Banach J. Math. Anal. 5 (2011), no. 2, 1–5

Banach Journal of Mathematical Analysis
ISSN: 1735-8787 (electronic)
www.emis.de/journals/BJMA/

LINEAR MAPS RESPECTING UNITARY CONJUGATION

B. V. RAJARAMA BHAT[1]

Communicated by M. S. Moslehian

Abstract. We characterize linear maps on von Neumann algebras which leave every unital subalgebra invariant. We use this characterization to determine linear maps which respect unitary conjugation, answering a question of M. S. Moslehian.

1. Introduction

Let $\mathcal{H}$ be a complex, separable Hilbert space and let $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded operators on $\mathcal{H}$. It was asked by M.S. Moslehian (private communication) as to what are linear maps α on $\mathcal{B}(\mathcal{H})$ which satisfy

$$\alpha(UXU^*) = U\alpha(X)U^* \ \forall X \in \mathcal{B}(\mathcal{H}),$$

for every unitary U on $\mathcal{H}$. We answer this question by first proving a theorem characterizing linear maps on von Neumann algebras which leave all subalgebras invariant.

2. Maps leaving subalgebras invariant

Theorem 2.1. *Let $\mathcal{A}$ be a von Neumann algebra and let I denote the identity element in $\mathcal{A}$. Let $\alpha : \mathcal{A} \to \mathcal{A}$ be a norm continuous linear map. Then the following are equivalent:*

(i) $\alpha(\mathcal{B}) \subseteq \mathcal{B}$ for every von Neumann subalgebra $\mathcal{B}$ of $\mathcal{A}$ with $I \in \mathcal{B}$.

(ii) $\alpha(\mathcal{B}) \subseteq \mathcal{B}$ for every abelian von Neumann subalgebra $\mathcal{B}$ of $\mathcal{A}$ with $I \in \mathcal{B}$.

(iii) $\alpha(x) = cx + \psi(x)I$ for some $c \in \mathbb{C}$ and some norm continuous linear functional $\psi : \mathcal{A} \to \mathbb{C}$.

Date: Received: 19 June 2010; Accepted: 8 October 2010.
2010 *Mathematics Subject Classification.* Primary 46L10; Secondary 47A65.
Key words and phrases. von Neumann algebras, unitary conjugation.

Before we prove this Theorem in general, we prove a special case as a Lemma and recall Halmos decomposition for pairs of generic projections. In the following for any projection p, $p^{\perp}$ denotes the projection $(I-p)$.

Lemma 2.2. *Let $\mathcal{A}$ be the algebra $M_2(\mathbb{C})$ of 2×2 complex matrices. Suppose $\alpha : \mathcal{A} \to \mathcal{A}$ is a linear map which leaves every unital $*$-subalgebra of $M_2(\mathbb{C})$ invariant. Then $\alpha(x) = cx + \psi(x)I$ for some $c \in \mathbb{C}$ and some linear functional ψ on $M_2(\mathbb{C})$.*

Proof. To begin with we assume that $\text{trace}(\alpha(X)) = 0$ for all X. As $\{cI : c \in \mathbb{C}\}$ is a unital commutative $*$-subalgebra of $M_2(\mathbb{C})$, $\alpha(I) = bI$ for some $b \in \mathbb{C}$. Combined with the trace assumption made now, $\alpha(I) = 0$.

Similarly since any rank one projection p generates a unital commutative algebra consisting of linear combinations of $p, p^{\perp}$, we get $\alpha(p) = c_p(p - p^{\perp})$ for some scalar c_p, for every rank one projection p. Hence $\alpha(p-p^{\perp}) = 2c_p(p-p^{\perp})$. Equivalently, every self-adjoint trace zero element of $M_2(\mathbb{C})$ is an eigenvector for α. In particular, there exist constants $c_1, c_2, \ldots, c_5$ such that $\alpha(A_i) = c_iA_i, 1 \leq i \leq 5$ where matrices A_i's are

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & i \\ -i & -1 \end{bmatrix}$$

respectively. Observing $A_1 + A_2 = A_4$ and $A_1 + A_3 = A_5$, linearity of α yields $c_1 = c_2 = c_4$ and $c_1 = c_3 = c_5$. Writing these matrices in the form $p - p^{\perp}$, we get a basis for $M_2(\mathbb{C})$ consisting of rank one projections and $\alpha(X) = c_1(X - \frac{1}{2}\text{trace}(X)I)$ for all X.

