Abstract. In this paper, we show a complement of Ando–Hiai inequality: Let $A$ and $B$ be positive invertible operators on a Hilbert space $H$ and $\alpha \in [0, 1]$. If $A \sharp_\alpha B \leq I$, then
\[ A^r \sharp_\alpha B^r \leq \| (A \sharp_\alpha B)^{-1} \|^{1-r} I \] for all $0 < r \leq 1$,
where $I$ is the identity operator and the symbol $\| \cdot \|$ stands for the operator norm.

1. Introduction

A (bounded linear) operator $A$ on a Hilbert space $H$ is said to be positive (in symbol: $A \geq 0$) if $(Ax, x) \geq 0$ for all $x \in H$. In particular, $A > 0$ means that $A$ is positive and invertible. For some scalars $m$ and $M$, we write $mI \leq A \leq MI$ if $m(x, x) \leq (Ax, x) \leq M(x, x)$ for all $x \in H$. The symbol $\| \cdot \|$ stands for the operator norm. Let $A$ and $B$ be two positive operators on a Hilbert space $H$. For each $\alpha \in [0, 1]$, the weighted geometric mean $A \sharp_\alpha B$ of $A$ and $B$ in the sense of Kubo–Ando [6] is defined by
\[ A \sharp_\alpha B = A^{\frac{1}{2}} \left( A^{-\frac{1}{2}} BA^{-\frac{1}{2}} \right)^\alpha A^{\frac{1}{2}} \]
if $A$ is invertible. In fact, the geometric mean $A \sharp_\frac{1}{2} B$ is a unique positive solution of $XA^{-1}X = B$.

Date: Received: 6 September 2009; Accepted: 7 February 2010.
2000 Mathematics Subject Classification. Primary 47A63; Secondary 47A30, 47A64.
Key words and phrases. Ando–Hiai inequality, positive operator, geometric mean.
To study the Golden-Thompson inequality, Ando–Hiai in [1] developed the following inequality, which is called Ando–Hiai inequality: Let $A$ and $B$ be positive invertible operators on a Hilbert space $H$ and $\alpha \in [0, 1]$. Then

$$A \#_{\alpha} B \leq I \quad \implies \quad A^{r} \#_{\alpha} B^{r} \leq I \quad \text{for all } r \geq 1,$$

(AH)

or equivalently

$$\|A^{r} \#_{\alpha} B^{r}\| \leq \|A \#_{\alpha} B\|^{r} \quad \text{for all } r \geq 1.$$

Löwner–Heinz inequality asserts that $A \geq B \geq 0$ implies $A^{r} \geq B^{r}$ for all $0 \leq r \leq 1$. As compared with Löwner–Heinz inequality, Ando–Hiai inequality is rephased as follows: For each $\alpha \in [0, 1]$

$$(A^{r/2}B^{r/2})^{\alpha} \leq A^{r} \quad \implies \quad (A^{1/2}BA^{1/2})^{\alpha} \leq A \quad \text{for all } 0 < r \leq 1. \quad (1.1)$$

Now, Ando–Hiai inequality does not hold for $0 < r \leq 1$ in general. In fact, put $r = 1/2$, $\alpha = 1/3$ and

$$A = \frac{1}{5} \begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix} \quad \text{and} \quad B = \frac{1}{25} \begin{pmatrix} 45 + 14\sqrt{5} & -5 - 7\sqrt{5} \\ -5 - 7\sqrt{5} & 50 - 14\sqrt{5} \end{pmatrix}.$$ 

Then we have

$$A^{1/3} B = \frac{1}{25} \begin{pmatrix} 15 + 2\sqrt{5} & -5 - \sqrt{5} \\ -5 - \sqrt{5} & 20 - 2\sqrt{5} \end{pmatrix} \leq I$$

since $\sigma(A^{1/3} B) = \{1, 0.4\}$. On the other hand, since

$$A^{2} \#_{1/3} B^{2} = \begin{pmatrix} 0.866032 & -0.187030 \\ -0.187030 & 0.770683 \end{pmatrix} \quad \text{and} \quad \sigma(A^{2} \#_{1/3} B^{2}) = \{1.01137, 0.625347\},$$

we have $A^{2} \#_{1/3} B^{2} \not\leq I$.

