Banach J. Math. Anal. 3 (2009), no. 2, 16–27

BANACH JOURNAL OF MATHEMATICAL ANALYSIS
ISSN: 1735-8787 (electronic)
http://www.math-analysis.org

E_0–SEMIGROUPS FOR CONTINUOUS PRODUCT SYSTEMS: THE NONUNITAL CASE

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Communicated by M. Frank

ABSTRACT. Let $\mathcal{B}$ be a σ–unital C^*–algebra. We show that every strongly continuous E_0–semigroup on the algebra of adjointable operators on a full Hilbert $\mathcal{B}$–module E gives rise to a full continuous product system of correspondences over $\mathcal{B}$. We show that every full continuous product system of correspondences over $\mathcal{B}$ arises in that way. If the product system is countably generated, then E can be chosen countably generated, and if E is countably generated, then so is the product system. We show that under these countability hypotheses there is a one-to-one correspondence between E_0–semigroups up to stable cocycle conjugacy and continuous product systems up to isomorphism. This generalizes the results for unital $\mathcal{B}$ to the σ–unital case.

1. INTRODUCTION

Factorizable families of Hilbert spaces are known since quite a while; see, for instance, Araki [1], Streater [24], and Parthasarathy and Schmidt [15]. Arveson [2, 3, 4, 5] developed this idea into a concise theory of *tensor product systems* of Hilbert spaces (*Arveson systems*, for short). Roughly speaking, Arveson's theory provides a classification of E_0–semigroups (unital endomorphism semigroups) on $\mathcal{B}(H)$ (H a Hilbert space) by Arveson systems up to *cocycle conjugacy*. It is comparably plain to associate with every E_0–semigroup an Arveson system, and to show that two E_0–semigroups have isomorphic Arveson systems, if and only if they are cocycle conjugate. All this and an index theory for E_0–semigroups is

Date: Received: 13 January 2009; Accepted: 7 April 2009.
2000 *Mathematics Subject Classification.* Primary 46L55; Secondary 46L53, 46L08.
Key words and phrases. Quantum probability, quantum dynamic, product system, Hilbert module, classification.

done in [2]. To show that every Arveson system comes from an E_0–semigroup, was done the remaining articles [3, 4, 5].

Liebscher [12] provided the second proof of this *fundamental theorem about Arveson systems*. This proof is still quite involved. But it adds the information that the E_0–semigroup having the given Arveson system may be chosen *pure*. Only recently, in [18] we provided a simple and self-contained proof. Shortly later, Arveson [6] presented another simple proof. In [19] we showed that the output of [6] and (a special case) of [18] are unitarily equivalent.

Meanwhile, several authors investigated tensor product systems of Hilbert bimodules or *correspondences*; see Bhat and Skeide [8], Muhly and Solel [14], and Hirschberg and Zacharias [9, 10]. A connection between E_0–semigroups on $\mathcal{B}^a(E)$, the algebra of all adjointable operators on a full Hilbert $\mathcal{B}$–module E, and product systems of correspondences over $\mathcal{B}$ (paralleling that of Arveson) has been established in Skeide [16] and, in its general version, in Skeide [23].

Our scope that resulted in the simple proof of [18], was to find a proof that works also for Hilbert modules. Funnily enough, in the two cases we could treat so far, namely when $\mathcal{B}$ is a unital C^*–algebra [20], or when $\mathcal{B}$ is a von Neumann algebra [22] (in preparation), we proceeded utilizing Arveson's idea [6] in an essential way. In these notes, we now add the nonunital case under countability assumptions. ($\mathcal{B}$ should be σ–unital. And for the complete classification result, the occurring modules should be countably generated.) For the discrete case in Section 3 we need the original idea of Skeide [18]. The correct adaptation of Arveson's idea [6] is a new crucial ingredient for the continuous time case. We also mention that our proof for unital $\mathcal{B}$ in [20] that the E_0–semigroup constructed there from a continuous product system induces the same continuous structure on that product system, contains a gap. The new Theorem 2.2 is far more general and fixes also the gap in [20].

2. The product system of an E_0–semigroup

Let $\mathbb{S}$ denote one of the semigroups $\mathbb{N}_0 = \{0, 1, \ldots\}$ and $\mathbb{R}_+ = [0, \infty)$. Fix a Hilbert $\mathcal{B}$–module E. In Skeide [16] we constructed the product system of a strict E_0–semigroup ϑ on $\mathcal{B}^a(E)$, following Bhat's construction in [7], starting from a *unit vector* $\xi \in E$, that is, from a vector with "length" $\langle \xi, \xi \rangle = \mathbf{1} \in \mathcal{B}$. This means, in particular, that $\mathcal{B}$ is unital and that E is full. The construction in [16] goes as follows.

Put $E_t := \vartheta_t(\xi\xi^*)E$. Turn it into a correspondence over $\mathcal{B}$ by defining the left action $bx_t := \vartheta_t(\xi b \xi^*)x_t$. (Note that this left action is unital.) Define a map $v_t \colon E \odot E_t \to E$ by setting $v_t(x \odot y_t) := \vartheta_t(x\xi^*)y_t$. It is easy to check that this map is *isometric* (that is, inner product preserving) and, therefore, well-defined. Surjectivity follows from strictness; see [16] for details. One easily verifies the following properties.