If α does not satisfy the assumption made above, consider β where,

$$\beta(X) = \alpha(X) - \frac{1}{2}\text{trace}(\alpha(X))I.$$

Proving the result for β is as good as proving the result for α. □

For any two projections p, q, denote the largest projection smaller than both p and q by $p \wedge q$. Recall that two projections p, q are said to be a generic pair if $p \wedge q = p \wedge q^{\perp} = p^{\perp} \wedge q = p^{\perp} \wedge q^{\perp} = 0$. The following result is well-known as Halmos decomposition ([1, 3]). If a pair of projections p, q on a Hilbert space $\mathcal{H}$ are generic, then $p(\mathcal{H})$ and $p^{\perp}(\mathcal{H})$ are isomorphic as Hilbert spaces and making use of this isomorphism, with respect to the decomposition $\mathcal{H} = p(\mathcal{H}) \oplus p^{\perp}(\mathcal{H})$, p and q have the form:

$$p = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \quad q = \begin{bmatrix} c^2 & cs \\ cs & s^2 \end{bmatrix}$$

with $0 < c, s < I, s = (I - c^2)^{\frac{1}{2}}$.

Proof of Theorem 2.1 : (i)$\Rightarrow$ (ii) and (iii) $\Rightarrow$ (i) are obvious. Now we show (ii) $\Rightarrow$ (iii).

If $\mathcal{A}$ has no non-trivial projection then $\mathcal{A}$ is isomorphic to $\mathbb{C}$ and there is nothing to show. Suppose p is a non-trivial projection in $\mathcal{A}$ and if only other non-trivial projection $\mathcal{A}$ has is $(I-p)$,then $\mathcal{A}$ is isomorphic to $\mathbb{C}^2$ and once again the result is obvious. In the following we exclude these two trivial cases.

Suppose p is a projection in $\mathcal{A}$ then the von Neumann algebra generated by p and I is $\{ap + bI : a, b \in \mathbb{C}\}$. It is abelian and hence left invariant by α. This shows that for any projection p in $\mathcal{A}$,

$$\alpha(p) = c_p p + d_p I \tag{2.1}$$

for some $c_p, d_p \in \mathbb{C}$. Note that scalars c_p, d_p are uniquely defined for non-trivial projections p. We wish to show that $c_p = c_q$ for any two non-trivial projections $p, q \in \mathcal{A}$.

Now suppose p_1, p_2, p_3 are three mutually orthogonal non-trivial projections in $\mathcal{A}$ such that $p_1 + p_2 + p_3 = I$. We have $\alpha(p_i) = c_{p_i} p_i + d_{p_i} I$ for $i = 1, 2, 3$. We also have $\alpha(p_1 + p_2) = c_{p_1+p_2}(p_1 + p_2) + d_{p_1+p_2} I$. But by linearity $\alpha(p_1 + p_2) = \alpha(p_1) + \alpha(p_2)$. So we get,

$$c_{p_1} p_1 + d_{p_1} I + c_{p_2} p_2 + d_{p_2} I = c_{p_1+p_2}(p_1 + p_2) + d_{p_1+p_2} I. \tag{2.2}$$

Multiplying this by p_3, yields, $d_{p_1} p_3 + d_{p_2} p_3 = d_{p_1+p_2} p_3$ or $d_{p_1} + d_{p_2} = d_{p_1+p_2}$. Substituting this back in (2.2) yields $c_{p_1} p_1 + c_{p_2} p_2 = c_{p_1+p_2}(p_1 + p_2)$, and then multiplications by p_1, p_2 show us $c_{p_1} = c_{p_2} = c_{p_1+p_2}$.

If p, q are two non-trivial projections in $\mathcal{A}$, such that $p \wedge q \neq 0$. Considering the triple $p \wedge q, p \ominus (p \wedge q), p^\perp$ we get $c_{p\wedge q} = c_p$, similarly $c_{p\wedge q} = c_q$, so $c_p = c_q$. It follows, that if p, q are non-trivial projections in $\mathcal{A}$, which are not in generic position and $q \neq p^\perp$, then $c_p = c_q$.

Suppose p, q are projections in $\mathcal{A}$ and are in generic position. If pqp is not a scalar multiple of p, then considering a non-trivial spectral projection p' of pqp, from the Halmos decomposition, we see that p', q are not in generic position as $(p')^\perp \wedge q \neq 0$. Hence $c_p = c_{p'} = c_q$. On the other hand, if p, q are in generic position and pqp is a scalar multiple of p, then by the Halmos decomposition it is clear that the algebra generated by p, q is $M_2(\mathbb{C})$ and we can apply Lemma 2.2 to get $c_p = c_q$.

Finally if $q = p^\perp$, on the one hand if there is a third non-trivial projection r different from p, q, we get $c_p = c_r = c_q$, and on the other hand if there is no such third projection then clearly $\mathcal{A}$ is isomorphic to $\mathbb{C}^2$ and we have already excluded this case.

