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ON PSEUDODIFFERENTIAL OPERATORS WITH SYMBOLS IN GENERALIZED SHUBIN CLASSES AND AN APPLICATION TO LANDAU-WEYL OPERATORS

FRANZ LUEF[1]* AND ZOHREH RAHBANI[2]

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ABSTRACT. The relevance of modulation spaces for deformation quantization, Landau–Weyl quantization and noncommutative quantum mechanics became clear in recent work. We continue this line of research and demonstrate that $Q_s(\mathbb{R}^{2d})$ is a good class of symbols for Landau–Weyl quantization and propose that the modulation spaces $M^p_{v_s}(\mathbb{R}^{2d})$ are natural generalized Shubin classes for the Weyl calculus. This is motivated by the fact that the Shubin class $Q_s(\mathbb{R}^{2d})$ is the modulation space $M^2_{v_s}(\mathbb{R}^{2d})$. The main result gives estimates of the singular values of pseudodifferential operators with symbols in $M^p_{v_s}(\mathbb{R}^{2d})$ for the standard Weyl calculus and for the Landau–Weyl calculus.

1. INTRODUCTION

Recently physicists have proposed an extension of quantum mechanics in one approach to quantum cosmology [1, 2] that goes under the name of noncommutative quantum mechanics. In [6, 7] we have used an extended Weyl calculus and modulation spaces to gain a deeper understanding of the mathematical structures underlying noncommutative quantum mechanics. These results are a continuation of a new approach to Moyal quantization in [16] and quantum mechanical systems of charged particles in strong uniform magnetic fields in [17, 18]. In this

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* Corresponding author.
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paper we focus on the relevance of modulation spaces as symbol classes of pseudodifferential operators, in particular we investigate the Shubin class $Q_s(\mathbb{R}^{2d})$. The Shubin class can be identified with the modulation spaces $M^2_{v_s}(\mathbb{R}^{2d})$. Our main results deal with decay estimates for singular values of pseudodifferential operators with symbols in general Shubin class $M^p_{v_s}(\mathbb{R}^{2d})$, which is based on Heil's method for the case $Q_s(\mathbb{R}^{2d})$. Our result extend the ones by Heil for $Q_s(\mathbb{R}^{2d})$. In [18] the extended Weyl calculus was called *Landau–Weyl* calculus. As a consequence of results in [18] we are able to derive singular value estimates for Landau–Weyl operators with symbols in the generalized Shubin class $M^p_s(\mathbb{R}^{2d})$. Finally we note that the Shubin class $Q_s(\mathbb{R}^{2d})$ is a Banach algebra with respect to convolution and point-wise multiplication for $s > d$ follows from Nikolskii's results on convolution algebras of weighted Lebesgue spaces $L^p_v(\mathbb{R}^{2d})$.

During the early 1980's Feichtinger introduced the class of *modulation spaces* in [9, 10]. Later he gave in joint work with Gröchenig a description of modulation spaces as coorbit spaces of the Heisenberg group [12]. In the last two decades modulation spaces have turned out to be the correct mathematical framework for a rigorous treatment of many problems in time-frequency analysis. In addition modulation spaces have been applied in a variety of areas, e.g. pseudodifferential operators, Schrödinger operators, quantum mechanics, harmonic analysis, sampling theory, stochastic processes etc., see [11] for an extensive list of references. We refer the interested reader to the excellent survey article [11] by Feichtinger for a discussion of the history and the manifold applications of modulation spaces. The proofs of our main results rely on a description of modulation spaces in terms of a Wilson basis that were constructed by Daubechies, Jaffard and Journe in [5]. A Wilson basis is an orthonormal basis of $L^2(\mathbb{R})$ build from a Gabor frame of redundancy 2. Since the physicist K.G. Wilson had suggested in an informal way a similar family of functions in [29], the basis was called Wilson basis. This groundbreaking result was extended by Feichtinger, Gröchenig and Walnut to the class of modulation spaces in [14]. We are convinced that modulation spaces and their Wilson basis description will be very useful for other problems in mathematical physics, e.g. the ones discussed in [26, 30, 31, 32]. The techniques and constructions in this paper are in terms of objects associated to the Heisenberg group.

The paper is organized as follows: In Section 2 we define the basic notions of time-frequency analysis and briefly discuss the Weyl calculus and the Landau–Weyl calculus. In Section 3 we introduce the modulation spaces $M^p_{v_s}(\mathbb{R}^{2d})$ for the standard radial symmetric weight $v_s(x,\omega) = (1 + x^2 + \omega^2)^{s/2}$ and the Shubin classes $Q_s(\mathbb{R}^{2d})$. We recall Nikolskii's result on convolution algebras and establish the Banach algebra properties of the Shubin class $Q_s(\mathbb{R}^{2d})$. In Section 4 we provide Schatten class results for pseudodifferential operators with symbols from generalized Shubin classes $M^p_{v_s}(\mathbb{R}^{2d})$ and provide an application to Landau–Weyl operators.