Thus, in [7], Nakamoto and Seo showed the following complement of Ando–Hiai inequality (AH):

**Theorem A.** Let $A$ and $B$ be positive operators such that $mI \leq A, B \leq MI$ for some scalars $0 < m < M$, $h = \frac{M}{m}$ and $\alpha \in [0, 1]$. Then

$$A \#_{\alpha} B \leq I \quad \implies \quad A^{r} \#_{\alpha} B^{r} \leq K(h^{2}, \alpha)^{-r}I \quad \text{for all } 0 < r \leq 1,$$

where the generalized Kantorovich constant $K(h, \alpha)$ is defined by

$$K(h, p) = \frac{h^{p} - h}{(p - 1)(h - 1)} \left( \frac{p - 1}{p} \frac{h^{p} - 1}{h^{p} - h} \right)^{p} \quad \text{for all } p \in \mathbb{R},$$

see [5, (2.79)].

We remark that $K(h^{2}, \alpha)^{-r} \neq 1$ in the case of $r = 1$, though $K(h^{2}, \alpha)^{-r} = 1$ in the case of $\alpha = 0, 1$ in Theorem A. Thereby, in this paper, we consider another complement of Ando–Hiai inequality (AH) which differ from Theorem A.
2. Main results

Theorem 2.1. Let $A$ and $B$ be positive invertible operators and $\alpha \in [0, 1]$. Then

$$A^\alpha B \leq I \implies A^rA^\alpha B^r \leq \|A^{-1} A^\alpha B^{-1}\|^{1-r} I \quad \text{for all } 0 < r \leq 1,$$

or equivalently

$$\|A^rA^\alpha B^r\| \leq \|A^{-1} A^\alpha B^{-1}\|^{1-r}\|A\| r \quad \text{for all } 0 < r \leq 1.$$

We remark that $\|A^{-1} A^\alpha B^{-1}\|^{1-r} = 1$ in the case of $r = 1$.

We need the following lemmas to give a proof of Theorem 2.1. Lemma 2.2 is regarded as a reversal of Löwner–Heinz inequality:

Lemma 2.2. Let $A$ and $B$ be positive invertible operators. Then

$$A \geq B \implies \|A^p B^{-p} A^\frac{p}{2}\| B^p \geq A^p \quad \text{for all } 0 < p \leq 1.$$

Proof. This lemma follows from Löwner–Heinz inequality. In fact, $A \geq B$ implies $A^p \geq B^p$ for all $0 < p \leq 1$ and then

$$I \geq A^{-\frac{p}{2}} B^p A^{-\frac{p}{2}} \geq \|A^p B^{-p} A^\frac{p}{2}\|^{-1}.$$ 

□

Lemma 2.3 ([3]). Let $A$ be a positive invertible operator and $B$ an invertible operator. For each real numbers $r$

$$(BAB^*)^r = BA^{\frac{r}{2}} (A^{\frac{1}{2}} B^* B A^{\frac{1}{2}})^{r-1} A^{\frac{1}{2}} B^*.$$

Proof of Theorem 2.1. If we put $C = A^{-\frac{1}{2}} B A^{-\frac{1}{2}}$, then the assumption implies $A^{-1} \geq C^\alpha$. By Lemma 2.2 and $0 < 1 - r < 1$, we have

$$A^r = A^{\frac{r}{2}} A^{1-r} A^{\frac{1}{2}} \leq \|A^{\frac{r}{2}} C^{\alpha(r-1)} A^{\frac{1}{2}}\| A^{\frac{1}{2}} C^{\alpha(1-r)} A^{\frac{1}{2}}.$$

On the other hand, it follows that $A \leq C^{-\alpha}$ implies $C^{\alpha-1} \leq (C^{\frac{1}{2}} A C^{\frac{1}{2}})^{-1}$. By Lemma 2.2, we have

$$\|(C^{\frac{1}{2}} A C^{\frac{1}{2}})^{\frac{r-1}{2}} C^{\alpha(1-r)(1-r)} (C^{\frac{1}{2}} A C^{\frac{1}{2}})^{\frac{1}{2}}\| C^{\alpha(1-r)} \geq (C^{\frac{1}{2}} A C^{\frac{1}{2}})^{r-1}.$$

Furthermore, by Lemma 2.3, we have

$$B^r = (A^{\frac{1}{2}} C A^{\frac{1}{2}})^r = A^{\frac{r}{2}} C^{\frac{r}{2}} (C^{\frac{1}{2}} A C^{\frac{1}{2}})^{r-1} C^{\frac{1}{2}} A^{\frac{1}{2}}$$

$$\leq \|(C^{\frac{1}{2}} A C^{\frac{1}{2}})^{\frac{r-1}{2}} C^{\alpha(1-r)(1-r)} (C^{\frac{1}{2}} A C^{\frac{1}{2}})^{\frac{1}{2}}\| A^{\frac{1}{2}} C^{\alpha(1-r)(1-r)} C^{\frac{1}{2}} A^{\frac{1}{2}}.$$

