Banach J. Math. Anal. 2 (2008), no. 2, 42–58

Banach Journal of Mathematical Analysis
ISSN: 1735-8787 (electronic)
http://www.math-analysis.org

SOME WEIGHTED SUM AND PRODUCT INEQUALITIES IN L^p SPACES AND THEIR APPLICATIONS

R. C. BROWN

This paper is dedicated to Professor Joseph E. Pečarić

Submitted by Th. M. Rassias

Abstract. We survey some old and new results concerning weighted norm inequalities of sum and product form and apply the theory to obtain limit-point conditions for second order differential operators of Sturm-Liouville form defined in L^p spaces. We also extend results of Anderson and Hinton by giving necessary and sufficient criteria that perturbations of such operators be relatively bounded. Our work is in part a generalization of the classical Hilbert space theory of Sturm-Liouville operators to a Banach space setting.

1. Introduction

Let w, v_0, v_1 be positive a.e. measurable or "weights" on the interval $I_a = [a, \infty)$, $a > -\infty$. We are interested in obtaining conditions which guarantee the validity of the weighted "sum" inequality:

$$\int_{I_a} w|y^{(j)}|^p \leq K_1(\epsilon) \int_{I_a} v_0|y|^p + \epsilon \int_{I_a} v_1|y^{(n)}|^p \tag{1.1}$$

for $0 \leq j < n$ where $1 \leq p \leq \infty$ and $\epsilon \in (0, \epsilon_0)$. The space of functions $\mathcal{D}^p(v_0, v_1; I_a)$ on which (1.1) holds is defined by

$$\mathcal{D}^p(v_0, v_1; I_a) := \left\{ y : y \in AC^{n-1}(I_a); \int_{I_a} v_0|y|^p, \int_{I_a} v_1|y^{(n)}|^p < \infty \right\}$$

Date: Received: 12 April 2008; Accepted 21 April 2008.

2000 *Mathematics Subject Classification.* Primary: 26D10, 47A30, 34B24; Secondary 47E05.

Key words and phrases. Weighted sum inequalities, weighted product inequalities, Sturm Liouville operators, limit-point conditions, relatively bounded perturbations.

where $AC^j(I_a)$ denotes the class of functions whose j-th derivative is locally absolutely continuous on I_a. We shall also show the inequality (1.1) often implies the "product" inequality

$$\int_{I_a} v_1|y^{(j)}|^p \leq K_2 \left(\int_{I_a} v_1|y|^p\right)^{\frac{n-j}{n}} \left(\int_{I_a} v_1|y^{(n)}|^p\right)^{\frac{j}{n}}.$$

It will turn out that (1.1) has a number of interesting applications to problems in concerning second-order differential operators determined by symmetric expressions of the form $-(ry')' + qy$ and defined in L^p spaces. The results generalize both some aspects of the Hilbert theory presented in the book of Naimark [23] and criteria obtained by Anderson and Hinton in [1] that perturbations of such operators in the L^2 setting be relatively bounded. We close this section with a few remarks on notation. Upper case letters such as K or C denote constants whose value may change from line to line. We distinguish between different constants by writing $K_1, K_2, C, C_1, \ldots,$ etc. $K(\cdot)$ indicates dependence on a parameter, e.g., $K_1(\epsilon)$. $L^p(I_a)$ signifies the (complex) L^p space on I_a having the norm

$$||u||_{p,I_a} := \left(\int_{I_a} |u|^p\right)^{1/p}.$$

$C^\infty(I_a)$ and $C^j(I_a)$ respectively denote the infinitely or j-fold differentiable functions having continuous j-th derivative on I_a and $C_0^\infty(I_a)$ or $C_0^j(I_a)$ consists of the subspace of $C^\infty(I_a)$ or $C^j(I_a)$ having compact support. A local property is indicated by the subscript "loc", e.g., $f \in L^p_{\text{loc}}(I)$, etc. Also, we write $AC^0(I_a)$ as $AC(I_a)$ and $L^1(I_a)$ as $L(I_a)$, and when the context is clear we abbreviate $\mathcal{D}^p(v_0, v_1; I_a)$ by "$\mathcal{D}^p$." If $T : X \to Y$ is an operator where X and Y are Banach spaces $\mathcal{D}(T)$, $\mathcal{R}(T)$, $\mathcal{G}(T)$, and $\mathcal{N}(T)$ respectively denote the domain, range, graph, and null space of T. Finally, if f and g are two functions the notation $f \approx g$ means that there are constants C_1 and C_2 such that $f \leq C_1 g$ and $g \leq C_2 f$.