Notation: For x and ω in $\mathbb{R}^d$ we denote their scalar product by $x \cdot \omega$ and for the sake of simplicity we abbreviate $\|x\|^2 = x \cdot x$ by x^2. The standard symplectic form on $\mathbb{R}^d \times \mathbb{R}^d$ is given by $\Omega(z_1, z_2) = x_2 \cdot \omega_1 - x_1 \cdot \omega_2$ for $z_1 = (x_1, \omega_1)$ and $z_2 = (x_2, \omega_2)$. For the Schwartz class we use the symbol $\mathcal{S}(\mathbb{R}^d)$, and for its dual, the space of tempered distributions, we write $\mathcal{S}'(\mathbb{R}^d)$. For a Banach space B of functions on $\mathbb{R}^d$ and a weight v we denote by B_v the weighted Banach space $\{f \in B : fv \in B\}$ with norm $\|f\|_{B_v} = \|fv\|_B$ for $f \in B$. In our discussion B_v is either the Lebesgue space $L^p_v(\mathbb{R}^{2d})$ or the sequence space $\ell^p_v(\mathbb{Z}^{2d})$. We denote the convolution of two compactly supported functions f and g on $\mathbb{R}^d$ by $(f * g)(x) = \int f(y)g(x-y)dy$. Finally $\mathcal{I}_p$ denotes the Schatten-von Neumann class of bounded linear operators on a Hilbert space $\mathcal{H}$ consisting of all compact operators T on $\mathcal{H}$ with singular values $(s_j(T))$ in ℓ^p.

2. Basics on time-frequency analysis and pseudodifferential operators

From a mathematical point of view time-frequency analysis is relying on the structure of the Heisenberg group. The Schrödinger representation of the Heisenberg group acts on $L^2(\mathbb{R}^d)$ by means of the unitary operators

$$\rho(x,\omega)f = e^{-\pi i x\cdot\omega}\pi(x,\omega)f(t) = e^{-\pi i x\cdot\omega}M_\omega T_x f,$$

where T_x and M_ω are defined by $T_x f(t) = f(t-x)$ and $M_\omega f(t) = e^{2\pi i x\cdot\omega} f(t)$ for $x, \omega \in \mathbb{R}^d$. The time-frequency/phase space shifts $\pi(z) = M_\omega T_x$ for $z = (x, \omega)$ yield a projective unitary representation of the time-frequency plane/phase space $\mathbb{R}^{2d}$ on $L^2(\mathbb{R}^d)$, which follows from the commutation relation between M_ω and T_x:

$$T_x M_\omega = e^{-2\pi i x\cdot\omega} M_\omega T_x.$$

Consequently the composition law of π's is as follows:

$$\pi(z)\pi(z') = e^{2\pi\Omega(z,z')}\pi(z')\pi(z),$$

where $\Omega(z, z')$ denotes the standard symplectic form on $\mathbb{R}^{2d}$.
Many researchers use in the definition of the Schrödinger representation ρ an alternative unimodular complex numbers τ instead of $e^{-\pi i x\cdot\omega}$. For the discussion of Weyl operators our particular choice appears to be most suitable.
Via integration the irreducible unitary Schrödinger representation of the Heisenberg group gives rise to a class of operators

$$A = \iint_{\mathbb{R}^{2d}} a(z)\rho(z)dz$$

for $a \in L^1(\mathbb{R}^{2d})$. Let a_Ω denote the symplectic Fourier transform of $a \in L^1(\mathbb{R}^{2d})$, i.e.

$$a_\Omega(z) = \iint_{\mathbb{R}^{2d}} a(z')\, e^{2\pi i\Omega(z,z')} dz'.$$

The operator associated to a_Ω

$$L_a = \iint_{\mathbb{R}^{2d}} a_\Omega(z)\rho(z)dz$$

is the famous Weyl correspondence and a is called the Weyl symbol of the pseudodifferential operator L_a. Pseudodifferential operators appear in many areas

of mathematics, physics, and engineering, see [15] for a discussion motivated by time-frequency analysis. There are many ways to associate to a symbol a pseudodifferential operator, e.g. the Kohn–Nirenberg correspondence, but for our purpose the Weyl calculus is best suited.
The Weyl correspondence associates the following integral operator to a symbol a

$$L_a f(x) = \iint a\big(\tfrac{x+y}{2}, \omega\big) e^{2\pi i (x-y)\omega} f(y) dy d\omega$$

A basic problem is to identify classes of functions or distributions a that give bounded operators on function spaces, e.g. Hilbert spaces. We propose that the generalized Shubin classes $M^p_{v_s}(\mathbb{R}^d)$ as good symbol classes for pseudodifferential operators.
Research on quantum mechanical systems modeling charged particles in uniform magnetic fields, deformation quantization and noncommutative quantum mechanics suggests new quantization rules that yields to a Weyl calculus on phase space, see [6, 16, 18], and de Gosson called it Landau–Weyl correspondence to emphasis the connection to the Landau levels in the quantum mechanics of charged particles. We briefly review the basic notions underlying Landau–Weyl calculus. The main idea is to look for a Weyl correspondence on double time-frequency plane $\mathbb{R}^{2d} \times \mathbb{R}^{2d}$. To this end we first introduce a Schrödinger representation of $\mathbb{R}^{2d}$ on $L^2(\mathbb{R}^{2d})$

$$\tilde{\rho}(z')F(z) = e^{2\pi i \Omega(z,z')} F(z - z') \text{ for } F \in L^2(\mathbb{R}^{2d}),$$

and by an elementary calculation we have that

$$\tilde{\rho}(z_1 + z_2) = e^{-2\pi i \Omega(z_1,z_2)} \tilde{\rho}(z_1)\tilde{\rho}(z_2) \text{ for all } z_1, z_2 \in \mathbb{R}^{2d}.$$