1. ϑ can be recovered from the unitaries v_t as $\vartheta_t(a) = v_t(a \odot \mathsf{id}_t)v_t^*$.
2. The restriction $u_{s,t}$ to $E_s \odot E_t \subset E \odot E_t$ defines a bilinear unitary onto $E_{s+t} \subset E$.

(3) $E_0 = \mathcal{B}$ and v_0, $u_{s,0}$, and $u_{0,t}$ are the canonical identifications, that is, right multiplication (in the case of v_0 and $u_{s,0}$) and left multiplication (in the case of $u_{0,t}$), with the elements in $E_0 = \mathcal{B}$.

(4) Both "multiplications" $(x, y_t) \mapsto xy_t := v_t(x \odot y_t)$ and $(x_s, y_t) \mapsto x_s y_t := u_{s,t}(x_s \odot y_t)$ iterate associatively, that is, $(xy_s)z_t = x(y_s z_t)$ and $(x_r y_s)z_t = x_r(y_s z_t)$.

A family $E^\odot = \bigl(E_t\bigr)_{t\in\mathbb{S}}$ of $\mathcal{B}$–correspondences with *structure maps* $u_{s,t}$ fulfilling 2 and the relevant part of 3 and 4, has been called *product system* in Bhat and Skeide [8]. Given a product system $E^\odot$ with structure maps $u_{s,t}$, a *full* Hilbert $\mathcal{B}$–module E (that is, the *range ideal* $\mathcal{B}_E := \overline{\mathsf{span}}\langle E, E\rangle$ of E coincides with $\mathcal{B}$) and a family of unitaries v_t fulfilling the relevant part of 3 and 4, has been called a *left dilation* of $E^\odot$ to E in Skeide [20]. Note that if there exists a left dilation of $E^\odot$, then $E^\odot$ is necessarily *full*, that is, E_t is full for every t. If the v_t form a left dilation of $E^\odot$ to E, then $\vartheta^v_t(a) := v_t(a \odot \mathsf{id}_t)v_t^*$ defines an E_0–semigroup ϑ^v. If E has a unit vector, then the product system constructed from ϑ^v is (isomorphic to) $E^\odot$. Recall that a *morphism* between product systems $E^\odot$ and $F^\odot$ is a family $w^\odot = \bigl(w_t\bigr)_{t\in\mathbb{S}}$ of bilinear adjointable maps $w_t\colon E_t \to F_t$ such that $(w_s x_s)(w_t y_t) = w_{s+t}(x_s y_t)$ and $w_0 = \mathsf{id}_{\mathcal{B}}$. An *isomorphism* is a morphism that consists of unitaries.

We say a strict E_0–semigroup ϑ and a product system $E^\odot$ are *associated*, if there exists a left dilation v_t of $E^\odot$ such that $\vartheta = \vartheta^v$. It is known that for each strict E_0–semigroup there is, up to isomorphism, only one product system that can be associated with that E_0–semigroup; see Skeide [21, Section 6].

We have just seen that every strict E_0–semigroup can be associated with a product system, provided that E has a unit vector. There is a general construction in Skeide [23] for arbitrary (full) E even if $\mathcal{B}$ is nonunital, based on the representation theory of $\mathcal{B}^a(E)$ from Muhly, Skeide, and Solel [13]. For the converse result, we have several stages:

(1) If $\mathcal{B}$ is unital, we have the existence result [23, Theorem 7.6] for the discrete case $\mathbb{S} = \mathbb{N}_0$.

(2) Without continuity conditions, we can prove the continuous time case $\mathbb{S} = \mathbb{R}_+$ by the method invented in Skeide [18] for the Hilbert space case, by reducing it to preceding result for the discrete case. Since, in the noncontinuous case, there are involved direct sums over the index set $[0, 1)$ and the shift on that set, the constructed E_0–semigroup is definitely noncontinuous.

(3) In [20] we resolved, still for unital $\mathcal{B}$, the continuous time case with continuity conditions both on the E_0–semigroup and on the product system.

(In Skeide [22] we deal with the general von Neumann case. But this is out of the scope of the present notes, where we restrict to the C^*–case.)

We see that in all three stages the case of nonunital $\mathcal{B}$ is still missing. As for all three stages it is crucial to find a good adaptation of Bhat's method of the construction of the product system from an E_0–semigroup (and not the abstract one based on [13]), we spend the present section to such find such a construction.

In the following sections we apply the new insight to adapt also the proof for three stages, well, not to the general nonunital case, but to the σ–unital case.

The crucial observation which gives the correct hint and resolves all problems that, so far, prevented us from dealing with the case of nonunital $\mathcal{B}$, is quite simple. What does it mean if E has a unit vector ξ? Well, it means that E has a direct summand $\xi\mathcal{B} \cong \mathcal{B}$. The projection onto that summand is $\xi\xi^*$. If E is full over a unital C^*–algebra, then a finite multiple E^n of E will have a unit vector; see [23, Lemma 3.2]. This was enough to treat the problems in the unital case. Now Lance [11, Proposition 7.4] asserts the following: If E is a full Hilbert module over a σ–unital C^*–algebra $\mathcal{B}$, then E^∞ has, well, not a unit vector, but a direct summand $\mathcal{B}$. And this turns out to be enough for all our purposes.

