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COMPOSITION OPERATORS FROM NEVANLINNA TYPE SPACES TO BLOCH TYPE SPACES

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Abstract. Let X and Y be complete metric spaces of analytic functions over the unit disk in the complex plane. A linear operator $T : X \to Y$ is a bounded operator with respect to metric balls if T takes every metric ball in X into a metric ball in Y. We also say that T is metrically compact if it takes every metric ball in X into a relatively compact subset in Y. In this paper we will consider these properties for composition operators from Nevanlinna type spaces to Bloch type spaces.

1. Introduction

Let $\mathbb{D}$ denote the open unit disk in the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ the space of all analytic functions on $\mathbb{D}$. For any analytic self-map φ of $\mathbb{D}$, the *composition operator* C_φ on $H(\mathbb{D})$ is defined by $C_\varphi f = f \circ \varphi$ for $f \in H(\mathbb{D})$. Some operator theoretic properties of C_φ acting on various analytic function spaces have been studied by many mathematicians. In recent progresses of studies on the operator C_φ, some authors have investigated the case C_φ acting between different function spaces. These studies on this setting have close connection with one about the weighted composition operator or the integral-type operator (*e.g.*[6, 7, 8, 12, 14, 15]). In particular, S. Li and S. Stević [6, 7, 8] introduced the generalized

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composition operator C_φ^g as follows:

$$C_\varphi^g f(z) = \int_0^z f'(\varphi(\zeta))g(\zeta)\, d\zeta,$$

and characterized operator-theoretic properties by using behaviors of φ and g. As a consequence of them, the characterizations for the operator C_φ acting between different function spaces are obtained. For some n-dimensional operators related to operators C_φ^g see, for example [13, 17, 18, 19] and the references therein. In this paper we will investigate operator-theoretic properties for composition operators from Nevanlinna-type spaces N^+, M and N^p $(p > 1)$ into Bloch-type spaces $\mathcal{B}_\omega$ and $\mathcal{B}_{\omega,0}$.

1.1. **Nevanlinna-type spaces N^+, M and N^p.** Let $\mathbb{T}$ denote the boundary of $\mathbb{D}$ and $d\sigma$ the normalized Lebesgue measure on $\mathbb{T}$. The Nevanlinna class N on $\mathbb{D}$ is defined as the set of all analytic functions f on $\mathbb{D}$ such that

$$\sup_{0\le r<1} \int_{\mathbb{T}} \log(1 + |f(r\zeta)|)\, d\sigma(\zeta) < \infty.$$

It is well-known that $f \in N$ has a finite nontangential limit, denoted by f^*, almost everywhere on $\mathbb{T}$.

The *Smirnov class* N^+ is a subclass of N such that f satisfies

$$\sup_{0\le r<1} \int_{\mathbb{T}} \log(1 + |f(r\zeta)|)\, d\sigma(\zeta) = \int_{\mathbb{T}} \log(1 + |f^*(\zeta)|)\, d\sigma(\zeta).$$

We define a metric $d_{N^+}(f, g) = \|f - g\|_{N^+}$ for $f, g \in N^+$ where

$$\|f\|_{N^+} = \int_{\mathbb{T}} \log(1 + |f^*(\zeta)|)\, d\sigma(\zeta).$$

The Smirnov class N^+ becomes an F-space with respect to the translation-invariant metric d_{N^+}. The subharmonicity of $\log(1 + |f|)$ implies that $f \in N^+$ has the following growth estimation:

$$\log(1 + |f(z)|) \le \frac{4\|f\|_{N^+}}{1 - |z|^2} \tag{1.1}$$

for $z \in \mathbb{D}$. Thus the convergence on the metric is stronger than the uniform convergence on compact subsets of $\mathbb{D}$.

The class M which is contained in N^+ is the set of all analytic functions f on $\mathbb{D}$ such that

$$\|f\|_M := \int_{\mathbb{T}} \log(1 + Mf(\zeta))\, d\sigma(\zeta) < \infty,$$

where $Mf(\zeta) = \sup_{0\le r<1} |f(r\zeta)|$ is the radial maximal function of f. This class M has been introduced by H.O. Kim and the study on M has been well established in [5].