The integrated representation of $\tilde{\rho}$ for a in $L^1(\mathbb{R}^{2d})$ gives a class of operators

$$\tilde{L}_a = \iint_{\mathbb{R}^{2d}} a_\Omega(z)\tilde{\rho}(z) dz$$

that are called *Landau–Weyl* operators and the mapping $a \mapsto \tilde{L}_a$ as Landau–Weyl correspondence. The kernel of the associated integral operator is

$$\tilde{K}(z, u) = e^{2\pi i \Omega(z,u)} a_\Omega(z, u).$$

The intertwiner between the two unitary representations ρ and $\tilde{\rho}$ of the Heisenberg group is the windowed wavepacket transform

$$\mathcal{U}_\varphi g(z) = 2^d W(g, \varphi)(\tfrac{z}{2}) \text{ for } \varphi, g \in L^2(\mathbb{R}^d),$$

where $W(g, \varphi)$ is the Wigner distribution

$$W(g, \varphi)(z) = \int_{\mathbb{R}^d} e^{-2\pi i \omega \cdot t} g(x + \tfrac{t}{2}) \overline{\varphi(x - \tfrac{t}{2})} dt.$$

An elementary computation establishes that $\mathcal{U}_\varphi$ is an intertwiner between the two representations. For later reference we collect these observations in the following lemma.

Lemma 2.1. *Let a be a Weyl symbol. Then we have*

$$\mathcal{U}_\varphi^\star \tilde{L}_a \mathcal{U}_\varphi = L_a.$$

The intertwiner allows one to transfer spectral properties of L_a to ones of $\tilde{L}_a$. For a proof we refer the interested reader to [17].

Proposition 2.2. *Let a be a Weyl symbol. Then (i) the discrete spectrum of L_a and $\tilde{L}_a$ are the same; (ii) If f is an eigenvector of L_a, i.e. $L_a f = \lambda f$, then $F = \mathcal{U}_\varphi f$ is an eigenvector of $\tilde{L}_a$ and $\tilde{L}_a F = \lambda F$; (ii) suppose that F is an eigenvector of $\tilde{L}_a$, i.e. $\tilde{L}_a F = \lambda F$, then $f = \mathcal{U}_\varphi^\star F$ is an eigenvector of L_a and $L_a f = \lambda f$.*

3. Modulation spaces

After the introduction of modulation spaces in [10] by Feichtinger his joint work with Gröchenig on coorbit theory [12] worked out the intrinsic relation between modulation spaces and time-frequency analysis via square-integrable representations of the Heisenberg group. In fact, coorbit theory specialized to the Schrödinger representation of the Heisenberg group provides the following definition of modulation spaces.

Let v_s be the radial symmetric weight $v_s(z) = (1 + x^2 + \omega^2)^{s/2}$ for $s \in \mathbb{R}$. Then for $p \geq 0$ the modulation space $M^p_{v_s}(\mathbb{R}^d)$ and a $g \in \mathscr{S}(\mathbb{R}^d)$ is defined as follows:

$$M^p_{v_s}(\mathbb{R}^d) = \{f \in \mathscr{S}'(\mathbb{R}^d) : \|f\|_{M^p_{v_s}} = \Big(\int_{\mathbb{R}^{2d}} |\langle f, \rho(x,\omega)g\rangle|^p v_s(x,\omega)^p dx d\omega\Big)^{1/p} < \infty\}.$$

This defines a Banach space and its norm is independent of the function g, any other $g \in \mathscr{S}(\mathbb{R}^d)$ yields an equivalent norm for $M^p_{v_s}(\mathbb{R}^d)$. Since the phase-factor $e^{-\pi i x\cdot\omega}$ is irrelevant in the definition of $M^p_{v_s}(\mathbb{R}^d)$ it is actually $V_g f(x,\omega) = \langle f, \pi(x,\omega)g\rangle$ that is of significance for the following discussion. In time-frequency analysis $V_g f$ is known as the Short-Time Fourier Transform of f with respect to a window g. Note that $V_g f(x,\omega)$ is the representation coefficient of the projective representation $(x,\omega) \mapsto \pi(x,\omega)$ of $\mathbb{R}^{2d}$, [4].

Modulation spaces may be defined in much greater generality, see [10], but for the present investigation the spaces $M^p_{v_s}(\mathbb{R}^d)$ provide the proper setting. A variety of function spaces may be identified with modulation spaces, e.g. Feichtinger's algebra $S_0(\mathbb{R}^d) = M^1_{v_0}(\mathbb{R}^d)$ [9], $M^2_{v_0}(\mathbb{R}^d) = L^2(\mathbb{R}^d)$, or more generally weighted L^2-spaces $L^2_s(\mathbb{R}^d) = \{f : \int_{\mathbb{R}^d} |f(x)|^2 (1+x^2)^{s/2} dx < \infty\}$, Bessel potential spaces $H_s = \{f : \int_{\mathbb{R}^d} |\widehat{f}(\omega)|^2 (1+\omega^2)^{s/2} d\omega < \infty\}$ and **Shubin classes** $Q_s(\mathbb{R}^d) = L^2_s(\mathbb{R}^d) \cap H_s(\mathbb{R}^{2d})$, which turns out to be $M^2_{v_s}(\mathbb{R}^d)$, see [3]. Throughout this paper we suppose $s > 0$.