To begin with, let $\mathcal{B}$ be an arbitrary C^*–algebra. Suppose E has a direct summand $\mathcal{B}$, that is, suppose $E = \mathcal{B} \oplus F$, so that also $\mathcal{B}^a(E)$ decomposes into $\begin{pmatrix} \mathcal{B}^a(\mathcal{B}) & \mathcal{B}^a(F,\mathcal{B}) \\ \mathcal{B}^a(\mathcal{B},F) & \mathcal{B}^a(F) \end{pmatrix}$. Let $p \in \mathcal{B}^a(E)$ denote the projection $(\beta, y) \mapsto (\beta, 0)$ onto $\mathcal{B} \subset E$. For $x \in E$ we define the element $xp \in \mathcal{B}^a(E)$ by setting $xp(\beta, y) := x\beta$. The adjoint map is $px^*\colon x' \mapsto (\langle x', x\rangle, 0)$. Observe that $x'ppx*$ is just the usual rank-one operator $x'x^*$. Note, too, that $\pi\colon \langle x, x'\rangle \mapsto px^*x'p$ defines nothing but the canonical embedding of $\mathcal{B}$ into the $\mathcal{B}^a(\mathcal{B})$–corner of $\mathcal{B}^a(E)$.

Let ϑ be a strict E_0–semigroup on $\mathcal{B}^a(E)$. Following the procedure in presence of a unit vector, we put $E_t := \vartheta_t(p)E$. It follows that $bx_t := \vartheta_t(\pi(b))x_t$ defines a nondegenerate (ϑ_t is strict!) left action of $\mathcal{B}$ on E_t turning, thus, E_t into a correspondence over $\mathcal{B}$. By

$$v_t(x \odot y_t) \;:=\; \vartheta_t(xp)y_t$$

we define a unitary $E \odot E_t \to E$. (By $\langle \vartheta_t(xp)y_t, \vartheta_t(x'p)y'_t\rangle = \langle y_t, \langle x, x'\rangle y'_t\rangle = \langle x \odot y_t, x' \odot y'_t\rangle$ we see that v_t is isometric. Surjectivity follows from $\vartheta_t(xy^*)z = \vartheta_t(xp)\vartheta_t(py^*)z = v_t(x \odot (\vartheta_t(py^*)z)$, existence of a bounded approximate unit of finite-rank operators for $\mathcal{K}(E)$ and strictness of ϑ_t, in precisely the same way as in [16].) Obviously, $\vartheta_t(a) = v_t(a \odot \mathsf{id}_t)v_t^*$. (Simply, apply both sides to $v_t(x \odot y_t)$.) The restriction $u_{s,t}$ of v_t to $E_s \odot E_t$ is surjective onto E_{s+t}. (It is into, because $\vartheta_{s+t}(p)v_t(x \odot y_t) = v_t(\vartheta_s(p)x \odot y_t)$. It is onto, because $(\mathbf{1} - \vartheta_{s+t}(p))v_t(x \odot y_t) = v_t((\mathbf{1} - \vartheta_s(p))x \odot y_t)$.) Also the marginal conditions for $t = 0$ or $s = 0$ are satisfied. So, $E^\odot = \bigl(E_t\bigr)_{t\in\mathbb{S}}$ is a product system and the v_t form a left dilation giving back ϑ as ϑ^v.

If E has no direct summand but $\mathcal{B}$ is σ–unital, then we know that E^∞ has a direct summand $\mathcal{B}$. It is known that ϑ and its amplification ϑ^∞ to $\mathcal{B}^a(E^\infty)$ have the same product system; see [21, Section 9]. We, thus, proved the following.

Theorem 2.1. *Let ϑ be a strict E_0–semigroup on $\mathcal{B}^a(E)$ where E is a full Hilbert module over a σ–unital C^*–algebra. Then the product system of ϑ can be obtained by the prescribed construction applied to the amplification of ϑ to $\mathcal{B}^a(E^\infty)$ based on any choice of a direct summand $\mathcal{B}$ of E^∞.*

If, in the continuous time case, ϑ is strongly continuous, then we would like that this property is reflected by a continuous structure of the product system. In Skeide [17, 20] a *continuous product system* is defined as a product system $E^\odot =$

$\big(E_t\big)_{t\in\mathbb{R}_+}$ with a family of isometric embeddings $i_t\colon E_t \to \widehat{E}$ into a right Hilbert $\mathcal{B}$–module $\widehat{E}$ (there is no left action on $\widehat{E}$) fulfilling the following conditions: Denote by

$$CS_i(E^\odot) \;=\; \Big\{ \big(x_t\big)_{t\in\mathbb{R}_+}\colon x_t \in E_t,\; t \mapsto i_t x_t \text{ is continuous} \Big\}$$

the set of *continuous sections* of $E^\odot$ (with respect to the embeddings i_t). Then, firstly,

$$\Big\{ x_s\colon \big(x_t\big)_{t\in\mathbb{R}_+} \in CS_i(E^\odot) \Big\} \;=\; E_s$$

for all $s \in \mathbb{R}_+$ (that is, $E^\odot$ has sufficiently many continuous sections), and, secondly,

$$(s,t) \;\longmapsto\; i_{s+t}(x_s y_t)$$

is continuous for all $\big(x_t\big)_{t\in\mathbb{R}_+}, \big(y_t\big)_{t\in\mathbb{R}_+} \in CS_i(E^\odot)$ (that is, the 'product' of continuous sections is continuous). A morphism between continuous product systems is *continuous*, if it sends continuous sections to continuous sections. An *isomorphism* of continuous product systems is a continuous isomorphism with continuous inverse. Clearly, an isomorphism provides a bijection between the sets of continuous sections.