For each $1 < p < \infty$, the *Privalov space* N^p on $\mathbb{D}$ is defined as

$$N^p = \left\{ f \in H(\mathbb{D}) \,:\, \|f\|_{N^p} = \sup_{0\le r<1} \left[\int_{\mathbb{T}} \{\log(1 + |f(r\zeta)|)\}^p\, d\sigma(\zeta) \right]^{1/p} < \infty \right\}.$$

Subbotin [20] studied linear space properties of N^p. For instance, N^p is an F-space with respect to the translation-invariant metric $d_{N^p}(f,g) = \|f-g\|_{N^p}$ or the convergence in N^p gives the uniform convergence on compact subsets of $\mathbb{D}$. Also Subbotin gave the following inclusions:

$$N^p \subsetneq M \subsetneq N^+ \subsetneq N.$$

Thus we will call N^+, M and N^p *Nevanlinna-type spaces*.

1.2. **Bloch-type spaces $\mathcal{B}_\omega$ and $\mathcal{B}_{\omega,0}$.** Let ω be a strictly positive continuous function on $\mathbb{D}$. If $\omega(z) = \omega(|z|)$ for every $z \in \mathbb{D}$, we call it a radial weight. A radial weight ω is called *typical* if it is nonincreasing with respect to $|z|$ and $\omega(z) \to 0$ as $|z| \to 1$. For a typical weight ω, the *Bloch-type space* $\mathcal{B}_\omega$ on $\mathbb{D}$ is the space of all analytic functions f on $\mathbb{D}$ such that

$$\sup_{z\in\mathbb{D}} \omega(z)|f'(z)| < \infty.$$

The *little Bloch-type space* $\mathcal{B}_{\omega,0}$ consists of all $f \in \mathcal{B}_\omega$ such that

$$\lim_{|z|\to 1} \omega(z)|f'(z)| = 0.$$

Both spaces $\mathcal{B}_\omega$ and $\mathcal{B}_{\omega,0}$ are Banach spaces with the norm

$$\|f\|_{\mathcal{B}_\omega} = |f(0)| + \sup_{z\in\mathbb{D}} \omega(z)|f'(z)|,$$

and $\mathcal{B}_{\omega,0}$ is a closed subspace of $\mathcal{B}_\omega$. For a closed subset $L \subset \mathcal{B}_{\omega,0}$, the compactness of it can be characterized as follows.

Lemma 1.1. *A closed set L in $\mathcal{B}_{\omega,0}$ is compact if and only if it is bounded with respect to the norm $\|\cdot\|_{\mathcal{B}_\omega}$ and satisfies*

$$\lim_{|z|\to 1} \sup_{f\in L} \omega(z)|f'(z)| = 0.$$

This result for the case $\omega(z) = (1-|z|^2)$ was proved by Madigan and Matheson [9]. By a modification of their proof, we can prove the above lemma. In the study on the compactness of operators which its range space is $\mathcal{B}_{\omega,0}$, this type of result is a very useful tool (see [6, 8, 9], *etc*).

1.3. **Boundedness and compactness.** Let X and Y be linear topological vector spaces. Recall that a linear operator $T : X \to Y$ is a bounded operator if the image of any topological bounded set in X under T is also a topological bounded set in Y. When X and Y are metric spaces with respect to suitable distance functions d_X and d_Y, we can define a metrical bounded operator from X into Y. Namely, a linear operator $T : X \to Y$ is a *metrical bounded operator* if there exists $C > 0$ such that $d_Y(Tf, 0) \le C d_X(f, 0)$ for $f \in X$. In general, the boundedness of T and the metrical boundedness of T do not coincide. When, however, X and Y are Banach spaces, the metrical boundedness of T coincides with the boundedness of T. For more informations on the metrical boundedness, we can refer to papers [2, 3]. Now we will introduce the third concept for the bounded operator from X into Y. We say that a linear operator $T : X \to Y$ is *bounded with respect to metric balls* if it takes every metric ball in X into a metric ball in

Y. Since a metric ball is also a bounded set if X and Y are Banach spaces, this boundedness coincides with the above two definitions for the boundedness.

A topological bounded subset L of N^+ is characterized by the following two conditions;

(i) there exists $K > 0$ such that $\|f\|_{N^+} \leq K$ for all $f \in L$,
(ii) the family $\{\log(1 + |f|)\}_{f \in L}$ is uniformly integrable.

The above condition (i) means that L is in a metric ball in N^+. This characterization was proved by Yanagihara [21]. For the class M and the space N^p, the same characterization for a bounded subset also holds (see [5, 20]). These mean that the topological boundedness of operators on Nevanlinna-type spaces is not equivalent to the metrical boundedness or the boundedness with respect to metric balls. To consider the boundedness for operators on Nevanlinna-type spaces, thus, we need another boudedness of operators. For the compactness for operators, we use the metrical compactness. Namely, $T : X \to Y$ is *metrically compact* if it takes every metric ball in X into a relatively compact subset in Y. Linear operators, for example composition operators or weighted composition operators, on Nevanlinna-type spaces have been studied by several authors (see [1, 2, 3, 10, 11, 16]). Motivated by these works, in this paper, we will give characterizations for the boundedness with metric balls and the metrical compactness of C_φ acting from Nevanlinna-type spaces into Bloch-type spaces.