In the sequel we need modulation spaces over $\mathbb{R}^{2d}$, which are defined in the same way as above. Before we continue the treatment of modulation spaces we interlude with a discussion of Banach algebra properties of $Q_s(\mathbb{R}^{2d})$ with respect to convolution and pointwise multiplication. We note that it is a consequence of Nikolskii's work on convolution algebras for weighted Lebesgue spaces $L^p_v(\mathbb{R}^d)$ and [8, 28].

A positive function v on $\mathbb{R}^{2d}$ is called a *Nikolskii–Wermer* weight, if for some $p \in [1,\infty]$ with conjugate exponent p' we have that

$$C_{p,v} = \sup_z \Big(\int_{\mathbb{R}^{2d}} \Big(\frac{v(z)}{v(z')v(z-z')}\Big)^{p'} dz'\Big)^{1/p'} < \infty. \tag{3.1}$$

Wermer proved that $L^p_v(\mathbb{R}^{2d})$ is a convolution algebra, if it satisfies the Nikolskii–Wermer condition (3.1). In the case of the $L^p_v(\mathbb{R}^{2d})$ Nikolskii demonstrated the sufficiency of (3.1) in [25], as well as the necessity of it in the case $p=\infty$. We give the elegant argument of Nikolskii in the case of $L^p_v(\mathbb{R}^{2d})$, since it appears to be forgotten by researcher in mathematics.

Proposition 3.1 (Nikolskii–Wermer)**.** *Let v be a Nikolskii–Wermer weight on $\mathbb{R}^{2d}$. Then $L^p_v(\mathbb{R}^{2d})$ is a Banach convolution algebra, i.e.*

$$\|F * G\|_{L^p_v} \le C_{p,v}\|F\|_{L^p_v}\|G\|_{L^p_v}.$$

Proof. Let $\tilde F$ and $\tilde G$ in $L^p_v(\mathbb{R}^{2d})$. Then we set $F = \tilde F v, G = \tilde G v$. An application of Hölder's inequality yields the assertion:

$$\begin{aligned}
\|(\tilde F * \tilde G)v\|_{L^p} &= \|v(\tfrac{F}{v} * \tfrac{G}{v})\|_{L^p} \\
&= \Big[\int_{\mathbb{R}^{2d}}\Big[\int_{\mathbb{R}^{2d}} \frac{|F(x,y)|}{v(x,y)}\frac{|G(x-z,y-t)|}{v(x-z,y-t)}dxdy\Big]^p v(z,t)^p dzdt\Big]^{1/p} \\
&= \Big[\int_{\mathbb{R}^{2d}}\Big[\int_{\mathbb{R}^{2d}} |F(x,y)||G(x-z,y-t)|\tfrac{v(z,t)}{v(x,y)v(x-z,y-t)}dxdy\Big]^p dzdt\Big]^{\frac{1}{p}} \\
&\le \Big[\int_{\mathbb{R}^{2d}}\Big[\Big(\int_{\mathbb{R}^{2d}}\Big(\frac{v(z,t)}{v(x,y)v(x-z,y-t)}\Big)^{p'}dxdy\Big)^{1/p'} \\
&\qquad \Big(\int_{\mathbb{R}^{2d}}(|F(x,y)||G(x-z,y-t)|)^p dxdy\Big)^{1/p}\Big]^p dzdt\Big]^{1/p} \\
&\le C_{p,v}\|F\|_{L^p}\|G\|_{L^p} \le C_{p,v}\|\tilde F\|_{L^p_v}\|\tilde G\|_{L^p_v}.
\end{aligned}$$

□

A straightforward computation demonstrates that v_s is a Nikolskii–Wermer weight of $L^p(\mathbb{R}^{2d})$ for $s > 2d/p'$.

Corollary 3.2. *If $s>d$, then $(L^2_{v_s}(\mathbb{R}^{2d}), *)$ is a Banach convolution algebra.*

Since the Bessel potential spaces $H^s(\mathbb{R}^{2d})$ are weighted function spaces on the Fourier transform side, we have that $H^s(\mathbb{R}^{2d})$ is a Banach algebra with respect to pointwise multiplication for $s>d$. Therefore we deduce that the Shubin class $Q^s(\mathbb{R}^{2d})$ is a Banach algebra with respect to convolution and pointwise multiplication for $s>d$.

Theorem 3.3. *$M^2_{v_s}(\mathbb{R}^d)$ is a Banach algebra with respect to convolution and pointwise multiplication if and only if $C_{2,v_s}<\infty$, i.e. for $s>d$. Consequently, $Q^s(\mathbb{R}^{2d})$ is in $S_0(\mathbb{R}^d)$.*

The final statement in the theorem is a consequence of results on embeddings of weighted L^p-spaces in [19].

3.1. **Wilson bases.** The great relevance of modulation spaces for time-frequency analysis is founded in a characterization of the spaces $M^p_{v_s}(\mathbb{R}^d)$ in terms of Gabor frames $\mathcal{G}(g,\Lambda)=\{\pi(\lambda)g:\lambda\in\Lambda\}$ for $L^2(\mathbb{R}^d)$, where g is called a Gabor atom and Λ is a lattice in $\mathbb{R}^{2d}$. Recall $\mathcal{G}(g,\Lambda)$ is a Gabor frame for $L^2(\mathbb{R}^d)$, if there exists two constants $A,B>0$ such that

$$A\|f\|_2^2\leq\sum_{\lambda\in\Lambda}\mid\langle f,\pi(\lambda)g\rangle\mid^2\leq B\|f\|_2^2$$

holds for all $f\in L^2(\mathbb{R}^d)$. If $A=B$ then we call $\mathcal{G}(g,\Lambda)$ a tight frame or a Parseval frame.