Hence, by Araki-Cordes inequality [2, Theorem IX.2.10], we have

$$\|(C^{\frac{1}{2}} A C^{\frac{1}{2}})^{\frac{r-1}{2}} C^{\alpha(1-r)(1-r)} (C^{\frac{1}{2}} A C^{\frac{1}{2}})^{\frac{1}{2}}\| \leq \|(C^{\frac{1}{2}} A C^{\frac{1}{2}})^{-\frac{1}{2}} C^{1-\alpha} (C^{\frac{1}{2}} A C^{\frac{1}{2}})^{-\frac{1}{2}}\|^{1-r}.$$
Therefore, it follows that
\[
\| (C^{\frac{1}{2}}AC^{\frac{1}{2}})^{\frac{1}{2}} C^{1-\alpha} (C^{\frac{1}{2}}AC^{\frac{1}{2}})^{\frac{1}{2}} \| = r ((C^{\frac{1}{2}}AC^{\frac{1}{2}})^{\frac{1}{2}} C^{\frac{1}{2}}AC^{\frac{1}{2}})^{\frac{1}{2}}
\]
\[
= r ((C^{\frac{1}{2}}AC^{\frac{1}{2}})^{\frac{1}{2}} C^{\frac{1}{2}}AC^{\frac{1}{2}})^{\frac{1}{2}}
\]
\[
= r (A^{-1/2} C^{\frac{1}{2}})
\]
\[
= r \left( A^{-\frac{1}{2}} C^{-\alpha} A^{-\frac{1}{2}} \right)
\]
\[
\leq \| A^{-\frac{1}{2}} C^{-\alpha} A^{-\frac{1}{2}} \|
\]

Therefore, it follows that

\[
A^r \#_\alpha B^r \leq \| A^r \| C^{\alpha(r-1)} A^{\frac{r-1}{2}} \| 1-\alpha \| (C^{\frac{1}{2}}AC^{\frac{1}{2}})^{\frac{r-1}{2}} C^{\alpha(r-1)(r-1)} (C^{\frac{1}{2}}AC^{\frac{1}{2}})^{\frac{r-1}{2}} \| 1-\alpha \|
\]
\[
\times \left( A^{\frac{r-1}{2}} C^{(r-1)(1-\alpha)} A^{\frac{r-1}{2}} \#_\alpha A^{\frac{r-1}{2}} C^{\alpha(r-1)(1-\alpha)} A^{\frac{r-1}{2}} \right)^{-1} A^r
\]
\[
\leq \| A^{-\frac{1}{2}} C^{-\alpha} A^{-\frac{1}{2}} \| 1-\alpha \| A \#_\alpha B \leq \| (A \#_\alpha B)^{-1} \| 1-\alpha \|
\]

by \( C^{(r-1)(1-\alpha)} \#_\alpha C^{(r-1)(1-\alpha)+1} = C^\alpha \) and the assumption of \( A \#_\alpha B \leq I \). Hence the
proof is complete.

By Theorem 2.1, we immediately have the following corollary in the case of \( r \geq 1 \).

**Corollary 2.4.** Let \( A \) and \( B \) be positive invertible operators on \( H \). Then

\[
\| A^{-r} \#_\alpha B^{-r} \| 1-\alpha \| A \#_\alpha B \| r \leq \| A^r \#_\alpha B^r \| \quad \text{for all } r \geq 1.
\]

Finally, Furuta [4] showed the following Knatorovich type operator inequality in terms of the condition number: Let \( A \) and \( B \) be positive invertible operators. Then

\[
B \leq A \quad \implies \quad B^r \leq (\| B \| 1-\alpha) A^r \quad \text{for all } r \geq 1 \tag{2.1}
\]

By Theorem 2.1, we have the following Kantorovich type inequality of (1.1) which corresponds to (2.1):

**Theorem 2.5.** Let \( A \) and \( B \) be positive invertible operators and \( \alpha \in [0,1] \). Then

\[
\left( A^{\frac{1}{2}} B A^{\frac{1}{2}} \right)^{\alpha} \leq A^r \quad \implies \quad \left( A^{\frac{1}{2}} B A^{\frac{1}{2}} \right)^{\alpha} \leq \| A^r \#_\alpha B^{-r} \| 1-\alpha \| A \|
\]

for all \( r \geq 1 \).

**References**


¹ Faculty of Engineering, Shibaura Institute of Technology, 307 Fukasaku, Minuma-ku, Saitama-city, Saitama 337-8570, Japan.
E-mail address: yukis@sic.shibaura-it.ac.jp