2. Some weighted norm inequalities of sum form

Suppose that f is a positive continuous function on I_a. Let $J_{t,\epsilon} := [t, t+\epsilon f(t)]$. For $1 < p < \infty$ set

$$S_1(t) := f^{-jp}\left[(\epsilon f)^{-1}\int_{J_{t,\epsilon}} w\right]\left[(\epsilon f)^{-1}\int_{J_{t,\epsilon}} v_0^{-p'/p}\right]^{p/p'} \tag{2.1}$$

$$S_2(t) := f^{(n-j)p}\left[(\epsilon f)^{-1}\int_{J_{t,\epsilon}} w\right]\left[(\epsilon f)^{-1}\int_{J_{t,\epsilon}} v_1^{-p'/p}\right]^{p/p'} \tag{2.2}$$

where $p' = p/(p-1)$. In the case that $p = 1$ or ∞ some modifications in these definitions are required. If $p = 1$ we substitute the L^∞ norm of v_i^{-1}, $i = 0, 1$, on $J_{t,\epsilon}$ for the integral term. For instance,

$$S_1(t) := f^{-j}\left[(\epsilon f)^{-1}\int_{J_{t,\epsilon}} w\right]\|v^{-1}\|_{\infty,J_{t,\epsilon}}.$$

And when $p = \infty$ we write

$$S_1(t) := f^{-j}\|w\|_{\infty,J_{t,\epsilon}} \left[(\epsilon f)^{-1}\int_{J_{t,\epsilon}} v_0^{-1}\right].$$

Similar changes apply to $S_2(t)$.

Theorem A. *Suppose* $1 \le p \le \infty$. *If there exists* f *and* $\epsilon_0 \in (0,\infty]$ *such that*

$$S_1(\epsilon_0) := \sup_{t\in I_a,\, 0<\epsilon\le\epsilon_0} S_1(t) < \infty \tag{2.3}$$

$$S_2(\epsilon_0) := \sup_{t\in I,\, 0<\epsilon\le\epsilon_0} S_2(t) < \infty, \tag{2.4}$$

then the inequality

$$\int_{I_a} w|y^{(j)}|^p \le K_1\left\{\epsilon^{-j\eta}\int_{I_a} v_0|y|^p + \epsilon^{(n-j)\eta}\int_{I_a} v_1|y^{(n)}|^p\right\} \tag{2.5}$$

where $\eta = p$ *if* $1 \le p < \infty$, $\eta = 1$ *if* $p = \infty$ *holds on* $\mathcal{D}^p(v_0, v_1; I_a)$ *for* $\epsilon \le \epsilon_0$, *and* $K_1 \approx \max\{S_1(\epsilon_0), S_2(\epsilon_0)\}$.

Proof. For $1 \le p < \infty$ this was shown in [4]. The main idea in the proof of Theorem A is to partition I_a in the following way. Let $t_0 := a$ and set $t_{j+1} = t_j + \epsilon f(t_j)$. On each interval $J_{t_j,\epsilon}$ we start with the basic interpolation inequality (see [4, Lemma 2.1])

$$|y^{(j)}(t)| \le K_1\left\{(\epsilon f)^{-(j+1)}\int_{J_{t_j,\epsilon}} |y| + (\epsilon f)^{n-(j+1)}\int_{J_{t_j,\epsilon}} |y^{(n)}|\right\}. \tag{2.6}$$

We then raise both sides to the p-th power, apply the inequality $(A+B)^p \le 2^{p-1}(A^p + B^p)$ to the right hand side, and use Hölder's inequality to introduce the weights v_0, v_1. Next, we multiply both sides by w and integrate over $J_{t,\epsilon}$. The functions $S_1(t)$ and $S_2(t)$ will naturally appear. We bound them by S_1 and S_2 and add the resulting inequalities over all the intervals to obtain (2.5). A requirement of this argument is that the sequence $\{t_j\}$ have no finite limit point. This is guaranteed by the continuity and positivity of f. For $p = \infty$ Hölder's inequality, multiplication by w in (2.6), and an easy estimate gives