The following theorem also settles a gap in the proof of [20, Proposition 4.9] and generalizes it considerably. We illustrate its applications in the end of Section 4.

Theorem 2.2. *Let $i_t\colon E_t \to E^i$ and $k_t\colon E_t \to E^k$ be two continuous structures on the product system $E^\odot = \big(E_t\big)_{\in\mathbb{R}_+}$. If the identity morphism is a continuous morphism from $E^\odot$ with respect to the embeddings i to $E^\odot$ with respect to the embeddings k, then the identity morphism is a continuous isomorphism.*

Proof. This statement means that if $x \in CS_i(E^\odot) \Longrightarrow x \in CS_k(E^\odot)$, then $x \in CS_i(E^\odot) \Longleftrightarrow x \in CS_k(E^\odot)$. Note that this is only a statement on the Banach bundle structure of $E^\odot$, while the product system structure does not play any role. Notice also that the notion of uniform convergence of a sequence of sections on any subset I of $\mathbb{R}_+$ depends only on the pointwise norms of E_t. It does not refer in any way to the embeddings i_t or k_t. Nevertheless, a uniform limit on I of sections that are continuous with respect to i (to k) is continuous on I with respect to i (to k). Therefore, if we can approximate a section $x \in CS_k(E^\odot)$ on each compact interval $I = [a,b] \subset \mathbb{R}_+$ uniformly by sections in $CS_i(E^\odot)$, then $x \in CS_i(E^\odot)$.

So let $x \in CS_k(E^\odot)$ and $I = [a,b]$ be as stated. For every $\beta \in I$ choose a section $y^\beta \in CS_i(E^\odot) \subset CS_k(E^\odot)$ such that $y^\beta_\beta = x_\beta$. Choose $\varepsilon > 0$. For every $\beta \in I$ choose an interval $I_\beta \subset I$ which is open in I and which contains β such that $\|x_\alpha - y^\beta_\alpha\| < \varepsilon$ for all $\alpha \in I_\beta$. (Since $\|x_\alpha - y^\beta_\alpha\| = \|k_\alpha x_\alpha - k_\alpha y^\beta_\alpha\|$ and since $y^\beta, x \in CS_k(E^\odot)$, such I_β exist.) So, we may choose $\beta_1, \ldots, \beta_m$ such that the union over I_{β_i} is $[a,b]$. By standard theorems about *partitions of unity* there exist continuous functions φ_i on $[a,b]$ with the following properties:

$$0 \;\le \varphi_i \;\le\; 1, \qquad \varphi_i \upharpoonright I_{\beta_i}^{\complement} \;=\; 0, \qquad \sum_{i=1}^{m} \varphi_i \;=\; 1.$$

From these properties one verifies easily that $\left\| x_\alpha - \sum_{i=1}^m \varphi_i(\alpha) y_\alpha^{\beta_i} \right\| < \varepsilon$ for all $\alpha \in I$. Since $\sum_{i=1}^m \varphi_i(\alpha) y_\alpha^{\beta_i} \in CS_i(E^\odot)$, the section $x^\odot$ is the locally uniform limit of section in $CS_i(E^\odot)$ and, therefore, in $CS_i(E^\odot)$ itself. □

Already in [17] we have shown that the product system $E^\odot$ of a strongly continuous strict E_0–semigroup on $\mathcal{B}^a(E)$ when E has a unit vector, can be equipped with a continuous structure in the following way: Put $\widehat{E} := E$. It is, then, easy to see that $E^\odot$ is a continuous product system with respect to the canonical embeddings i_t of the submodules $E_t \subset E$ into E. We now, simply, do the same for the product system constructed above in the case of general $\mathcal{B}$ (and full E, of course).

Construct the amplification ϑ^∞ of ϑ on $\mathcal{B}^a(E^\infty)$, so that E^∞ has now a direct summand $\mathcal{B}$ with projection $p \in \mathcal{B}^a(E^\infty)$ onto that summand. Put $E_t := \vartheta_t^\infty(p)E^\infty$ and choose for i_t the canonical embeddings of E_t into E^∞. Precisely as in Skeide [21] (where the unital case has been treated, so that $p = \xi\xi^*$ for some unit vector $\xi \in E^\infty$) one shows that $E^\odot$ is continuous, that the continuous structure does not depend on the choice of the summand $\mathcal{B}$ in E^∞, and that, if E has already a direct summand $\mathcal{B}$, then the continuous structure is the same as if we had proceeded without amplifying ϑ. We do not repeat the proof from [21] as it generalizes word by word.

This concludes the description of the construction of full product systems from strict E_0–semigroups $\vartheta = \left(\vartheta_t\right)_{t\in\mathbb{S}}$ and of continuous full product systems from strongly continuous strict E_0–semigroups $\vartheta = \left(\vartheta_t\right)_{t\in\mathbb{R}_+}$. The remainder of these notes is dedicated to the reverse direction.