This proves that for any two non-trivial projections in $\mathcal{A}$ we have $c_p = c_q$ (call this constant as c). Now if $p_1, p_2, \ldots, p_k$ are mutually orthogonal projections in $\mathcal{A}$ then for $x = \sum_{i=1}^k a_i p_i$ with scalars $a_1, a_2, \ldots, a_k$, $\alpha(x) = \sum_i a_i \alpha(p_i) = \sum_i a_i (c p_i + d_{p_i} I) = cx + d_x I$, for some scalar d_x. By spectral theorem every self-adjoint element of $\mathcal{A}$ can be approximated in norm by elements of the form $\sum_i a_i p_i$. It follows that, for every self-adjoint element $x \in \mathcal{A}$, α has the form,

$$\alpha(x) = cx + \psi(x) I.$$

for some $\psi(x) \in \mathbb{C}$. By continuity and linearity of α, it is clear that ψ is a continuous linear functional. □

Remark 2.3. No continuity assumption is needed in Theorem 2.1 in certain situations. For instance if the algebra $\mathcal{A} = \mathcal{B}(\mathcal{H})$, then as every bounded operator is a finite linear combination of projections (See [2, 4]), Theorem 2.1 follows without any continuity assumption (of course, then the functional ψ also need not be continuous).

Remark 2.4. It is a natural question as to whether in Theorem 2.1 (ii), we can replace 'abelian' by 'maximal abelian'. Clearly the answer is no, if the algebra $\mathcal{A}$ itself is abelian, as in this case every map α would satisfy (ii). However, this can be done if the algebra $\mathcal{A}$ is $\mathcal{B}(\mathcal{H})$. To see this consider any rank one projection p in $\mathcal{B}(\mathcal{H})$. Looking at maximal abelian subalgebras of $\mathcal{B}(p^{\perp}(\mathcal{H}))$, one has $\alpha(p) = c_p p + \beta_p$, where $c_p \in \mathbb{C}$ and β_p is in every maximal abelian subalgebra of $\mathcal{B}(p^{\perp}(\mathcal{H}))$. This of course, means that $\alpha(p)$ has the form (2.1). Now one can continue as in the proofs of Lemma 2.2 and Theorem 2.1 to get $c_p = c_q$ for every rank one projections p, q and that suffices to obtain (iii), under continuity assumption on α.

3. Unitary Conjugation

Finally, we have the result we were looking for.

Theorem 3.1. *Let $\alpha : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ be a linear map. Then the following are equivalent.*

(i) $\alpha(UXU^*) = U\alpha(X)U^*$ $\forall X, U$ in $\mathcal{B}(\mathcal{H})$ with $UU^* = U^*U = I$.

(ii) $\alpha(X) = cX + d$ trace $(X)I$ for some $c, d \in \mathbb{C}$ if $\mathcal{H}$ is finite dimensional, $\alpha(X) = cX$ for some $c \in \mathbb{C}$ if $\mathcal{H}$ is infinite dimensional.

Proof. Clearly (ii)$\Rightarrow$ (i). Now suppose $\mathcal{A}$ is a von Neumann subalgebra of $\mathcal{B}(\mathcal{H})$ with $I \in \mathcal{A}$. If U is a unitary in the commutant von Neumann algebra $\mathcal{A}'$ we get $\alpha(X) = \alpha(UXU^*) = U\alpha(X)U^*$ for $X \in \mathcal{A}$. So

$$\alpha(X)U = U\alpha(X).$$

However every element in a unital C^* is algebra is a linear combination of at most four unitaries. Hence

$$\alpha(X)Y = Y\alpha(X)\ \forall X \in \mathcal{A}, Y \in \mathcal{A}'.$$

Then by von Neumann's double commutant theorem $\alpha(X) \in \mathcal{A}$. Now with Remark 2.3, Theorem 2.1 is applicable, and we have $\alpha(X) = cX + \psi(X)I$ for some $c \in \mathbb{C}$ and some linear functional ψ. Further, by (i), for every unitary U,

$$cUXU^* + \psi(UXU^*)I = U[cX + \psi(X)I]U^*.$$

So, $\psi(UXU^*) = \psi(X)$ for all X. Taking $X = YU$, we get $\psi(UY) = \psi(YU)$ for every Y. Once again, since every operator is a linear combination of at most four unitaries, $\psi(XY) = \psi(YX)$. So ψ is a trace. It is well-known that if $\mathcal{H}$ is infinite dimensional $\mathcal{B}(\mathcal{H})$ does not admit a non-trivial finite trace. □

Acknowledgements: We thank an anonymous referee for several useful suggestions and for asking the question in Remark 2.4

References

1. A. Böttcher and I.M. Spitkovsky, *A gentle guide to the basics of two projections theory*, Linear Algebra Appl. **432** (2010), no. 6, 1412-1459.
2. P.A. Fillmore, *Sums of operators with square zero*, Acta Sci. Math. (Szeged) **28** (1967), 285–288.
3. P.R. Halmos, *Two subspaces*, Trans. Amer. Math. Soc. **144** (1969), 381-389.
4. K. Matsumoto, *Selfadjoint operators as a real span of* 5 *projections*, Math. Japon. **29** (1984), no. 2, 291–294.

[1] Indian Statistical Institute, R. V. College Post, Bangalore-560059, India.
E-mail address: bhat@isibang.ac.in