$$\begin{aligned}
w|y^{(j)}(t)| \le K_1 \Bigg\{ &(\epsilon f)^{-j}\|w\|_{\infty,J_{t_j,\epsilon}}\left[(\epsilon f)^{-1}\int_{J_{t_j,\epsilon}} v_0^{-1}\right]\|v_0 y\|_{\infty,J_{t_j,\epsilon}} \\
&+ (\epsilon f)^{(n-j)}\|w\|_{\infty,J_{t_j,\epsilon}}\left[(\epsilon f)^{-1}\int_{J_{t_j,\epsilon}} v_1^{-1}\right]\|v_1 y^{(n)}\|_{\infty,J_{t_j,\epsilon}}\Bigg\} \\
\le K_1 &\max\{S_1, S_2\}\left(\|v_0 y\|_{\infty,I_a} + \|v_0 y^{(n)} u\|_{\infty,I_a}\right).
\end{aligned}$$

Taking the L^∞ norm of the left side completes the argument. ☐

Since the integrals or L^∞ norms in the definitions of $S_1(t)$ and $S_2(t)$ may be difficult to handle we can replace $S_1(t)$ and $S_2(t)$ by simpler expressions provided the weights satisfy a certain condition.

Theorem 2.1. *Let f and $J_{t,\epsilon}$ be as above. Suppose that there is a constant K not depending on t (but possibly on ϵ) such that*

$$\frac{v_0(s)}{v_0(t)}, \frac{v_1(s)}{v_1(t)} \geq K \tag{2.7}$$

a.e. on $J_{t,\epsilon}$. If $p = 1, \infty$ assume also that

$$\frac{w(s)}{w(t)} \leq K_1 \tag{2.8}$$

a.e. on $J_{t,\epsilon}$. Then the sum inequality (2.5) *holds on $\mathcal{D}^p$ for $1 \leq p < \infty$ if*

$$T_1(\epsilon_0) := \sup_{t \in I_a, 0<\epsilon\leq\epsilon_0} f^{-jp} w v_0^{-1} < \infty \tag{2.9}$$

$$T_2(\epsilon_0) := \sup_{t \in I_a, 0<\epsilon\leq\epsilon_0} f^{(n-j)p} w v_1^{-1} < \infty. \tag{2.10}$$

Proof. To prove this for $1 < p < \infty$ we proceed as in the proof of Theorem A beginning with the basic interpolation inequality (2.6). We then raise both sides of this inequality to the p-th power, etc., and multiply by w. Next, using (2.7) to move $v_0^{-1/p}$ and $v_1^{-1/p}$ out of the integrals we get that

$$w|y^{(j)}|^p \leq K_2 \max\{T_1(\epsilon_0), T_2(\epsilon_0)\} 2^{p-1} \left\{ (\epsilon^{-jp} \left((\epsilon f)^{-1} \int_{J_{t,\epsilon}} v_0^{1/p} |y| \right)^p \right.$$
$$\left. + \epsilon^{(n-j)p} \left((\epsilon f)^{-1} \int_{J_{t,\epsilon}} v_1^{1/p} |y^{(n)}| \right)^p \right\}.$$

Finally, we integrate both sides over I_a and apply the Hardy-Littlewood Maximal Theorem (cf. [21, Theorem 21.76] to the two integral terms on the right-hand side. This gives (2.5). The cases $p = 1, \infty$ amount to special cases of Theorem A, where we use (2.7) and (2.8) to replace $S_1(\epsilon_0)$ and $S_2(\epsilon_0)$ by $T_1(\epsilon_0)$ and $T_2(\epsilon_0)$. □

Remark 2.2. Using a different argument it was shown in [4] that (2.5) also remains true if f is nondecreasing and the "semi-pointwise" averages

$$R_1(\epsilon_0) := \sup_{t \in I_a,\, 0<\epsilon\leq\epsilon_0} f(t)^{-pj} w(t) \left[(\epsilon f)^{-1} \int_{J_{t,\epsilon}} v_0^{-p'/p} \right]^{p/p'} \tag{2.11}$$

$$R_2(\epsilon_0) := \sup_{t \in I_a,\, 0<\epsilon\leq\epsilon_0} f(t)^{p(n-j)} w(t) \left[(\epsilon f)^{-1} \int_{J_{t,\epsilon}} v_1^{-p'/p} \right]^{p/p'} \tag{2.12}$$

are finite.