3. Discrete case and algebraic continuous time case

Let F be a full correspondence over $\mathcal{B}$. We seek a full Hilbert $\mathcal{B}$–module E such that $E \cong E \odot F$, for in that case this induces a unital strict endomorphism $\theta\colon a \mapsto a \odot \mathsf{id}_F \in \mathcal{B}^a(E \odot F) \cong \mathcal{B}^a(E)$ of $\mathcal{B}^a(E)$ and the discrete product system $E^\odot$ associated with the discrete E_0–semigroup $\vartheta = \left(\vartheta_n\right)_{n\in\mathbb{N}_0}$ with $\vartheta_n := \theta^n$ is $E^\odot = \left(E_n\right)_{n\in\mathbb{N}_0}$ with $E_n := F^{\odot n}$ and the canonical identifications $F^{\odot m} \odot F^{\odot n} = F^{\odot(m+n)}$.

If $E_1 = F$ has a unit vector $\xi_1 = \zeta$, then $\xi^\odot = \left(\xi_n\right)_{n\in\mathbb{N}_0}$ with $\xi_n := \zeta^{\odot n}$ is a *unital unit*. In general, a *unit* $\xi^\odot$ for a product system is a family $\xi^\odot$ of elements $\xi_t \in E_t$ such that $\xi_s\xi_t = \xi_{s+t}$ $(s,t \in \mathbb{S})$ and $\xi_0 = \mathbf{1} \in \mathcal{B} = E_0$. A unit is *unital*, if it consists of unit vectors. Already Arveson [2] noted that in presence of a unital unit in a product system it is easy to construct an E_0–semigroup associated with that product system. Simply, embed E_t as $\xi_s E_t$ into E_{s+t}. These embeddings form an inductive system and the factorization $u_{s,t}\colon E_s \odot E_t \to E_{s+t}$ "survive" the inductive limit as $v_t\colon E_\infty \odot E_t \to E_{\text{“}\infty+t\text{”}} = E_\infty$. Clearly, all associativity conditions are preserved so that the v_t define a left dilation of $E^\odot$ to E_∞ and the induced E_0–semigroup ϑ^v has $E^\odot$ as product system.

The basic idea in Skeide [23] was: Even if F has no unit vector, then F^n has one for suitably big $n \in \mathbb{N}$. The same is true *cum grano salis* for the correspondence $M_\infty(F^n) \cong M_{n\cdot\infty,\infty}(F) \cong M_{\infty,\infty}(F) = M_\infty(F)$ over $M_\infty(\mathcal{B})$. The *cum grano salis* refers to that $M_\infty(\mathcal{B}) = \mathcal{K}(\mathcal{B}^\infty)$ is always nonunital, and $M_\infty(F) = \mathcal{K}(\mathcal{B}^\infty, F^\infty)$ cannot contain a unit vector. What we need are both *strict completions*, the multiplier algebra $\mathcal{B}^a(\mathcal{B}^\infty)$ of $\mathcal{K}(\mathcal{B}^\infty)$ and the correspondence $\mathcal{B}^a(\mathcal{B}^\infty, F^\infty)$ over $\mathcal{B}^a(\mathcal{B}^\infty)$. This correspondence **does** have a unit vector. (Simply observe that the map that takes $\mathcal{B}$ to the direct summand $\mathcal{B}$ that exists in in F^∞ is a unit vector in $\mathcal{B}^a(\mathcal{B}, F^\infty)$. Then use this and the property $\infty \cdot \infty = \infty$ to construct a unit vector in $\mathcal{B}^a(\mathcal{B}^\infty, F^\infty)$ as described in [23].) With this unit vector we find an inductive limit E_∞ (a Hilbert $\mathcal{B}^a(\mathcal{B}^\infty)$–module!) and a strict E_0–semigroup ϑ^∞. If we define the (full) Hilbert $\mathcal{B}$–module $E := E_\infty \odot \mathcal{B}^\infty$, the E_0–semigroup ϑ^∞ gives rise to a strict E_0–semigroup ϑ on $\mathcal{B}^a(E) \cong \mathcal{B}^a(E_\infty)$, and ϑ has $E^\odot$ as product system.

All this has been described in [23, Section 7] in a very detailed manner for unital $\mathcal{B}$. The point is now that everything goes through precisely as in [23], if $\mathcal{B}$ is σ–unital. Just that now instead of F^n for $n \in \mathbb{N}$ we have to start with F^∞. ([11, Proposition 7.4] guarantees that F^∞ has a direct summand $\mathcal{B}$ and from this it follows that the correspondence $\mathcal{B}^a(\mathcal{B}^\infty, F^\infty)$ over $\mathcal{B}^a(\mathcal{B}^\infty)$ has a unit vector.) The rest goes precisely as in [23]. We, thus, proved:

Theorem 3.1. *Let $E^\odot = \bigl(E_n\bigr)_{n\in\mathbb{N}_0}$ be a full product system of correspondences over a σ–unital C^*–algebra. Then there exist a full Hilbert $\mathcal{B}$–module E and a strict E_0–semigroup ϑ on $\mathcal{B}^a(E)$ such that the product system of ϑ is $E^\odot$.*