Furthermore all our arguments can be applied to studies on generalized composition operators C_φ^g from Nevanlinna-type spaces into Bloch-type spaces. Hence we will formulate the corresponding results on operators C_φ^g without giving proofs.

2. The operator $C_\varphi : X \to \mathcal{B}_\omega$, when $X = N^+$, M or N^p

First we will give one of sufficient conditions for which C_φ is bounded with respect to metric balls.

Proposition 2.1. *If an analytic self-map φ of $\mathbb{D}$ satisfies*

$$\sup_{z \in \mathbb{D}} \frac{\omega(z)|\varphi'(z)|}{1 - |\varphi(z)|^2} \exp\left\{\frac{c}{1 - |\varphi(z)|^2}\right\} < \infty \tag{2.1}$$

for any $c > 0$, then $C_\varphi : N^+$ or $M \to \mathcal{B}_\omega$ is bounded with respect to metric balls.

Proof. Since $\|f\|_{N^+} \leq \|f\|_M$, it is enough to prove the case $C_\varphi : N^+ \to \mathcal{B}_\omega$. For $f \in N^+$ the inequality (1.1) gives

$$|f(z)| \leq \exp\left\{\frac{4\|f\|_{N^+}}{1 - |z|^2}\right\} \qquad \text{for } z \in \mathbb{D}.$$

By Cauchy's integral formula, we obtain that

$$(1 - |z|^2)|f'(z)| \leq \frac{2}{\pi} \int_{\mathbb{T}} |f(z + (1 - |z|)\zeta/2)|\,|d\zeta| \leq 4 \exp\left\{\frac{16\|f\|_{N^+}}{1 - |z|^2}\right\} \tag{2.2}$$

for each $z \in \mathbb{D}$. Hence we have that

$$\|C_\varphi f\|_{\mathcal{B}_\omega} \leq \exp\left\{\frac{4\|f\|_{N^+}}{1 - |\varphi(0)|^2}\right\} + 4 \sup_{z \in \mathbb{D}} \frac{\omega(z)|\varphi'(z)|}{1 - |\varphi(z)|^2} \exp\left\{\frac{16\|f\|_{N^+}}{1 - |\varphi(z)|^2}\right\}.$$

Combining this inequality and the condition (2.1), we see that C_φ takes every metric ball in N^+ into a metric ball in $\mathcal{B}_\omega$, namely $C_\varphi : N^+ \to \mathcal{B}_\omega$ is bounded with respect to metric balls. □

Remark 2.2. According to the above proof, we also see that the condition (2.1) implies $C_\varphi(N^+) \subset \mathcal{B}_\omega$. Since N^+ is an F-space, hence, the closed graph theorem shows that $C_\varphi : N^+ \to \mathcal{B}_\omega$ is bounded. But the condition (2.1) does not imply the metrical boundedness of it.

Lemma 2.3. *Let $X \in \{N^+, M, N^p\}$ $(1 < p < \infty)$. We assume that an analytic self-map φ of $\mathbb{D}$ satisfies $C_\varphi(X) \subset \mathcal{B}_\omega$. Then $C_\varphi : X \to \mathcal{B}_\omega$ is metrically compact if and only if for any sequences $\{f_j\}$ in X with $\|f_j\|_X \leq K$ and converge to zero uniformly on compact subsets of $\mathbb{D}$, $\{C_\varphi f_j\}$ converges to zero in $\mathcal{B}_\omega$.*

Proof. This is an extension of a well-known result on the compactness of composition operators on analytic function spaces. We see that any metrical bounded sequence in X form a normal family. Hence an argument by using the Montel theorem also proves this lemma. □

Theorem 2.4. *Let φ be an analytic self-map of $\mathbb{D}$. Then the following conditions are equivalent;*

(i) *$C_\varphi : N^+ \to \mathcal{B}_\omega$ is bounded with respect to metric balls,*
(ii) *$C_\varphi : M \to \mathcal{B}_\omega$ is bounded with respect to metric balls,*
(iii) *$C_\varphi : N^+ \to \mathcal{B}_\omega$ is metrically compact,*
(iv) *$C_\varphi : M \to \mathcal{B}_\omega$ is metrically compact,*
(v) *$\varphi \in \mathcal{B}_\omega$ and φ satisfies*

$$\lim_{|\varphi(z)|\to 1} \frac{\omega(z)|\varphi'(z)|}{1-|\varphi(z)|^2} \exp\left\{\frac{c}{1-|\varphi(z)|^2}\right\} = 0 \tag{2.3}$$

for any $c > 0$.