Remark 2.3. Another possibility lies in the application of the Besicovitch covering theorem. Let I be some finite or infinite interval. Suppose that each $t \in I$ is the center of an interval $\Delta_{t,\epsilon} := [t - \epsilon f(t)/2, t + \epsilon f(t)/2]$ contained in I_a where f is bounded if I is infinite. Let $\mathfrak{J}_\epsilon$ denote the collection of these intervals. It is then possible (see [19, Theorem 1.1, p.2]) to extract from $\mathfrak{J}_\epsilon$ finitely many families $\Gamma_1, \ldots, \Gamma_l$ of disjoint intervals in $\mathfrak{J}_\epsilon$ whose union covers I. If we redefine $S_1(t)$

and $S_2(t)$ by replacing the intervals $J_{t,\epsilon}$ by $\Delta_{t,\epsilon}$, then (2.5) is readily seen to hold on each Γ_j, $1 \le j \le l$. Hence,

$$\int_{\Gamma_j} w|y^{(j)}|^p \le K_1 \left\{ \epsilon^{-j\eta} \int_I v_0|y|^p + \epsilon^{(n-j)\eta} \int_I v_1|y^{(n)}|^p \right\},$$

so that

$$\int_I w|y^{(j)}|^p \le lK_1 \left\{ \epsilon^{-j\eta} \int_I v_0|y|^p + \epsilon^{(n-j)\eta} \int_I v_1|y^{(n)}|^p \right\}.$$

How can conditions like (2.3), (2.4), (2.9), (2.10), or (2.11), (2.12) be verified? Essentially, as we have already in part done in Theorem 2.1, we will want to choose f so that $v_i(s) \approx v_i(t)$, $i = 0, 1$, and $w(s) \approx w(t)$ on $J_{t,\epsilon}$. In a very general case this can always be done as we now demonstrate.

Proposition 2.4. *Suppose that w, v_0, and v_1 are continuous on I_a. Then there exists a positive function f^* depending on t and possibly ϵ such*

$$\frac{1}{2} \le \frac{w(s)}{w(t)}, \frac{v_0(s)}{v_0(t)}, \frac{v_1(s)}{v_1(t)} \le \frac{3}{2} \tag{2.13}$$

on $J_{t,\epsilon}$. Moreover, the sequence $\{t_j\}$ defined as in Theorem A using f^* has no finite limit point.

Proof. Given $t \ge a$, $\epsilon > 0$, and for $i = 0, 1$ let $f_i(t, \epsilon) := (s_i(t) - t)/\epsilon$ where

$$s_i(t) = \min\{t + \epsilon, \sup\{z > t : 3v_i(t)/2 \ge v_i(u) \ge v_i(t)/2 \text{ for } u \in (t, z]\}. \tag{2.14}$$

Define $s_2(t)$ and $f_2(t, \epsilon)$ similarly for w and set $f^*(t, \epsilon) = \min\{f_i(t, \epsilon)\}$, $i = 0, 1, 2$. With this construction of f^* (2.13) follows. To prove the second assertion Set

$$s_{*,0} := a < \cdots < s_{*,j+1} := s_{*,j} + \epsilon f_*(s_{*,j}, \epsilon) \equiv s_0(s_{*,j})$$

and suppose that $\{s_{*,j}\}$ converges to $\bar{s}_* < \infty$. We show that for all sufficiently large j and for $u \in (s_{*,j}, \bar{s}_*]$ that $3v_i(s_{*,j})/2 \ge v_i(u) \ge v_i(s_{*,j})/2$. If this is not so then for every j there is a $k > j$ and a $u^* \in [s_{*,k}, \bar{s}_*]$ such that for one of the weights, say, v_0 either (i) $v_0(s_{*,k})/2 > v_0(u^*)$ or (ii) $3v_0(s_{*,k})/2 < v_0(u^*)$. But from the continuity and positivity of v_0, given, say, $1/10 > \mu > 0$ there is a j such that for any $k > j$ and all $u \in [s_{*,k}, \bar{s}_*]$ we have that

$$(1 - \mu)v_0(\bar{s}_*) < v_0(u) < (1 + \mu)v_0(\bar{s}_*).$$

If (i) is true then

$$(1 - \mu)v_0(\bar{s}_*) < v_0(u^*) < v_0(s_{*,k})/2 < (1 + \mu)v_0(\bar{s}_*)/2$$

so that $9/10 < (1/2)(11/10)$ which is false. Similarly, if (ii) holds we have

$$(3/2)(1 - \mu)v_0(\bar{s}_*) < (3/2)v_0(s_{*,k}) < v_0(u^*) < (1 + \mu)v_0)(\bar{s}_*)$$

so that $27/20 < 11/10$ which is also false. This argument shows that

$$\bar{s}_* \le s_*(s_{*,k}) = s_{*,k+1} < s_*(s_{*,k+1}),$$

and so $\bar{s}_*$ cannot be a limit point of the sequence $\{s_{*,j}\}$. $\square$

