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BOUNDED STRUCTURES OF UNIFORMLY A-CONVEX ALGEBRAS

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Abstract. We examine the uniqueness of the bounded structure of semisimple and Mackey complete uniformly A-convex algebras. We also consider the particular locally C^*-case and the uniform one.

1. Introduction

Numerous examples show that there is no a Johnson's theorem type for locally uniformly A-convex algebras; that is the uniqueness (up to an equivalence) of the topological structure. However, there is a similar theorem for the von Neumann bounded structure (the bound structure, in short; or also the bornology) of such unital semi-simple and Mackey complete algebras. It is shown that the bounded structure of a unital, semi-simple and Mackey complete locally uniformly A-convex algebra is unique up to a bornological isomorphism (Proposition 4.1). We also examine particular cases where the algebra has additional properties. Using an Allan's result, we show that if it moreover owns the C^*-property, then its bounded structure is the one of a C^* -algebra (Proposition 5.3). In case it is uniform, a result of S.T. Bhatt ([2]; see also [1]) allows to obtain that it is the one of a uniform Banach algebra (Proposition 6.1).

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2. Definitions

Let (E, τ) be a locally convex algebra (*l.c.a.*), with a separately continuous multiplication, whose topology τ is given by a family $(p_\lambda)_{\lambda\in\Lambda}$ of seminorms. The algebra (E, τ) is said to be locally A-convex (*l-A-c.a.*; [4, 5]) if, for every x and every λ, there is $M(x, \lambda) > 0$ such that

$$\max\left[p_\lambda(xy), p_\lambda(yx)\right] \leq M(x, \lambda)p_\lambda(y); \forall y \in E.$$

In the case of a single space norm, $(E, \|.\|)$ is called an A-normed algebra. If $M(x, \lambda) = M(x)$ depends only on x, we say that (E, τ) is a locally uniformly A-convex algebra (*l.u-A -c.a.*; [5]). If it happens that, for every λ,

$$p_\lambda(xy) \leq p_\lambda(x)p_\lambda(y); \forall x, y \in E,$$

then (E, τ) is named a locally m-convex algebra (*l.m.c.a.*; cf. [9, 10]). Recall also that an *l.c.a.* has a continuous multiplication if, for every λ, there is λ'such that

$$p_\lambda(xy) \leq p_{\lambda'}(x)p_{\lambda'}(y); \forall x, y \in E.$$

If $(E, (p_\lambda)_\lambda)$ is a unital *l-A-c.a.*, then it can be endowed with a stronger m-convex topology $M(\tau)$ (cf. [11]), where τ is the topology on E. It is determined by the family $(q_\lambda)_{\lambda\in\Lambda}$ of seminorms given by

$$q_\lambda(x) = \sup\{p_\lambda(xu) : p_\lambda(u) \leq 1\}.$$

If $(E, (p_\lambda)_\lambda)$ is an *l.u-A-c.a.*, then there is yet ([12, 13]) an algebra norm $\|.\|_0$ which induces a topology $\tau_{\|.\|_0}$ stronger than $M(\tau)$. It is given by

$$\|x\|_0 = \sup\{q_\lambda(x) : \lambda\}.$$

Following the terminology of [8, pp. 101-102], if E is an involutive algebra and p a vector space seminorm on E, we say that p is a C^*-seminorm if $p(x^*x) = [p(x)]^2$, for every x. An involutive topological algebra whose topology is defined by a (saturated) family of C^*-seminorms is called a C^*-convex algebra. A complete C^*-convex algebra is called a locally C^*-algebra (by Inoue). A Fréchet C^*-convex algebra is a metrizable C^*-convex algebra, that is equivalently a metrizable locally C^*-algebra, or also a Fréchet locally C^*-algebra.

The bounded structure (bornology) of a locally convex space (*l.c.s.*) (E, τ) is the collection $\mathbb{B}\tau$ of all the subsets B of E which are bounded in the sense of von Neumann, that is B is absorbed by every neighborhood of the origin. If $\tau_{\|.\|}$ is the topology induced by a norm $\|.\|$, we write $\mathbb{B}\tau_{\|.\|}$. We say that an *l.c.s.* (E, τ) is Mackey complete if its bounded structure $\mathbb{B}\tau$ admits a fundamental system $\mathcal{B}$ of Banach discs that is, for every B in $\mathcal{B}$, the vector space generated by B is a Banach space when endowed with the gauge $\|.\|_B$ of B.