Proof. First note that the relation $\|f\|_{N^+} \leq \|f\|_M$ implies that every metric ball in M is also a metric ball in N^+. Thus implications (i) ⇒ (ii) and (iii) ⇒ (iv) are trivial. A relatively compact set in $\mathcal{B}_\omega$ is a bounded set in it. Since $\mathcal{B}_\omega$ is a Banach space, each bounded set is also in a metric ball in $\mathcal{B}_\omega$. Hence the metrically compactness of C_φ mapping into $\mathcal{B}_\omega$ gives the boundedness with respect to metric balls. So implications (iii) ⇒ (i) and (iv) ⇒ (ii) are true.

Now we will prove (v) ⇒ (iii). Assume that the condition (2.3) holds and take an $\varepsilon > 0$ arbitrary. Then we can choose an $0 < r_0 < 1$ such that

$$\sup_{|\varphi(z)|>r_0} \frac{\omega(z)|\varphi'(z)|}{1-|\varphi(z)|^2} \exp\left\{\frac{c}{1-|\varphi(z)|^2}\right\} < \varepsilon$$

for any $c > 0$. Choose any sequence $\{f_j\}$ in N^+ with $\|f_j\|_{N^+} \leq K$ for all j converging to zero uniformly on compact subsets of $\mathbb{D}$. Since $\varphi \in \mathcal{B}_\omega$ and (2.3) imply (2.1), by Proposition 2.1 or its remark, we see that $C_\varphi(N^+) \subset \mathcal{B}_\omega$. The assumption $\varphi \in \mathcal{B}_\omega$ also implies that

$$\sup_{|\varphi(z)|\leq r_0} \omega(z)|(C_\varphi f)'(z)| \leq \sup_{z\in\mathbb{D}} \omega(z)|\varphi'(z)| \cdot \max_{|w|\leq r_0} |f_j'(w)| \to 0$$

as $j \to \infty$. On the other hand, it follows from (2.2) that

$$\sup_{|\varphi(z)|>r_0} \omega(z)|(C_\varphi f)'(z)| \leq 4 \sup_{|\varphi(z)|>r_0} \frac{\omega(z)|\varphi'(z)|}{1-|\varphi(z)|^2} \exp\left\{\frac{16K}{1-|\varphi(z)|^2}\right\} < \varepsilon.$$

Thus we see

$$\limsup_{j\to\infty} \|C_\varphi f_j\|_{\mathcal{B}_\omega} \leq \varepsilon.$$

Since ε is arbitrary, we get $\|C_\varphi f_j\|_{\mathcal{B}_\omega} \to 0$ as $j \to \infty$. Lemma 2.3 shows that $C_\varphi : N^+ \to \mathcal{B}_\omega$ is metrically compact.

Finally we prove the implication (ii) $\Rightarrow$ (v). Fix $c > 0$ and put $w = \varphi(z)$. We define the following functions:

$$f_w(v) = \exp\left\{c\frac{1-|w|^2}{(1-\overline{w}v)^2}\right\} - 1 \quad \text{and} \quad g_w(v) = c\frac{1-|w|^2}{(1-\overline{w}v)^2}.$$

Then g_w belongs to the Hardy space H^1 and $\|g_w\|_{H^1} = c$. By an application of the nontangential complex maximal theorem for H^1 functions (see [4, Theorem II.3.1]), we have that

$$\int_{\mathbb{T}} \log(1 + N_\alpha f_w(\zeta))\, d\sigma(\zeta) \leq \int_{\mathbb{T}} N_\alpha g_w(\zeta)\, d\sigma(\zeta) \leq C\|g_w\|_{H^1} = C, \tag{2.4}$$

where $N_\alpha f$ denotes the nontangential maximal function:

$$N_\alpha f(\zeta) = \sup\{|f(z)| : z \in \Gamma_\alpha(\zeta)\}$$

and $\Gamma_\alpha(\zeta)$ $(0 < \alpha < 1)$ is the open convex hull of the set $\{\zeta\} \cup \alpha\mathbb{D}$. Inequalities (2.4) show that $\{f_w\}$ forms a metric ball in M. Also we have that

$$f_w'(v) = 2c\frac{\overline{w}(1-|w|^2)}{(1-\overline{w}v)^3} \exp\left\{c\frac{1-|w|^2}{(1-\overline{w}v)^2}\right\}.$$