We have the following important fact, that is crucial for the modern theory of Gabor frames, see [22].

Theorem 3.4. *If $\mathcal{G}(g,\Lambda)$ is a Gabor frame for $L^2(\mathbb{R}^d)$ with $g\in M^1_{v_s}(\mathbb{R}^d)$, then $\mathcal{G}(g,\Lambda)$ is a Banach frame for $M^p_{v_s}(\mathbb{R}^d)$, $1\leq p\leq\infty$. Consequently, each f in $M^p_{v_s}(\mathbb{R}^d)$ has a discrete representation with respect to a dual window $\gamma\in M^1_{v_s}(\mathbb{R}^d)$, i.e.*

$$f=\sum_{\lambda\in\Lambda}\langle f,\pi(\lambda)\gamma\rangle\pi(\lambda)g$$

that converges unconditional for $p\in[1,\infty)$ and with weak*-converge in $M^\infty_{1/v_s}(\mathbb{R}^d)$. Furthermore, we have for all $f\in M^p_{v_s}(\mathbb{R}^d)$

$$A\|f\|_{M^p_{v_s}}\leq\big(\sum_{\lambda\in\Lambda}|\langle f,\pi(\lambda)g\rangle|^p v_s(\lambda)^p\big)^{1/p}\leq B\|f\|_{M^p_{v_s}}.$$

In general Gabor frames are redundant systems, but Daubechies, Jaffard and Journe found a modification of Gabor frames of redundancy 2 that provides an orthonormal basis for $L^2(\mathbb{R})$, [5]. This construction goes under the name *Wilson basis*, since they followed some suggestions of the physicist K. Wilson [29]. Wilson bases are a useful gadget in various proofs of basic properties of modulation spaces, e.g. in the isomorphism theorem for modulation spaces, the kernel theorem for $M^1_{v_s}(\mathbb{R}^d)$, or in the approach of Feichtinger and Kozek to quantization [15].

Let $\mathcal{G}(g,\frac{1}{2}\mathbb{Z}\times\mathbb{Z})$ be a Gabor system of redundancy 2 in $L^2(\mathbb{R})$. Then the associated *Wilson system*, $\mathcal{W}(g)$, consists of the functions

$$\begin{aligned}\psi_{k0}&=T_kg\\ \psi_{kn}&=\sqrt{2}\ \mathrm{Re}(T_{k/2}M_ng)\quad k,n\in\mathbb{Z}\ ,k\neq0\\ \psi_{kn}&=\sqrt{2}\ \mathrm{Im}(T_{k/2}M_ng)\quad k,n\in\mathbb{Z}\ ,k\neq0\end{aligned}$$

or briefly

$$\psi_{kn}=c_nT_{k/2}(M_n+(-1)^{k+n}M_{-n})g\ \ k,n\in\mathbb{Z},n\geq0$$

where $c_0=1/2$, $c_n=1/\sqrt{2}$ for $n\geq0$.

Theorem 3.5 (Daubechies-Jaffard-Journe)**.** *Let g be in $L^2(\mathbb{R})$ such that $g=g^*$ and $\|g\|_2=1$. If $\mathcal{G}(g,\frac{1}{2}\mathbb{Z}\times\mathbb{Z})$ is a tight Gabor frame for $L^2(\mathbb{R})$, then $\mathcal{W}(g)$ is an orthonormal Wilson basis of $L^2(\mathbb{R})$.*

In [5] it was shown that the Gabor atom g in the preceding theorem can be chosen in the Schwartz class $\mathscr{S}(\mathbb{R})$ or even g and its Fourier transform $\widehat{g}$ may be exponentially decaying.
Shortly after this breakthrough Feichtinger, Gröchenig and Walnut extended the result of Theorem 3.5 to a characterization of modulation spaces in [14], i.e. they showed that $\mathcal{W}(g)$ is an unconditional basis for $M^p_{v_s}(\mathbb{R})$.

Theorem 3.6 (Feichtinger-Gröchenig-Walnut)**.** *Assume that $\mathcal{W}(g)$ is an orthonormal basis for $L^2(\mathbb{R})$ with $g \in M^1_{v_s}(\mathbb{R})$. Then we have that*

$$C^{-1}\|f\|_{M^p_{v_s}} \leq \big(\sum_{n=0}^{\infty}\sum_{k\in\mathbb{Z}} |\ \langle f, \psi_{kn}\rangle\ |^p\ v_s(\tfrac{k}{2}, n)^p\big)^{1/p} \leq C\|f\|_{M^p_{v_s}},$$

for a constant $C \geq 1$. Furthermore, the orthogonal expansion

$$f = \sum_{n=0}^{\infty}\sum_{k\in\mathbb{Z}}\langle f, \psi_{kn}\rangle\psi_{kn}$$

converges unconditionally in the $M^p_{v_s}(\mathbb{R})$-norm if $p \in [1, \infty)$ and weak in $M^\infty_{1/v_s}(\mathbb{R})$ otherwise.*

A proof of this theorem is given in Chapter 12 of [21] and was first given in [14].
So far we have considered Wilson bases in $\mathbb{R}$, but we can generalize them to higher dimensions by means of a tensor product construction.
Given an orthonormal basis $\mathcal{W}(g) = \{\psi_{kn} : k, n \in \mathbb{Z}, n \geq 0\}$ for $L^2(\mathbb{R})$ one defines the basis functions on $L^2(\mathbb{R}^d)$ as the (tensor) products

$$\Psi_{rs}(x) = \sqcap_{i=1}^{d}\psi_{r_i s_i}(x_i)$$

where $r, s \in \mathbb{Z}^d, s \geq 0$. Then $\{\Psi_{rs}, r, s \in \mathbb{Z}^d, s \geq 0\}$ is an orthonormal basis for $L^2(\mathbb{R}^d)$ and more generally an unconditional basis for $M^p_{v_s}(\mathbb{R}^d)$.