If the topology of an *l.c.s.* is given by a family $(p_\lambda)_{\lambda\in\Lambda}$ of seminorms, with Λ a directed set, we will often, for simplicity, write only $(p_\lambda)_\lambda$, especially when there is no risk of confusion.

An algebra, with an involution $x \longrightarrow x^*$ and a unit e, is said to be hermitian if every hermitian element (i.e., $x = x^*$) has a real spectrum. It is said to be symmetric if $e + x^*x$ is invertible for every x.

3. Examples

Here are different examples of $l.u$-A-$c.a.$'s.

Example 3.1. Let Ω be the first non countable ordinal and endow the set $[0,\Omega[$ with the order topology. Consider $E_1 = C([0,\Omega[)$ the complex algebra of continuous functions, on $[0,\Omega[$, endowed with the topology of uniform convergence on compacta. It is a commutative complete $l.m.c.a.$ with identity and continuous involution. It is not a Q-algebra but it is an $l.u$-A-$c.a.$.

Example 3.2. Let $E_2 = C_b(\mathbb{R})$ be the algebra of complex continuous bounded functions on the real field $\mathbb{R}$ with the usual pointwise operations and the complex conjugation as an involution. Denote by $C_0^+(\mathbb{R})$ the srictly positive elements of $C_b(\mathbb{R})$. Consider the family $\{p_\varphi : \varphi \in C_0^+(\mathbb{R})\}$ of seminorms given by

$$p_\varphi(f) = \sup\{f(x)\varphi(x) : x \in \mathbb{R}\}; f \in C_b(\mathbb{R}).$$

They determine a locally convex topology β. The space $(C_b(\mathbb{R}), \beta)$ is a complete locally convex * -algebra. It is not an $l.m.c.a.$ ([4]), nor a Q-algebra. But it is an $l.u$ -A-$c.a.$ with a continuous multiplication.

Example 3.3. Let $E_3 = C([0,1])$ be the algebra of complex valued continuous functions on the segment $[0,1]$. A vector space norm p is defined on this algebra by

$$p(f) = \sup\{\,|f(x)\varphi(x)| : x \in [0,1]\,\}; f \in C([0,1]).$$

where

$$\varphi(x) = \begin{cases} x, 0 \le x \le \frac{1}{2} \\ 1-x, \frac{1}{2} \le x \le 1 \end{cases}$$

Then $(C([0,1]), p)$ is not complete nor an $l.m.c.a.$. It is a pseudo-complete $l.u$-A-$c.a.$.

Example 3.4. Here is a similar example where the family of seminorms is not a singleton. Let $E_4 = L^\infty[0,1]$ be the complex algebra of essentially bounded measurable functions endowed with the L^ω -topology given by the L^p-norms, $p \geqslant 1$ i.e., $\|f\|_p = (\int_0^1 |f(t)|\,dt)^{\frac{1}{p}}$. It becomes a metrizable locally *-algebra. It is neither complete, nor an $l.m.c.a.$ nor a Q-algebra. It is an $l.u$-A-$c.a.$.

Example 3.5. In [14], it is given an example of an $l.u$-A-$c.a.$ the multiplication of which is not (jointly) continuous.

4. Comparison of bounded structures.

It is known (Johnson's theorem) that, in a semisimple algebra, there is -up to an equivalence- a unique Banach algebra norm. The same for commutative Fréchet m-convex algebras ([3]). Numerous examples show that without completeness or metrizability, these results do not remain valid. The uniqueness of bounded structures -up to an equivalence- require also particular conditions. In the uniformly A-convex case, we do have the following.

