\{\|\langle x, x\rangle + \langle tx, tx\rangle + \langle z, z\rangle - 2\mathrm{Re}\big(\langle x, Q_sz\rangle + \langle tx, F_sz\rangle\big)\|\Big\} \\ &= \inf_{z \in E} \Big\{\|\langle x, x\rangle + \langle tx, tx\rangle + \langle z, z\rangle - 2\mathrm{Re}\big(\langle x, Q_sz\rangle + \langle F_s^*tx, z\rangle\big)\|\Big\} \\ &= \inf_{z \in E} \Big\{\|\langle x, x\rangle + \langle tx, tx\rangle + \langle z, z\rangle - 2\mathrm{Re}\big(\langle (Q_s + F_{s^*}t)x, z\rangle\big)\|\Big\}.\end{aligned}$$

Let $A = Q_s + F_{s^*}t$, $a = \langle x, x\rangle + \langle tx, tx\rangle - \langle Ax, Ax\rangle$ and $b = \langle z - Ax, z - Ax\rangle$. Then

$$d(w, G(s))^2 = \inf_{b\in\mathcal{A}} \{\|a + b\|\}.$$

Since $b \geq 0$, it follows that $d(w, G(s))^2 \leq \|a\|$. Next we show that $a \geq 0$. To do this consider

$$\begin{aligned}
a &= \langle x, x\rangle + \langle tx, tx\rangle - \langle (Q_s + F_{s^*}t)x, (Q_s + F_{s^*}t)x\rangle\big) \\
&= \langle x, x\rangle + \langle tx, tx\rangle - \langle Q_sx, Q_sx\rangle - \langle Q_sx, F_{s^*}tx\rangle - \langle F_{s^*}tx, Q_sx\rangle - \langle F_{s^*}tx, F_{s^*}tx\rangle \\
&= \langle x, x\rangle + \langle tx, tx\rangle - \langle Q_s^2x, x\rangle - \langle F_sQ_sx, tx\rangle - \langle tx, F_sQ_sx\rangle - \langle F_{s^*}tx, F_{s^*}tx\rangle \\
&= \langle F_sx, F_sx\rangle + \langle Q_{s^*}tx, Q_{s^*}tx\rangle - 2\mathrm{Re}\big(\langle F_sx, Q_{s^*}tx\rangle\big) \\
&= \langle F_sx, F_sx\rangle + \langle Q_{s^*}tx, Q_{s^*}tx\rangle - 2\mathrm{Re}\big(\langle Q_{s^*}tx, F_sx\rangle\big) \\
&= \langle Q_{s^*}tx - F_sx, Q_{s^*}tx - F_sx\rangle \geq 0.
\end{aligned}$$

Since $0 \leq a \leq a + b$, we have $\|a\| \leq \|a + b\|$ and hence $\|a\| \leq d(w, G(s))^2$, concluding $d(w, G(s))^2 = \|a\|$. Hence $d(w, G(s)) = \|\langle Q_{s^*}tx - F_sx, Q_{s^*}tx - F_sx\rangle\|^{\frac{1}{2}} = \|Q_{s^*}tx - F_sx\|$.

□

Remark 3.12. Using Theorem 3.3 and Proposition 3.11 we can get an alternative proof of Theorem 3.5.

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