4. Schatten class results for Weyl and Landau–Weyl calculus

In this section we present and prove results on singular values of pseudodifferential operators for the Weyl and Weyl-Landau calculus with symbols in $M^p_{v_s}(\mathbb{R}^{2d})$. Our main theorem extends the ones by Heil in [23] for the case of Weyl quantization and symbols in the Shubin class $M^2_{v_s}(\mathbb{R}^{2d})$. The usefulness of modulation spaces as symbol classes for pseudodifferential operators was demonstrated for the first time in [27] and was later continued in [20].

The method of Heil relies on the fact that the pseudodifferential operators have a kernel and that a finite rank approximation of the kernel allows one to deduce estimates of its singular values. Instead of a frame expansion we invoke Wilson bases to obtain a finite rank approximation for the pseudodifferential operator. One of the reasons for this choice is that it provides a straightforward treatment of Feichtinger's kernel theorem for the modulation spaces $M^1_{v_s}(\mathbb{R}^d)$ for $s \geq 0$, see [15, 13]. The Schwartz kernel theorem is central in the treatment of distributions. Feichtinger was able to deduce a kernel theorem for $M^1_v(\mathbb{R}^d)$, where v is a weight on $\mathbb{R}^{2d}$. We restrict our discussion to the radial symmetric weight v_s, see [21] for the general case.

If L is a bounded operator between $M^1_{v_s}(\mathbb{R}^d)$ and $M^\infty_{1/v_s}(\mathbb{R}^d)$, then we can associate to A a matrix $(a_{(k,m),(l,n)})$ with respect to the Wilson basis by

$$a_{(k,m),(l,n)} = \langle L\psi_{l,n}, \psi_{k,m}\rangle \text{ for } k,l,m,n \in \mathbb{Z}^d, m,n \geq 0.$$

The fact that the Wilson description of modulation spaces yields an elementary proof of the kernel theorem for $M^1_{v_s}(\mathbb{R}^d)$, see [21].

Theorem 4.1 (Feichtinger). (1) *If L is a bounded operator from $M^1_{v_s}(\mathbb{R}^d)$ to $M^\infty_{1/v_s}(\mathbb{R}^d)$, then there exists a unique kernel $k \in M^\infty_{1/v_s}(\mathbb{R}^{2d})$ such that*

$$\langle Lf, g\rangle = \langle k, g \otimes \overline{f}\rangle \quad f,g \in M^1_{v_s}(\mathbb{R}^d).$$

(2) *An element $k \in M^\infty_{1/v_s}(\mathbb{R}^{2d})$ defines a bounded operator $L : M^1_{v_s}(\mathbb{R}^d) \longrightarrow M^\infty_{1/v_s}(\mathbb{R}^d)$ by*

$$\langle Lf, g\rangle = \langle k, g \otimes \overline{f}\rangle \quad f,g \in M^1_{v_s}(\mathbb{R}^d).$$

Let $\{\Psi_{k,l} : k,l \in \mathbb{R}^d\}$ be a Wilson basis for $L^2(\mathbb{R}^d)$. Then the matrix entries for the operator L are given by $a_{(k,l),(m,n)} = \langle L\Psi_{k,l}, \Psi_{m,n}\rangle$ and the associated kernel is

$$k = \sum_{k,l,m,n} a_{(k,l),(m,n)} \Psi_{k,l} \otimes \overline{\Psi_{m,n}}. \tag{4.1}$$

Therefore integral operators with kernels in $M^p_{v_s}(\mathbb{R}^{2d})$ are bounded operators from $M^1_{v_s}(\mathbb{R}^d)$ to $M^\infty_{1/v_s}(\mathbb{R}^d)$. Since pseudodifferential operators with symbol $a \in M^p_{v_s}(\mathbb{R}^{2d})$ may be viewed as integral operators A_k with kernel in $M^p_{v_s}(\mathbb{R}^{2d})$, we are interested in their compactness properties. The invariance of $M^p_{v_s}(\mathbb{R}^{2d})$ under symplectic Fourier transform yields that a pseudodifferential L_a with $a \in M^p_{v_s}(\mathbb{R}^{2d})$ has a kernel in $M^p_{v_s}(\mathbb{R}^{2d})$.

There is a vast literature on this circle of ideas. We make use of a theorem in [24], where Labate invokes some results about absolutely summing operators to prove statements about the summability of the singular values of pseudodifferential operators. Let T be a positive compact operator on the Hilbert space $L^2(\mathbb{R}^d)$. Then the singular values $\{s_j(T)\}_{j=1}^\infty$ are the square roots of the eigenvalues of the positive, selfadjoint operator T^*T.