Since $\{C_\varphi f_w\}$ is a metric ball in $\mathcal{B}_\omega$, there is a positive constant C which independent of $w = \varphi(z)$ such that $\|C_\varphi f_w\|_{\mathcal{B}_\omega} \leq C$. Thus we obtain that

$$\begin{aligned} C &\geq \omega(z)|(C_\varphi f_w)'(z)| \\ &= 2c\frac{\omega(z)|\varphi'(z)||\varphi(z)|}{(1-|\varphi(z)|^2)^2} \exp\left\{\frac{c}{1-|\varphi(z)|^2}\right\}, \end{aligned}$$

and so

$$\frac{\omega(z)|\varphi'(z)|}{1-|\varphi(z)|^2} \exp\left\{\frac{c}{1-|\varphi(z)|^2}\right\} \leq \frac{C(1-|\varphi(z)|^2)}{2c|\varphi(z)|}.$$

Taking limit as $|\varphi(z)| \to 1$ on both sides of the above inequality, we get (v). This completes the proof. □

Each $f \in N^p$ has the estimation:

$$\log(1 + |f(z)|) \leq \frac{4^{1/p}\|f\|_{N^p}}{(1-|z|^2)^{1/p}}. \tag{2.5}$$

Thus we obtain the following sufficient condition for the boundedness with respect to metric balls of $C_\varphi : N^p \to \mathcal{B}_\omega$ by using the inequality (2.5) instead of (1.1).

Proposition 2.5. *Let $1 < p < \infty$. If an analytic self-map φ of $\mathbb{D}$ satisfies*

$$\sup_{z\in\mathbb{D}} \frac{\omega(z)|\varphi'(z)|}{1-|\varphi(z)|^2} \exp\left\{\frac{c}{(1-|\varphi(z)|^2)^{1/p}}\right\} < \infty$$

for any $c > 0$, then $C_\varphi : N^p \to \mathcal{B}_\omega$ is bounded with respect to metric balls.

Moreover we consider the following test functions

$$f_w(v) = \exp\left\{c\left(\frac{1-|w|^2}{(1-\overline{w}v)^2}\right)^{1/p}\right\} - 1,$$

where $w = \varphi(z)$ and $c > 0$. Then we see that the family $\{f_w\}$ becomes a metic ball in N^p converging to 0 uniformly on compact subsets on $\mathbb{D}$ as $|w| \to 1$. Hence the inequality (2.5) and the above functions $\{f_w\}$ give an analogue of Theorem 2.4 as follows.

Theorem 2.6. *Let $1 < p < \infty$ and φ be an analytic self-map of $\mathbb{D}$. Then the following conditions are equivalent;*

(i) *$C_\varphi : N^p \to \mathcal{B}_\omega$ is bounded with respect to metric balls,*
(ii) *$C_\varphi : N^p \to \mathcal{B}_\omega$ is metrically compact,*
(iii) *$\varphi \in \mathcal{B}_\omega$ and φ satisfies*

$$\lim_{|\varphi(z)|\to 1} \frac{\omega(z)|\varphi'(z)|}{1-|\varphi(z)|^2} \exp\left\{\frac{c}{(1-|\varphi(z)|^2)^{1/p}}\right\} = 0$$

for any $c > 0$.

Proof. Since the proof is similar to that of Theorem 2.4, we omit the details of it. □

3. The operator $C_\varphi : X \to \mathcal{B}_{\omega,0}$, when $X = N^+$, M or N^p

In this section we will investigate operators $C_\varphi : N^+$ or $M \to \mathcal{B}_{\omega,0}$ and $C_\varphi : N^p \to \mathcal{B}_{\omega,0}$. As in Section 2, it is enough only to prove the case $C_\varphi : N^+$ or $M \to \mathcal{B}_{\omega,0}$.

Proposition 3.1. *If an analytic self-map φ of $\mathbb{D}$ satisfies*

$$\lim_{|z|\to 1} \frac{\omega(z)|\varphi'(z)|}{1-|\varphi(z)|^2} \exp\left\{\frac{c}{1-|\varphi(z)|^2}\right\} = 0 \tag{3.1}$$

for all $c > 0$, then $C_\varphi : N^+$ or $M \to \mathcal{B}_{\omega,0}$ is bounded with respect to metric balls. Also we obtain that $C_\varphi(N^+) \subset \mathcal{B}_{\omega,0}$ or $C_\varphi(M) \subset \mathcal{B}_{\omega,0}$