Remark 2.5. With this definition of f^* we see that

$$S_1(t) \approx f^{*-jp} w v_0^{-1} \tag{2.15}$$

$$S_2(t) \approx f^{*(n-j)p} w v_1^{-1}. \tag{2.16}$$

It is even simpler to define f^* so that (2.7) holds in Theorem 2.1 since we can omit w and only consider the lower bounds in (2.14). We omit the details. In particular, this means that if the weights are continuous then the integral expressions (2.1) and (2.2) in Theorem A can in theory always be replaced by the point evaluation expressions (2.15) and (2.16). Also, in Theorem 2.1 if the weights are continuous then the conditions (2.7) will be satisfied if f^* is chosen. But while Proposition 2.4 is of some theoretical interest it is usually not of much practical use since it is difficult to characterize f^* in a convenient fashion from its definition. Fortunately, a satisfactory substitute for f^* is often suggested by the particular weights w, v_0, and v_1.

Example 2.6. Let $a = 1$, $w(t) = t^\beta$, $v_0(t) = t^\gamma$, $v_1(t) = t^\alpha$ and $f(t) = t^\delta$ where $\delta \leq 1$. Then

$$1 \leq \sup_{s \in J_{t,\epsilon}} st^{-1} \leq 1 + \epsilon t^{\delta - 1} \leq 1 + \epsilon.$$

A calculation shows that $S_1(\epsilon_0)$, $S_2(\epsilon_0)$ are finite if

$$\beta \leq \min\left\{\delta pj + \gamma, -\delta(n-j)p + \alpha\right\}, \tag{2.17}$$

and any fixed ϵ_0 (say $\epsilon_0 = 1$). In (2.17) β will be as large as possible relative to α and γ if δ is chosen by "equality", i.e.,

$$\delta = (\alpha - \gamma)/np \leq 1. \tag{2.18}$$

With this choice of δ

$$\beta \leq \gamma\left(\frac{n-j}{n}\right) + \alpha\left(\frac{j}{n}\right).$$

Example 2.7. Let $w(t) = v_0(t) = v_1(t) = \mathrm{e}^t$, $a \geq 0$, and $f(t) = 1$. Then $1 \leq \mathrm{e}^s/\mathrm{e}^t \leq \epsilon$ on $J_{t,\epsilon}$. (2.5) follows.

It was demonstrated in [7, Theorem 3.2] that either of the conditions (2.3) or (2.4) is necessary as well as sufficient for (2.5) provided the weights are chosen so that $S_1(t) \approx S_2(t)$. The choice of δ according to (2.18) forces this in Example 2.6.

Example 2.8. Let $v_0 = v_1 = w$ and take $f(t) = 1$. Then (2.5) is true if and only if

$$\sup_{t \in I_a, 0 < \epsilon \leq \epsilon_0} \left(\epsilon^{-1} \int_t^{t+\epsilon} w\right)\left(\epsilon^{-1} \int_t^{t+\epsilon} w^{-p'/p}\right)^{p/p'} < \infty.$$

A necessary and sufficient condition for (2.5) can also be stated in a more general setting if the weights satisfy certain growth conditions.

Theorem 2.9. *Let w, v_0, v_1 be weights such that $w, v_0^{-p'/p}, v_1^{-p'/p} \in L_{loc}(I_a)$ and the following conditions are satisfied:*

$$\begin{aligned} |v_1'| &\le K v_1^{1-1/np} v_0^{1/np}, \\ |v_0'| &\le np v_1^{-1/np} v_0^{1+1/np} \end{aligned} \tag{2.19}$$

on I_a. Then the sum inequality (2.5) holds for $1 < p < \infty$ and $j = 0, \dots, n-1$ if and only if

$$S(\epsilon_0) := \sup_{t \in I_a, \epsilon \in (0,\epsilon_0)} \frac{1}{\epsilon f(t)} \int_t^{t+\epsilon f(t)} w v_1^{-j/n} v_0^{(j-n)/n} < \infty \tag{2.20}$$

where $f(t) = (v_1/v_0)^{1/np}$.