Proposition 4.2 (Labate). *Let $1 \leq p < \infty$ and p' is the conjugate exponent of p.*

(i) *If $1 \leq p \leq 2$ and $k \in M^p(\mathbb{R}^d)$, then the singular values of A_k are 2-summable and*

$$\|A\|_{\mathcal{I}_2} = \|A_k\|_{\mathcal{I}_2} = \big(\sum s_j(A_k)^2\big)^{1/2} \leq C\,\|k\|_{M^p}.$$

A_k is a compact (for $p = 1$, A_k is weakly compact and completely continuous) operator which maps $M^p(\mathbb{R}^d)$ into itself.

(ii) *If* $2 \leq p < \infty$ *and* $k \in M^{p'}(\mathbb{R}^d)$, *then* A_k *is a compact and completely continuous operator which maps* $M^{p'}(\mathbb{R}^d)$ *into itself. Singular values of* A_k *are p-summable such that*

$$\|A\|_{\mathcal{I}_p} = \|A_k\|_{\mathcal{I}_p} = \left(\sum s_j(A_k)^p\right)^{1/p} \leq C\,\|k\|_{M^{p'}}.$$

The singular values $s_j(T)$ of T have a geometric description as the distance of T from the operators of rank less than j:

$$s_j(T) = \inf\{\|T - A\|_{L^2} :\ A \operatorname{rank}(A) < j\}.$$

We state this inequality in the following lemma and give its short proof. This description of singular values allows one to derive an elementary statement about singular values of a sum of two compact operators.

Lemma 4.3. *Let* S, T *be a compact operators on the Hilbert space* $L^2(\mathbb{R}^d)$. *Then we have that*

$$s_{j+k-1}(S+T) \leq s_j(S) + s_k(T.)$$

Proof. If $\operatorname{rank} A < j$ and $\operatorname{rank} B < k$ then $\operatorname{rank}(A+B) < j+k-1$ and

$$s_{j+k-1}(S+T) \leq \|S - A + T - B\|_{L^2} \leq \|S - A\|_{L^2} + \|T - B\|_{L^2}.$$

Therefore we establish the desired assertion after taking the infimum over the operators of finite rank less than m and n, respectively. □

The main idea is to expand the symbol of a pseudodifferential operators in terms of a Wilson basis. Actually we will prove that:

Theorem 4.4. *For integral kernel* $k \in M^p_{v_s}(\mathbb{R}^{2d})$ *and corresponding integral operator* $A_k f(x) = \int k(x,y) f(y) dy$ *and* $s > 0$, *the following statements hold for singular values* $\{s_j(A_k)\}$:

(i) *If* $1 \leq p \leq 2$ *and* $k \in M^p_{v_s}(\mathbb{R}^{2d})$, *then* $s_N(A_k) = \mathcal{O}(N^{-s/2d-1/p})$

(ii) *If* $2 \leq p < \infty$ *and* $k \in M^{p'}_{v_s}(\mathbb{R}^{2d})$ *then* $s_N(A_k) = \mathcal{O}(N^{-s/p'd-1/p'})$.

Corollary 4.5. *If the kernel* k *is in* $M^p_{v_s}(\mathbb{R}^{2d})$, *then*

(i) *for* $1 \leq p \leq 2$, $A_k \in \mathcal{I}_r$, *if* $r > \frac{2d}{s+d}$.
(ii) *for* $2 \leq p < \infty$, $A_k \in \mathcal{I}_r$, *if* $r > \frac{2p'd}{sp'+2d}$.

Proof. (i) By definition A_k is in $\mathcal{I}_r$ if $\sum s_j(A_k)^r \leq \infty$. Since k is in $M^p_s(\mathbb{R}^{2d})$ so by Theorem 4.4 we have

$$\sum s_j(A_k)^r \leq \sum (j^{\frac{-s-d}{2d}})^r.$$

This series is convergence if $\frac{r(s+d)}{2d} > 1$ and therefore $r > \frac{2d}{s+d}$.

(ii) Follows similar lines as the proof of (i). □

The proof of Theorem 4.4 will be based on a few lemmas. The first lemma enables us to approximate singular values of our pseudodifferential operators in $\mathcal{I}_p$.

Lemma 4.6. *If an operator A is in $\mathcal{I}_p$, then*

$$s_{2N}(A)^p \leq \tfrac{1}{N}\sum_{l>N} s_l(A)^p \leq \tfrac{1}{N}\inf\{\|A-T\|^p_{\mathcal{I}_p} \ : \mathrm{rank}(T) < N\}$$

Remark 4.7. In our discussion of pseudodifferential operators, the existence of a matrix representation with respect to a multivariate Wilson basis is the crucial tool in our proofs. Let kernel k be in $M^p(\mathbb{R}^{2d})$ and consider the Wilson basis ψ_{mn} for $M^p(\mathbb{R}^d)$. Then by the tensor products, $\Psi_{mn}(x,y) = \psi_{mn}(x)\overline{\psi_{mn}(y)}$ yield a basis for $M^p(\mathbb{R}^{2d})$, so by (4.1), $k = \sum_{m,n\in\mathbb{Z}^{2d}}\langle k,\Psi_{mn}\rangle\Psi_{mn}$ and $A_k = \sum_{m,n\in\mathbb{Z}^{2d}}\langle k,\Psi_{mn}\rangle A_{\Psi_{mn}}$ such that

$$A_{\Psi_{mn}}(f) = \int \Psi_{mn}(x,y)f(y)dy = \int \psi_{mn}(x)\overline{\psi_{mn}(y)}f(y)dy = \psi_{mn}(x)\langle f,\psi_{mn}\rangle.$$

The last equality arises from duality and shows that $A_{\Psi_{mn}}$ is a rank-one operator.