Proof. Since $C_\varphi : N^+$ or $M \to \mathcal{B}_\omega$ is bounded with respect to balls by Proposition 2.1, we only prove that $C_\varphi(L) \subset \mathcal{B}_{\omega,0}$ for any metric balls L in N^+. However the inequality (2.2) shows that

$$\omega(z)|(C_\varphi f)'(z)| \le 4\frac{\omega(z)|\varphi'(z)|}{1-|\varphi(z)|^2} \exp\left\{\frac{16\|f\|_{N^+}}{1-|\varphi(z)|^2}\right\}$$

for each $f \in L$. Thus the condition (3.1) implies $\omega(z)|(C_\varphi f)'(z)| \to 0$ as $|z| \to 1$. The case $C_\varphi : M \to \mathcal{B}_{\omega,0}$ is verified by the relation $\|f\|_{N^+} \le \|f\|_M$. □

Theorem 3.2. *Let φ be an analytic self-map of $\mathbb{D}$. Then the following conditions are equivalent;*

(i) $C_\varphi : N^+ \to \mathcal{B}_{\omega,0}$ *is bounded with respect to metric balls,*
(ii) $C_\varphi : N^+ \to \mathcal{B}_\omega$ *is bounded with respect to metric balls and* $\varphi \in \mathcal{B}_{\omega,0}$*,*
(iii) $C_\varphi : N^+ \to \mathcal{B}_{\omega,0}$ *is metrically compact,*
(iv) $C_\varphi : M \to \mathcal{B}_{\omega,0}$ *is bounded with respect to metric balls,*
(v) $C_\varphi : M \to \mathcal{B}_\omega$ *is bounded with respect to metric balls and* $\varphi \in \mathcal{B}_{\omega,0}$*,*
(vi) $C_\varphi : M \to \mathcal{B}_{\omega,0}$ *is metrically compact,*
(vii) φ *satisfies the condition* (3.1)*.*

Proof. By the same reasons as in the proof of Theorem 2.4, implications (i) $\Rightarrow$ (iv), (iii) $\Rightarrow$ (vi), (iii) $\Rightarrow$ (i) and (vi) $\Rightarrow$ (iv) are true. Also we can easily see that (i) $\Rightarrow$ (ii) and (iv) $\Rightarrow$ (v) are hold. In fact, we may consider the function $f(z) = z$ in N^+ or M. This one satisfies $\|f\|_{N^+} \le \|f\|_M \le \log 2$, and so f is in some metric balls in N^+ or M. This shows that $\varphi \in \mathcal{B}_{\omega,0}$.

To prove (ii) $\Rightarrow$ (vii), we take a sequence $\{z_j\}$ in $\mathbb{D}$ with $|z_j| \to 1$ as $j \to \infty$ and

$$\begin{aligned} &\limsup_{|z|\to 1} \frac{\omega(z)|\varphi'(z)|}{1-|\varphi(z)|^2} \exp\left\{\frac{c}{1-|\varphi(z)|^2}\right\} \\ &= \lim_{j\to\infty} \frac{\omega(z_j)|\varphi'(z_j)|}{1-|\varphi(z_j)|^2} \exp\left\{\frac{c}{1-|\varphi(z_j)|^2}\right\}. \end{aligned} \tag{3.2}$$

If $\sup_{j\ge 1} |\varphi(z_j)| < 1$, then the assumption $\varphi \in \mathcal{B}_{\omega,0}$ implies that the right limit in the equation (3.2) is equal to 0, and so we obtain the condition (3.1). If $\sup_{j\ge 1} |\varphi(z_j)| = 1$, then we can choose a subsequence $\{z_{j_k}\} \subset \{z_j\}$ such that $|\varphi(z_{j_k})| \to 1$ as $k \to \infty$. Since $C_\varphi : N^+ \to \mathcal{B}_\omega$ is bounded with respect to metric balls, by Theorem 2.4, φ satisfies

$$\lim_{|\varphi(z)|\to 1} \frac{\omega(z)|\varphi'(z)|}{1-|\varphi(z)|^2} \exp\left\{\frac{c}{1-|\varphi(z)|^2}\right\} = 0 \tag{3.3}$$

for any $c > 0$. By (3.2) and (3.3) we have that

$$\begin{aligned} &\limsup_{|z|\to 1} \frac{\omega(z)|\varphi'(z)|}{1-|\varphi(z)|^2} \exp\left\{\frac{c}{1-|\varphi(z)|^2}\right\} \\ &= \lim_{k\to\infty} \frac{\omega(z_{j_k})|\varphi'(z_{j_k})|}{1-|\varphi(z_{j_k})|^2} \exp\left\{\frac{c}{1-|\varphi(z_{j_k})|^2}\right\} \\ &\le \limsup_{|\varphi(z)|\to 1} \frac{\omega(z)|\varphi'(z)|}{1-|\varphi(z)|^2} \exp\left\{\frac{c}{1-|\varphi(z)|^2}\right\} = 0. \end{aligned}$$

This implies that (3.1) holds. (v) $\Rightarrow$ (vii) is also verified by the same argument in the above.