Proposition 4.1. *Let E be a complex unital algebra such that (E,τ) and $(E,\tau^{'})$ are l.u-A-c.a.'s.*

(i) *If $\mathbb{B}\tau = \mathbb{B}\tau^{'}$, then $\|.\|_0$ is equivalent to $\|.\|_0^{'}$.*

(ii) *If E is semisimple and both of (E,τ) and $(E,\tau^{'})$ are M-complete, then $\mathbb{B}\tau = \mathbb{B}\tau^{'}$.*

Proof. (i) Since τ (resp. $\tau^{'}$) is coarser than $\tau_{\|.\|}$ (resp. $\tau_{\|.\|^{'}}$), the unit balls of $\|.\|_0$ and $\|.\|_0^{'}$ are bounded for τ and $\tau^{'}$ respectively. But $\mathbb{B}\tau = \mathbb{B}\tau^{'}$. Thus the unit ball of each norm is bounded for the other. Whence the equivalence.

(ii) Since (E,τ) and $(E,\tau^{'})$ are M-complete, one has $\mathbb{B}\tau = \mathbb{B}_{\tau_{\|.\|_0}}$ and $\mathbb{B}\tau^{'} = \mathbb{B}_{\tau_{\|.\|_0^{'}}}$, with $(E,\|.\|_0)$ and $(E,\|.\|_0^{'})$ Banach algebras (cf. [12] or [13]). But E is semisimple, hence $\|.\|_0$ and $\|.\|_0^{'}$ are equivalent (Johnson's theorem). Therefore $\mathbb{B}\tau^{'} = \mathbb{B}_{\tau_{\|.\|_0^{'}}} = \mathbb{B}_{\tau_{\|.\|_0}} = \mathbb{B}\tau$. □

Remark 4.2. The converse is not true in (i) of the previous proposition. Take, for example, the algebra $C([0,1])$ endowed with the single seminorm $\|.\|_1$ i.e., $\|f\|_1 = \int_0^1 |f(t)|\,dt$, and by the topology induced by the Arens algbra $L^{\omega}[0,1]$ i.e., $\cap L^p[0,1]$ with the topology given by the family of seminorms $(\|.\|_p)_p$, where $\|f\|_p = (\int_0^1 |f(t)|\,dt)^{\frac{1}{p}}$ and $p = 1,2,....$ The associated norm to both topologies is $\|.\|_{\infty}$.

Remark 4.3. A locally C^*-algebra is always semisimple. So in the case of such algebras, semisimplicity is superfluous in the previous proposition. Actually one can say more in that situation. As an illustration, take Example 3.1 and Example 3.4 above. We examine the general case in the next section.

5. Locally C^*-algebras.

Now we consider the involutive case. First observe that if an *l.u-A-c.a.* (E,τ) is endowed with a continuous involution, then one may suppose that the topology is given by a family $(p_\lambda)_\lambda$ of seminorms such that

(1) $\max[p_\lambda(xy), p_\lambda(yx)] \leq M(x)p_\lambda(y); \forall y \in E$.

(2) $p_\lambda(x^*) = p_\lambda(x)$.

Indeed, if $(r_\lambda)_\lambda$ is a family of seminorms defining λ and satisfying (1), then put $p_\lambda(x) = \max(r_\lambda(x), r_\lambda(x^*))$.

Moreover, one has $\|x^*\|_0 = \|x\|_0$. But if τ is given by p_λ's satisfying the C^*-equality, it is not clear wether or not $\|.\|_0$ inherits this property. The following result is an answer to this question. To proceed, we need to specify a structure result ([15, Theorem 2.1]) according to the particular situation we deal with.

Proposition 5.1. *Let $(E,(p_\lambda)_\lambda)$ be an involutive M-complete l.c.a. . If the seminorms satisfy the C^*-equality i.e., $p_\lambda(x^*x) = p_\lambda^2(x)$, for every x, then $(E,(p_\lambda)_\lambda)$ is a bornological inductive limit of Fréchet locally C^*-algebras; hence it is hermitian.*