Once the discrete case (Theorem 3.1) is known, we may use it to construct a solution for the algebraic (without continuity conditions) continuous time case $\mathbb{S} = \mathbb{R}_+$. This can be done by the procedure invented in Skeide [18] for the Hilbert space case, as we pointed out in [20] for modules over unital $\mathcal{B}$. The idea is the following: To find a left dilation of a full product system $E^\odot = \bigl(E_t\bigr)_{t\in\mathbb{R}_+}$ we start with a left dilation of the discrete subsystem $\bigl(E_t\bigr)_{t\in\mathbb{N}_0}$ to a Hilbert module $\breve{E}$, that is, with a family of unitaries $\breve{v}_n \colon \breve{E} \odot E_n \to \breve{E}$ that fulfill the necessary associativity conditions. (If $\mathcal{B}$ is σ–unital, then existence of such a dilation is granted by Theorem 3.1.) We put $E := \breve{E} \odot \int_0^1 E_\alpha\, d\alpha$. The following identifications

$$
\begin{aligned}
E \odot E_t \;&=\; \breve{E} \odot \Bigl(\int_0^1 E_\alpha\, d\alpha\Bigr) \odot E_t \;=\; \breve{E} \odot \int_t^{1+t} E_\alpha\, d\alpha \\
&\cong\; \Bigl(\breve{E} \odot E_n \odot \int_{t-n}^1 E_\alpha\, d\alpha\Bigr) \oplus \Bigl(\breve{E} \odot E_{n+1} \odot \int_0^{t-n} E_\alpha\, d\alpha\Bigr) \\
&\cong\; \Bigl(\breve{E} \odot \int_{t-n}^1 E_\alpha\, d\alpha\Bigr) \oplus \Bigl(\breve{E} \odot \int_0^{t-n} E_\alpha\, d\alpha\Bigr) \;=\; E \qquad (3.1)
\end{aligned}
$$

suggest, then, a family of unitaries $v_t\colon E \odot E_t \to E$. The slightly tedious thing in [18] was to show associativity, that is, to show that the v_t form a dilation to E.

By that method, whenever we are able to dilate the discrete subsystem $\bigl(E_t\bigr)_{t\in\mathbb{N}_0}$ of $E^\odot$ and to give a meaning to the direct integrals with respect to a translation (mod 1) invariant measure, we are also able to dilate the whole product system $E^\odot$. In absence of continuity conditions, this translation invariant measure can only be the counting measure, so that the direct integrals are simply direct sums. We find:

Theorem 3.2. *Let $E^\odot = \bigl(E_t\bigr)_{t\in\mathbb{R}_+}$ be a full product system of correspondences over a σ–unital C^*–algebra. Then there exist a full Hilbert $\mathcal{B}$–module E and a strict E_0–semigroup ϑ on $\mathcal{B}^a(E)$ such that the product system of ϑ is $E^\odot$.*

Remark 3.3. Note that both theorems remain true whenever for one member E_t of $E^\odot$ with $t \neq 0$, a suitable multiple $E_t^{\mathfrak{n}}$ ($\mathfrak{n}$ a cardinal number) of E_t has a direct summand $\mathcal{B}$, also if $\mathcal{B}$ is not σ–unital.

4. The continuous case

We now switch to the problem when, in the situation of Theorem 3.2, the product system is also continuous. Following the same idea as described in (3.1), we simply could pass to the Lebesgue measure, show that the direct integrals make sense, and convince ourselves that the resulting E_0–semigroup is strongly continuous and gives back the continuous structure on $E^\odot$ we started with. This is possible, but the operations mod 1 create quite horrible problems to write it down.

In Skeide [20], in the case of unital $\mathcal{B}$, we followed a different idea due to Arveson [6] in the case of Hilbert spaces. Suppose E_1 has unit vector ξ_1. (For Hilbert spaces this is not a problem. For continuous full product systems and unital $\mathcal{B}$ this is automatic; see [20, Lemma 3.2].) Consider those sections $x = \bigl(x_\alpha\bigr)_{\alpha\in\mathbb{R}_+}$ of $E^\odot$ that are *stable* with respect to ξ_1 in the sense that there exists $\alpha_0 \in \mathbb{R}_+$ such that $\xi_1 x_\alpha = x_{\alpha+1}$ for all $\alpha \geq \alpha_0$. Then one may define a semiinner product on the space of stable sections by setting $\langle x, y\rangle = \lim_{T\to\infty} \int_T^{T+1} \langle x_\alpha, y_\alpha\rangle \, d\alpha$. (This limit is over a function of T which is eventually constant.) One may divide out the kernel of that inner product and complete. On that space of sections mod $\langle\bullet,\bullet\rangle$ the product system "acts" from the right as $xy_t = \bigl(x_{\alpha-t}y_t\bigr)_{\alpha\in\mathbb{R}_+}$ (where we put $x_\alpha = 0$ for negative α).

This approach yields the same result, as if we constructed the dilation of the discrete subsystem based on the unit vector ξ_1 and used it as input for (3.1). (We proved that in Skeide [19] for Hilbert spaces.) But it does not have anymore the problems with addition on $[0, 1)$ mod 1. It is not too much trouble to prove the desired continuity results. But here we shall see how Arveson's approach can be safed for the nonunital case.