Finally we will prove the implication (vii) $\Rightarrow$ (iii). Take any metric ball L_{N^+} in N^+. Then there is a constant $K > 0$ such that $\|f\|_{N^+} \le K$ for any $f \in L_{N^+}$. For any $f \in L_{N^+}$ and $z \in \mathbb{D}$ we have that

$$\omega(z)|(C_\varphi f)'(z)| \le 4\frac{\omega(z)|\varphi'(z)|}{1-|\varphi(z)|^2} \exp\left\{\frac{16K}{1-|\varphi(z)|^2}\right\}.$$

Combining this with (3.1), we obtain

$$\lim_{|z|\to 1} \sup_{f\in L_{N^+}} \omega(z)|(C_\varphi f)'(z)| = 0,$$

and so Lemma 1.1 shows that $C_\varphi(L_{N^+})$ is compact in $\mathcal{B}_{\omega,0}$ for any metric balls L_{N^+}. This means that $C_\varphi : N^+ \to \mathcal{B}_{\omega,0}$ is metrically compact. The proof is accomplished. □

Finally we will state results for the case $X = N^p$. Proofs of these ones are similar to those of Proposition 3.1 and Theorem 3.2. Hence they will be omitted.

Proposition 3.3. *Let $1 < p < \infty$. If an analytic self-map φ of $\mathbb{D}$ satisfies*

$$\lim_{|z|\to 1} \frac{\omega(z)|\varphi'(z)|}{1-|\varphi(z)|^2} \exp\left\{\frac{c}{(1-|\varphi(z)|^2)^{1/p}}\right\} = 0 \tag{3.4}$$

for all $c > 0$, then $C_\varphi : N^p \to \mathcal{B}_{\omega,0}$ is bounded with respect to metric balls. Also we obtain $C_\varphi(N^p) \subset \mathcal{B}_{\omega,0}$.

Theorem 3.4. *Let $1 < p < \infty$ and φ be an analytic self-map of $\mathbb{D}$. The following conditions are equivalent;*

(i) *$C_\varphi : N^p \to \mathcal{B}_{\omega,0}$ is bounded with respect to metric balls,*
(ii) *$C_\varphi : N^p \to \mathcal{B}_{\omega}$ is bounded with respect to metric balls and $\varphi \in \mathcal{B}_{\omega,0}$,*
(iii) *$C_\varphi : N^p \to \mathcal{B}_{\omega,0}$ is metrically compact,*
(iv) *φ satisfies the condition (3.4).*

4. On generalized composition operators

For $g \in H(\mathbb{D})$ and an analytic self-map φ of $\mathbb{D}$ the generalized composition operator C_φ^g introduced by Li and Stević is defined by

$$C_\varphi^g f(z) = \int_0^z f'(\varphi(\zeta))g(\zeta)\, d\zeta.$$

Since $(C_\varphi^g f)'(z) = f'(\varphi(z))g(z)$ and $C_\varphi^g f(0) = 0$, by using the inequality (2.2) or (2.5), we have

$$\|C_\varphi^g f\|_{\mathcal{B}_\omega} \le 4 \sup_{z\in\mathbb{D}} \frac{\omega(z)|g(z)|}{1-|z|^2} \exp\left\{\frac{16\|f\|_X}{1-|\varphi(z)|^2}\right\}$$

for $f \in X$ where $X \in \{N^+, M\}$, or

$$\|C_\varphi^g f\|_{\mathcal{B}_\omega} \le 4 \sup_{z\in\mathbb{D}} \frac{\omega(z)|g(z)|}{1-|z|^2} \exp\left\{\frac{16^{1/p}\|f\|_{N^p}}{(1-|\varphi(z)|^2)^{1/p}}\right\}$$

for $f \in N^p$. Hence if we replaced $|\varphi'(z)|$ with $|g(z)|$ in arguments in the above section 2, then we obtain the characterizetions for the boundedness with respect to metric ball of opreators $C_\varphi^g : N^+$, M or $N^p \to \mathcal{B}_\omega$. Moreover we can easily see that Lemma 2.3 holds for the operator C_φ^g. Thus we also obtain the characterizations for the metrically compactness of C_φ^g. Since the proofs of the following results are quite similar to the proofs of Theorem 2.4 and 2.6, we have decided to state only the results.