Hints. By a result of Sebestyèn ([16]), the p_λ's are submultilicative. Take $(B_i)_i$ a basis of the bounded structure of E . For B_i and every λ, put $p_\lambda(B_i) = \sup\{p_\lambda(x) : x \in B_i\}$ and $\Lambda_n^i = \{\lambda \in \Lambda : p_\lambda(B_i) < n\}$. One has $\Lambda = \cup\{\Lambda_n^i : n = 1, 2, ...\}$. Now, for every n, put $q_n^i(x) = \sup\{p_\lambda(x) : \lambda \in \Lambda_n^i\}$ and $E_i = \{x : q_n^i(x) < \infty; n = 1, 2, ...\}$. Then $(E_i, (q_n^i)_n)$ is a Fréchet *l.m.c.a.*. Moreover the seminorms satisfy the C^* -equality. The algebras $(E_i, (q_n^i)_n)$ are the locally C^*-algebras looked for. The hermiticity follows immediately. □

Remark 5.2. Hermiticity can also be obtained by a result of A. Mallios (see the proof of Proposition 6.1 below).

Proposition 5.3. *Let $(E, (p_\lambda)_\lambda)$ be an involutive M-complete l.u-A -c.a., the topology of which can be given by seminorms satisfying the C^*-equality. Then*

(i) *If E is commutative (not necessairily unital), then $(E, (p_\lambda)_\lambda)$ is a bornological inductive limit of C^* -algebras.*
(ii) *If E is unital (not necessairily commutative), then its bounded structure is the one of a C^* -algebra.*

Proof. (i) The E_i's in Proposition 5.1 are also *l.u-A-c.a.*'s, hence every element is bounded. They are then Q-algebras ([10, Proposition 13.6]). Being a locally C^*-algebra with the Q -property, every E_i is actually a C^* -algebra (See [8, Section 8, Chap. II] for a full discussion about the latter result) .

(ii) One checks that $(E, \|.\|_0)$ is with a continuous involution. It is then hermitian, by the previous proposition. Moreover, its closed unit ball is the geatest member of the family of self-adjoint closed and idempotent discs containing the unit element. Hence, by an Allan's result (cf. [2, Lemma (i)]), it is a C^*-algebra for an equivalent norm. So it is semisimple. Conclude by (ii) of Proposition 4.1. □

Remark 5.4. The converse is not always true in the previous proposition. Take e.g., the algebra $C_b(\mathbb{R})$ of Example 3.2.

Remark 5.5. The C^*-property is not necessary in (i) of Proposition 5.3. Let X be a non compact, locally compact and metrizable space such that $X = \cup K_n$ where $(K_n)_n$ is an exhaustive sequence of compact subsets of X. Take the complex algebra $K(X)$ of continuous functions with compact support and $E_n = K(X, K_n)$ the subalgebra of functions with support in K_n. It is known that $K(X)$ is algebraically the inductive limit of the E_n's. Take the strict inductive limit topology τ of the C^* -algebra norms $\|.\|_n$, where $\|.\|_n$ is the supremum norm on E_n. Then (E, τ) is a *l.m.c.a.* which is also an *l.u-A-c.a.* with a continuous involution. It can not be a locally C^*-algebra for it is a Q-algebra and so it should be a C^*-algebra (cf. [8, Corollary 8.2, p. 111]). But then $E = E_{n_0}$, for some n_0; which is not the case.

Remark 5.6. The uniform convexity is not necessary in (i) of Proposition 5.3. Take e.g., the unitizaton $K_1(X)$ of $K(X)$ in the previous remark, endowed with the usual topology. It is a locally C^*-algebra which is not an *l.u-A- c.a.*. However the conclusion of the assertion in question holds.

Remark 5.7. For the importance of the unit element, see [7].

We provide a criterion to check that a locally C^*-algebra is an *l.u -A-c.a.*. Actually completeness is not necessary in the hypotheses.