Suppose $E^\odot$ is a full continuous product system, and suppose (following Remark 3.3) that for some $t \neq 0$ and for some cardinal number $\mathfrak{n}$ the multiple $E_t^{\mathfrak{n}}$ has a direct summand $\mathcal{B}$. By rescaling, we may assume that $t = 1$. Once more, the correspondence $\mathcal{B}^a(\mathcal{B}^{\mathfrak{n}}, E_1^{\mathfrak{n}})$ over $\mathcal{B}^a(\mathcal{B}^{\mathfrak{n}})$ has a unit vector Ξ_1, say. This vector cannot act on E_α. It can, however, act on $E_\alpha^{\mathfrak{n}}$. Let S be a set of cardinality $\#S = \mathfrak{n}$ and denote the elements of $E_\alpha^{\mathfrak{n}}$ as $X_\alpha = \bigl(X_\alpha^s\bigr)_{s\in S}$. Then put $\Xi_1 X_\alpha := \bigl(\sum_{s'\in S}(\Xi_1)_{ss'}X_\alpha^{s'}\bigr)_{s\in S}$.

We start by defining the direct integrals we need. Let the continuous structure of $E^\odot$ be determined by the family i of embeddings $i_t\colon E_t \to \widehat{E}$. This gives rise to embeddings $i_t^{\mathfrak{n}}\colon E_t^{\mathfrak{n}} \to \widehat{E}^{\mathfrak{n}}$. Every section $X = \bigl(X_t\bigr)_{t\in\mathbb{R}_+}$ with $X_t \in E_t^{\mathfrak{n}}$ gives rise to a function $t \mapsto X(t) := i_t^{\mathfrak{n}} X_t$ with values in $\widehat{E}^{\mathfrak{n}}$. We denote by

$$CS_i^{\mathfrak{n}}(E^\odot) \;=\; \Bigl\{\, X\colon t \mapsto X(t) \text{ is continuous} \,\Bigr\}$$

the set of all such sections that are continuous. Let $0 \le a < b < \infty$. By $\int_a^b E_\alpha^{\mathfrak{n}}\,d\alpha$ we understand the norm completion of the pre-Hilbert $\mathcal{B}$–module that consists of continuous sections $X \in CS_i^{\mathfrak{n}}(E^\odot)$ restricted to $[a,b)$ with inner product

$$\langle X, Y\rangle_{[a,b]} \;:=\; \int_a^b \langle X_\alpha, Y_\alpha\rangle\,d\alpha \;=\; \int_a^b \langle X(\alpha), Y(\alpha)\rangle\,d\alpha.$$

Note that all continuous sections are bounded on the compact interval $[a,b]$ and, therefore, square integrable. The following proposition is proved precisely as [20, Proposition 4.2] (which holds for arbitrary subbundles of Banach bundles).

Proposition 4.1. *$\int_a^b E_\alpha^{\mathfrak{n}}\,d\alpha$ contains the space $\mathfrak{R}_{[a,b)}$ of restrictions to $[a,b)$ of those sections X for which $t \mapsto X(t)$ is right continuous with finite jumps (this implies by the definition of jump that there exists a left limit) in finitely many points and bounded on $[a,b)$, as a pre-Hilbert submodule.*

Let $\mathcal{S}$ denote the right $\mathcal{B}$–module of all sections X that are *locally* $\mathfrak{R}$, that is, for every $0 \le a < b < \infty$ the restriction of X to $[a,b)$ is in $\mathfrak{R}_{[a,b)}$, and which are *stable* with respect to the unit vector Ξ_1 in $\mathcal{B}^a(\mathcal{B}^{\mathfrak{n}}, E_1^{\mathfrak{n}})$, that is, there exists an $\alpha_0 \ge 0$ such that

$$X_{\alpha+1} \;=\; \Xi_1 X_\alpha$$

for all $\alpha \ge \alpha_0$. By $\mathcal{N}$ we denote the subspace of all sections in $\mathcal{S}$ which are eventually 0, that is, of all sections $X \in \mathcal{S}$ for which there exists an $\alpha_0 \ge 0$ such that $X_\alpha = 0$ for all $\alpha \ge \alpha_0$. A straightforward verification shows that

$$\langle X, Y\rangle \;:=\; \lim_{m\to\infty} \int_m^{m+1} \langle X(\alpha), Y(\alpha)\rangle\,d\alpha$$

defines a semiinner product on $\mathcal{S}$ and that $\langle X, X\rangle = 0$ if and only if $X \in \mathcal{N}$. Actually, we have

$$\langle X, Y\rangle \;=\; \int_T^{T+1} \langle X(\alpha), Y(\alpha)\rangle\,d\alpha$$

for all sufficiently large $T > 0$; see [6, Lemma 2.1]. So, $\mathcal{S}/\mathcal{N}$ becomes a pre-Hilbert module with inner product $\langle X + \mathcal{N}, Y + \mathcal{N}\rangle := \langle X, Y\rangle$. By E we denote its completion.

The following proposition is proved as [20, Proposition 4.3].

Proposition 4.2. *For every section X and every $\alpha_0 \geq 0$ define the section X^{α_0} as*

$$X^{\alpha_0}_\alpha := \begin{cases} 0 & \alpha < \alpha_0 \\ \Xi_1^n X_{\alpha-n} & \alpha \in [\alpha_0+n, \alpha_0+n+1), n \in \mathbb{N}_0. \end{cases}$$

If X is in $CS_i^{\mathfrak{n}}(E^\odot)$, then X^{α_0} is in $\mathcal{S}$. Moreover, the set $\{X^{\alpha_0} + \mathcal{N} \colon X \in CS_i^{\mathfrak{n}}(E^\odot), \alpha_0 \geq 0\}$ is a dense submodule of E.