Theorem 4.1. *Let $g \in H(\mathbb{D})$ and φ be an analytic self-map of $\mathbb{D}$. Then the following conditions are equivalent;*

(i) $C_\varphi^g : N^+ \to \mathcal{B}_\omega$ *is bounded with respect to metric balls,*
(ii) $C_\varphi^g : M \to \mathcal{B}_\omega$ *is bounded with respect to metric balls,*
(iii) $C_\varphi^g : N^+ \to \mathcal{B}_\omega$ *is metrically compact,*
(iv) $C_\varphi^g : M \to \mathcal{B}_\omega$ *is metrically compact,*
(v) *g and φ satisfy* $\sup_{z\in\mathbb{D}} \omega(z)|g(z)| < \infty$ *and*

$$\lim_{|\varphi(z)|\to 1} \frac{\omega(z)|g(z)|}{1-|\varphi(z)|^2} \exp\left\{\frac{c}{1-|\varphi(z)|^2}\right\} = 0$$

for any $c > 0$.

Theorem 4.2. *Let $1 < p < \infty$, $g \in H(\mathbb{D})$ and φ be an analytic self-map of $\mathbb{D}$. Then the following conditions are equivalent;*

(i) $C_\varphi^g : N^p \to \mathcal{B}_\omega$ *is bounded with respect to metric balls,*
(ii) $C_\varphi^g : N^p \to \mathcal{B}_\omega$ *is metrically compact,*
(iii) *g and φ satisfy* $\sup_{z\in\mathbb{D}} \omega(z)|g(z)| < \infty$ *and*

$$\lim_{|\varphi(z)|\to 1} \frac{\omega(z)|g(z)|}{1-|\varphi(z)|^2} \exp\left\{\frac{c}{(1-|\varphi(z)|^2)^{1/p}}\right\} = 0$$

for any $c > 0$.

Operators $C_\varphi^g : N^+, M$ or $N^p \to \mathcal{B}_{\omega,0}$ have the same characterization as Theorem 3.2 or 3.4. By considering the test function $f(z) = z$, we see that the boundedness with metric balls of $C_\varphi^g : X \to \mathcal{B}_{\omega,0}$ where $X \in \{N^+, M, N^p\}$ implies $\omega(z)|g(z)| \to 0$ as $|z| \to 1$. Hence we obtain the following characterizations for $C_\varphi^g : X \to \mathcal{B}_{\omega,0}$.

Theorem 4.3. *Let $g \in H(\mathbb{D})$ and φ be an analytic self-map of $\mathbb{D}$. Then the following conditions are equivalent;*

(i) $C_\varphi^g : N^+ \to \mathcal{B}_{\omega,0}$ *is bounded with respect to metric balls,*
(ii) $C_\varphi^g : N^+ \to \mathcal{B}_\omega$ *is bounded with respect to metric balls and* $\omega(z)|g(z)| \to 0$ *as* $|z| \to 1$,
(iii) $C_\varphi^g : N^+ \to \mathcal{B}_{\omega,0}$ *is metrically compact,*
(iv) $C_\varphi^g : M \to \mathcal{B}_{\omega,0}$ *is bounded with respect to metric balls,*
(v) $C_\varphi^g : M \to \mathcal{B}_\omega$ *is bounded with respect to metric balls and* $\omega(z)|g(z)| \to 0$ *as* $|z| \to 1$,
(vi) $C_\varphi^g : M \to \mathcal{B}_{\omega,0}$ *is metrically compact,*
(vii) *g and φ satisfy*

$$\lim_{|z|\to 1} \frac{\omega(z)|g(z)|}{1-|\varphi(z)|^2} \exp\left\{\frac{c}{1-|\varphi(z)|^2}\right\} = 0$$

for all $c > 0$.

Theorem 4.4. *Let $1 < p < \infty$, $g \in H(\mathbb{D})$ and φ be an analytic self-map of $\mathbb{D}$. The following conditions are equivalent;*

(i) $C_\varphi^g : N^p \to \mathcal{B}_{\omega,0}$ *is bounded with respect to metric balls,*

(ii) $C_\varphi^g : N^p \to \mathcal{B}_\omega$ *is bounded with respect to metric balls and* $\omega(z)|g(z)| \to 0$ *as* $|z| \to 1$,

(iii) $C_\varphi^g : N^p \to \mathcal{B}_{\omega,0}$ *is metrically compact,*

(iv) g *and* φ *satisfy*

$$\lim_{|z|\to 1} \frac{\omega(z)|g(z)|}{1-|\varphi(z)|^2} \exp\left\{\frac{c}{(1-|\varphi(z)|^2)^{1/p}}\right\} = 0$$

for all $c > 0$.

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