Proposition 5.8. *A locally C^*-algebra $(E, (p_\lambda)_\lambda)$ every element of which is bounded is an l.u-A-c.a..*

Proof. It is known that there is a greatest self-adjoint closed bounded and idempotent disc B_0, containing the unit, in $(E, (p_\lambda)_\lambda)$. By hypothesis, one has $E = E_{B_0}$ the algebra generated by B_0. Also the gauge $\|.\|_{B_0}$ defines a topology stronger than the initial one. So

$$\forall \lambda, \exists k_\lambda \geq 1 : p_\lambda(x) \leq k_\lambda \|x\|_{B_0}, \forall x.$$

But then

$$\forall \lambda, p_\lambda(xy) \leq \|x\|_{B_0} k_\lambda p_\lambda(y); \forall x, y \in E.$$

The topology is also given by the family of seminorms $(k_\lambda p_\lambda)_{\lambda, k \geq 1}$for which

$$\forall x, \exists M(x) = \|x\|_{B_0} : kp_\lambda(xy) \leq \|x\|_{B_0} kp_\lambda(y); \forall y \in E.$$

□

The examples in Section 3 concern only commutative algebras. Here is a non commutative algebra to which the previous proposition applies. First, we recall the construction of the matrix algebra (Mallios) as quoted in [8, pp. 109-110]. Let $(E, (p_\lambda)_\lambda)$ be a unital locally C^*-algebra and take the unital complex algebra of $n \times n$ matrices with entries in E

$$M_n(E) := \{x \equiv (x_{ij}) : (x_{ij}) \in E; i, j = 1, ..., n\}.$$

Algebraic operations and involution in $M_n(E)$ are defined as in the case of complex matrices. Consider $M \equiv E^n$ the finitely generated free (left) E-module associated with E, i.e.,

$$\begin{aligned} M &= \{m = \Sigma_1^n m_i e_i : m_i \in E \text{ and } e_i := (\delta_{ij})_{1 \leq j \leq n} \in E^n, \\ \text{with } \delta_{ii} &= e, \text{ the identity element of } E, \text{ and } \delta_{ij} = 0 \text{ for } i \neq j\}. \end{aligned}$$

Endow M with the family of the following seminorms

$$\widetilde{p_\lambda}(m) := \Sigma_1^n p_\lambda(m_i), \forall m \in M, \forall \lambda.$$

It becomes a locally convex E-module. Now, to each $x \in M_n(E)$ is associated a continous E-linear operator T_x, on M, given by

$$T_x(m) := (\Sigma_1^n x_{1i} m_i, ..., \Sigma_1^n x_{ni} m_i) \in M; m \in M.$$

Finally, for each λ, put

$$q_\lambda(x) := \sup\{\widetilde{p_\lambda}(T_x(m)) : \widetilde{p_\lambda}(m) \leq 1\}, x \in M.$$

Then the family $(q_\lambda)_\lambda$ of seminorms makes $M_n(E)$ a unital locally C^*-algebra.

Example 5.9. Let E be a unital commutative locally C^*-algebra every element of which is bounded, and $M_n(E)$ the associated Matrix algebra (as above) . The latter is a unital non commutative locally C^*-algebra. Moreover every element is bounded. Indeed, take $x \equiv (x_{ij})$ with $x_{ij} \in E$ and $i,j = 1,...,n$. Each x_{ij} is absorbed by an idempotent bounded disc B_{ij}. But one can take an $\alpha > 0$ such that $x_{ij} \in \alpha B_{ij}$, for every i and every j. Now, since E is commutative the idempotent hull of $\{B_{ij}\}_{ij}$ is bounded.

Remark 5.10. If the locally C^*-algebra E in Example 5.9 is not commutative, the arguments used there are no more valid. So we do not know, in that case, if every element of $M_n(E)$ is still bounded.

The arguments in Proposition 5.3 suggested the following result which seems to have some interest for its own.

Proposition 5.11. *Let E be a complex algebra such that (E,τ) is a locally C^* -algebra and $(E, \|.\|)$ is a normed space. If $\mathbb{B}\tau = \mathbb{B}_{\tau_{\|.\|}}$, then $(E,\|.\|)$ is (up to an isomorphism) a C^*-algebra.*

Proof. Without loss of generality, we consider the unital case. Since (E,τ) is complete and $\mathbb{B}\tau = \mathbb{B}_{\tau_{\|.\|}}$, the space $(E,\|.\|)$ is a Banach space. Now, the multiplication and the involution are bounded with respect to the norm. Hence $(E,\|.\|)$ is (up to an isomorphism) a Banach algebra. Also, since (E,τ) is a locally C^*-algebra, the involutive algebra E is symmetric. Now take the subset $B = \{x \in E : \ p_\lambda(x) \leq 1; \forall\lambda\}$. It is the geatest member of the family of self-adjoint closed and idempotent discs containing the unit element. Hence, by an Allan's result (cf. [2, Lemma (i)]), it is a C^*-algebra for an equivalent norm. □