After these preparations it is completely plain to see that for every $t \in \mathbb{R}_+$ the map $X \odot y_t \mapsto Xy_t$, where

$$(Xy_t)_\alpha = \begin{cases} X_{\alpha-t} y_t & \alpha \geq t, \\ 0 & \text{else}, \end{cases}$$

and where $X_\alpha y_t = \left(X_\alpha^s y_t\right)_{s\in S}$, defines an isometry $v_t \colon E \odot E_t \to E$, and that these isometries iterate associatively.

Proposition 4.3. *Each v_t is surjective.*

Proof. By Proposition 4.2 it is sufficient to approximate every section of the form X^{α_0} with $X \in CS_i^{\mathfrak{n}}(E^\odot), \alpha_0 \geq 0$ in the (semi-)inner product of $\mathcal{S}$ by finite sums of sections of the form Yz_t for $Y \in \mathcal{S}, z_t \in E_t$. As what the section does on the finite interval $[0,t)$ is not important for the inner product, we may even assume that $\alpha_0 \geq t$. And as in the proof of Proposition 4.2 the approximation can be done by approximating X in $\mathfrak{R}_{[\alpha_0,\alpha_0+1)}$ and then extending the restriction to $[\alpha_0, \alpha_0+1)$ stably to the whole axis. (This stable extension is the main reason why we worry to introduce the subspace of right continuous sections.)

Proposition 4.3 for $\mathfrak{n} = 1$ and unital $\mathcal{B}$ is done in [20, Proposition 4.6]. The restriction that $\mathcal{B}$ be unital can be omitted without affecting the proof. One may either repeat the proof word by word (for functions with values in $\widehat{E}^{\mathfrak{n}}$ instead of $\widehat{E}$). Or one may note that the proof goes through for any finite $\mathfrak{n}$, and that the approximation maybe done (with one more ε) by restricting to a suitable finite subset S (of course, depending on the section to be approximated). □

So, the v_t form a dilation of $E^\odot$ to E. Like in [20, Proposition 4.7], we show that the dilation is *continuous* in the following sense.

Proposition 4.4. *For every $x \in E$ and every continuous section $y \in CS_i(E^\odot)$ the function $t \mapsto xy_t$ is continuous.*

In the proof of [20, Proposition 4.7] just replace the section $z \in CS_i(E^\odot)$ by $Z \in CS_i^{\mathfrak{n}}(E^\odot)$.

Corollary 4.5. *The E_0–semigroup ϑ^v is strictly continuous.*

Proof. This proof is almost identical to that of [20, Corollary 4.8]. The only problem is that in our context here we do not have available a continuous section ζ of unit vectors that would fulfill $x\zeta_\varepsilon \to x\mathbf{1} = x$ for all $x \in E$. Instead, for a given $x \in E$ we choose $\beta \in \mathcal{B}$ with $\|\beta\| \leq 1$ and $x\beta$ sufficiently close to x. Then we choose a continuous section $\zeta \in CS_i(E^\odot)$ with $\zeta_0 = \beta$. With that section ζ everything goes exactly like in the proof of [20, Corollary 4.8]. □

Corollary 4.6. *The continuous structure induced by the E_0–semigroup ϑ^v coincides with the continuous structure of $E^\odot$.*

Proof. This is Proposition 4.4 together with Theorem 2.2. □

Remark 4.7. For unital $\mathcal{B}$, this corollary is [20, Proposition 4.9]. In the proof of [20, Proposition 4.9] we showed, however, only right continuity. Theorem 2.2 settles this gap.

Remark 4.8. Note that the proofs of the two preceding corollaries do not depend on the concrete form of the left dilation. We, therefore, showed the following more general statement: If v_t is a left dilation of a continuous product system $E^\odot$ that is continuous in the sense of Proposition 4.4, then the induced E_0–semigroup ϑ^v is strongly continuous and the continuous structure induced by that E_0–semigroup upon $E^\odot$ coincides with the original one.

We summarize.

Theorem 4.9. *Every full continuous product system of correspondences over a σ–unital C^*–algebra $\mathcal{B}$ is the continuous product system associated with a strictly continuous E_0–semigroup that acts on the algebra of all adjointable operators on a full Hilbert $\mathcal{B}$–module.*

For the sake of completeness, we state in the σ–unital case the classification theorem for E_0–semigroup by product systems as stated in Skeide [21] for unital C^*–algebras. We refer to [21] for the definition of stable cocycle conjugacy. Precisely under the same conditions as in [21, Section 9] for unital C^*–algebras $\mathcal{B}$, we obtain the following theorem for σ–unital C^*–algebras. Recall that a continuous product system $E^\odot$ is *countably generated*, if there is a countable subset S of $CS_i(E^\odot)$ such that $CS_i(E^\odot)$ is the locally uniform closure of the linear span of S.

Theorem 4.10. *Let $\mathcal{B}$ be a σ–unital C^*–algebra. Then there is a one-to-one correspondence between equivalence classes (up to stable cocycle conjugacy with strongly continuous cocycles) of strongly continuous strict E_0–semigroups acting on the operators of countably generated full Hilbert $\mathcal{B}$–modules and isomorphism classes of countably generated continuous product systems of full correspondences over $\mathcal{B}$.*

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