Lemma 4.8. *Let k be a kernel in $M^p_{v_s}(\mathbb{R}^{2d})$. Then singular values of the corresponding integral operator A_k behave as follow:*
$s_j(A_k) = \mathcal{O}(j^{-\frac{s}{2d}-\frac{1}{p}})$.

Proof. For each $N > 0$ and $k \in M^p_{v_s}(\mathbb{R}^{2d})$, set $k_N = \sum_{|m|,|n|\leq N}\langle k,\Psi_{mn}\rangle\Psi_{mn}$ then $A_{k_N} = \sum_{|m|,|n|\leq N}\langle k,\Psi_{mn}\rangle A_{\Psi_{mn}}$. For $m = (m_1,m_2)$ and $n = (n_1,n_2)$ define

$$J_N = \{m_1,m_2,n_1,n_2 \in \mathbb{Z}^d \ : \ |m_1|,|m_2|,|n_1|,|n_2| < N\} = \{-N,...N\}^{4d}.$$

By previous remark A_{k_N} is a finite-rank operator and its range is the line through $\psi_{mn}(x)$. So we have

$$\mathrm{rank}(A_{k_N}) \leq (2N+1)^{2d} \leq 10^d N^{2d}.$$

In view of the Wilson basis description of modulation spaces, Proposition 4.2 and Lemma 4.6 we obtain the following:

$$\begin{aligned}
Ms_{2M}(A_k)^p \leq \sum_{l=M+1}^{2M} s_l(A_k)^p &\leq \sum_{l>\mathrm{rank}(A_{k_N})} s_l(A_k)^p \leq \|A_k - A_{k_N}\|^p_{\mathcal{I}_p} \\
&\leq \|k-k_N\|^p_{M^p} = \left\|\sum_{m,n\notin J_N}\langle k,\Psi_{mn}\rangle\Psi_{mn}\right\|^p_{M^p} \\
&\leq C\sum_{m,n\notin J_N}|\langle k,\Psi_{mn}\rangle|^p\,\tfrac{v_s(m,n)^p}{v_s(m,n)^p} \\
&\leq C\sum_{m,n\notin J_N}|\langle k,\Psi_{mn}\rangle|^p\,v_s(m,n)^p(\sup_{m,n\notin J_N}\tfrac{1}{v_s(m,n)^p}) \\
&\leq C(\sup_{m,n\notin J_N}\tfrac{1}{v_s(m,n)^p})\,\|k\|^p_{M^p_s}.
\end{aligned}$$

Since $m,n \notin J_N$, so

$$v_s(m_1,m_2,n_1,n_2) = (1+m_1^2+m_2^2+n_1 62+n_2^2)^{s/2} \geq (1+4N^2)^{s/2} \geq (1+N^2)^{s/2} \geq (N^2)^{s/2}$$

and therefore

$$\frac{1}{v_s(m_1,m_2,n_1,n_2)^p} \leq \frac{1}{N^{sp}} = N^{-sp}.$$

This means that

$$Ms_{2M}(A_k)^p \leq C'N^{-sp}$$

where C' is a constant independent of k. Now set $M = 10^d N^{2d}$, then

$$\begin{aligned} Ms_{2M}(A_k)^p &\leq C'N^{-sp} \Rightarrow \\ s_{2M}(A_k)^p &\leq \frac{C'}{10^d}N^{-sp-2d} = DN^{-sp-2d} \Rightarrow \\ s_{2M}(A_k) &\leq DN^{\frac{-sp-2d}{p}}. \end{aligned}$$

Now by reindexing $j = 2M = 20^d N^{2d}$ we have $N = (\frac{j}{20^d})^{\frac{1}{2d}}$. If we substitute this expression into the above inequality, then we get the desired formula for s_j

$$s_j \leq D((\frac{j}{20^d})^{\frac{1}{2d}})^{\frac{-sp-2d}{p}},$$

i.e. $s_j(A_k) = \mathcal{O}(j^{-s/2d-1/p})$. □

The proof of the main result Theorem 4.4 is a straightforward consequence of the preceding lemma.

Proof of Theorem 4.4

(i) Let $1 \leq p \leq 2$ then by Proposition 4.2(i), it is enough to set $p = 2$.
(ii) Proposition 4.2-(ii) and substituting p' instead of p in Lemma 4.8 imply the result.

All the results so far are also true for the Kohn–Nirenberg quantization due to the invariance of the modulation spaces with radial symmetric weights under metaplectic transformations [23]. The final result states that the main result remains valid for operators of the Landau–Weyl calculus. This is a consequence of the discussion at the end of Section 2.

Theorem 4.9. *If $a \in M^p_{v_s}(\mathbb{R}^{2d})$, then we have the following results for the Weyl operator L_a and the Landau–Weyl operator $\tilde{L}_a$:*
(i) *For $1 \leq p \leq 2$ the operators L_a and $\tilde{L}_a$ are in $\mathcal{I}_r$ if $r > \frac{2d}{s+d}$.*
(ii) *For $2 < p \leq \infty$ the operators L_a and $\tilde{L}_a$ are in $\mathcal{I}_r$ if $r > \frac{2p'd}{sp'+2d}$.*

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[1]Department of Mathematics, University of California, Berkeley CA 94720-3840, USA.
E-mail address: luef@math.berkeley.edu

[2] Faculty of Mathematics, VALI-e-ASR University, Rafsanjan, Iran.
E-mail address: zrahban@mail.vru.ac.ir