6. Uniform algebras.

Let (E,τ) be an *l.u-A-c.a.* which is also uniform that is its topology can be given by two families $(p_\lambda)_\lambda$ and (r_i) of seminorms such that
(1) $\max[p_\lambda(xy), p_\lambda(yx)] \leq M(x)p_\lambda(y); \forall y \in E$.
(2) $r_i(x^2) = r_i^2(x); \forall x \in E$.
One considers the canonical norm $\|.\|_0$ associated to $(p_\lambda)_\lambda$. It is not granted that it is uniform i.e., square preserving. However, we will see that this is true up to an equivalence.

Proposition 6.1. *Let (E,τ) be a unital M-complete l.u-A-c.a.. If its topology can be given by a family of square preserving seminorms, then its bounded structure is the one of a uniform Banach algebra.*

Proof. By a result of H.V. Dedania ([6]), each r_i is submultiplicative. Thus (E,τ) is an *l.m.c.a.*. It is then semi-simple (cf. [9, Lemma 5.1., p. 275]). So Proposition 4.1 appplies. Now $\mathbb{B}\tau = \mathbb{B}_{\tau_{\|.\|_0}}$. But then $B_1 = \{x : \|x\|_0 \leq 1\}$ is the greatest closed bounded and idempotent disc, containing the unit, of $(E,\|.\|_0)$. Hence it is a uniform algebra for an equivalent norm, by a result of S.T. Bhatt ([2, Lemma (ii)]). □

Remark 6.2. The converse is not always true in the previous proposition. Take e.g., the algebra $C_b(\mathbb{R})$ of Example 3.2.

Now here is the analogue of Proposition 5.8, in the context of this section.

Remark 6.3. A unital uniform *l.m.c.a.* every element of which is bounded is an *l.u-A- c.a.*.

Proof. It is known that there is a greatest closed bounded and idempotent disc B_0, containing the unit, in $(E,(p_\lambda)_\lambda)$. By hypothesis, one has $E = E_{B_0}$ the algebra generated by B_0. Also the gauge $\|.\|_{B_0}$ defines a topology stronger than the initial one. So

$$\forall\lambda, \exists k_\lambda \geq 1 : p_\lambda(x) \leq k_\lambda \|x\|_{B_0}, \forall x.$$

But then

$$\forall\lambda, p_\lambda(xy) \leq \|x\|_{B_0} k_\lambda p_\lambda(y); \forall x, y \in E.$$

The topology is also given by the family of seminorms $(k_\lambda p_\lambda)_{\lambda,k\geq 1}$for which

$$\forall x, \exists M(x) = \|x\|_{B_0} : kp_\lambda(xy) \leq \|x\|_{B_0} kp_\lambda(y); \forall y \in E.$$

□

The analogue of Proposition 5.11 is the following.

Proposition 6.4. *Let E be a unital complex algebra such that (E,τ) is a uniform locally m-convex algebra and $(E,\|.\|)$ is a normed space. If $\mathbb{B}\tau = \mathbb{B}_{\tau_{\|.\|}}$, then $(E,\|.\|)$ is (up to an isomorphism) a uniform Banach algebra.*

Proof. Since (E,τ) is complete and $\mathbb{B}\tau = \mathbb{B}_{\tau_{\|.\|}}$, the space $(E,\|.\|)$ is a Banach space. Now, the multiplication is bounded with respect to the norm. Hence $(E,\|.\|)$ is (up to an isomorphism) a Banach algebra. Also, since (E,τ) is uniform it is semisimple (cf. [9, Lemma 5.1., p. 275]) and the subset $B = \{x \in E : p_\lambda(x) \leq 1; \forall\lambda\}$ is the geatest member of the family of closed and idempotent discs containing the unit element. Hence it is a uniform algebra for an equivalent norm, by a result of S.T. Bhatt ([2, Lemma (ii)]). □

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