Before proving this we need a lemma showing that our choice of f works like f^* in Proposition 2.4.

Lemma 2.10. *There are constants $K_2, K_3 > 0$ possibly depending on ϵ so that*

$$K_2 \le \frac{f(s)}{f(t)}, \frac{v_1(s)}{v_1(t)}, \frac{v_0(s)}{v_0(t)} \le K_3 \tag{2.21}$$

on the intervals $J_{t,\epsilon}$ for sufficiently small $\epsilon > 0$.

Proof. First, we note that

$$\begin{aligned} |f'| &= \left| \left((v_1/v_0)^{1/np} \right)' \right| = \left| (1/np)(v_1/v_0)^{1/np-1} v_0^{-2} (v_0 v_1' - v_1 v_0') \right| \\ &\le (1/np) v_1^{1/np-1} v_0^{-1/np} |v_1'| + (1/np)(v_1/v_0)^{1/np-1} v_0^{-2} v_1 |v_0'| \\ &\le K/np + 1. \end{aligned}$$

Hence for $t < s \le t + \epsilon f(t)$,

$$|f(s) - f(t)| = \left| \int_t^s f' \right| \le \int_t^s |f'| \le \epsilon f(t)(K/np + 1),$$

so that $f(s)/f(t) \le 1 + \epsilon(K/np + 1)$ and $f(s)/f(t) \ge 1 - \epsilon(K/np + 1)$. Next, consider v_0. Since

$$\begin{aligned} |f v_0'| &\le (v_1/v_0)^{1/np} np v_0^{1+1/np} v_1^{-1/np}, \\ &= np v_0 \end{aligned}$$

and by the previous result we obtain that

$$v_0'/v_0 \le |v_0'/v_0| \le np(1 - \epsilon(K/np + 1)) f(t))^{-1}.$$

Integrating this over $J_{t,\epsilon}$ implies that

$$\begin{aligned} v_0(s)/v_0(t) &\le \exp(K_4(\epsilon)(s - t)/f(t)) \\ &\le \exp(K_4(\epsilon)). \end{aligned} \tag{2.22}$$

But also since

$$-v_0'/v_0 \le |v_0'/v_0| \le np(1 - \epsilon(K/np + 1)) f(t))^{-1},$$

we get by integration that

$$v_0(s)/v_0(t) \geq (\exp(K_4(\epsilon)))^{-1}. \tag{2.23}$$

(2.21) for v_0 follows from (2.22) and (2.23). That (2.21) holds also for v_1 is a consequence of the identity

$$v_1(s)/v_1(t) = (v_0(s)/v_0(t)(f(s)/f(t))^{np}$$

and (2.21) for v_0 and f. □

Proof of Theorem 2.9. We know by Theorem A that the sum inequality (2.5) holds if (2.3) and (2.4) hold. However because of the conditions (2.21) allowing f, v_0, and v_1 to be taken in and out of integrals over the interval $J_{t,\epsilon}$ both conditions are found to be equivalent to (2.20). Since the assumptions on the weights guarantee that $S_1(t) = S_2(t)$ we could apply [7, Theorem 3.2] to conclude that (2.20) is a necessary condition; but we choose to give an explicit argument. Let $\phi \geq 0$ be a C_0^∞ function such that $\phi(t) = 1$ on $[0, 1]$ and ϕ has support on $(-2, 2)$. Define

$$H_j(t) = \phi(t)(t^j/j!).$$

Then $H_j^{(j)}(t) = 1$ on $[0, 1]$ and $H_j(t)$ has support on $(-2, 2)$. Set $u = (s-t)/\epsilon f(t)$ for $t - 2\epsilon f(t) \leq s \leq t + 2\epsilon f(t)$ and define

$$H_{j,t}(s) = (\epsilon f(t))^j H_j(u).$$

Note that

$$H_{j,t}^{(k)}(s) = H_j^{(k)}(u)(\epsilon f(t))^{j-k}$$

so that in particular $H_{j,t}^{(j)}(s) = 1$. Next, choose t and sufficiently small ϵ such that $t - 2\epsilon f(t) > a$ and consider the function

$$\begin{aligned} S(t) &:= (\epsilon f(t))^{-1} \int_t^{t+\epsilon f(t)} w v_1^{-j/n} v_0^{j/n-1} \\ &= (\epsilon f(t))^{-1} \int_t^{t+\epsilon f(t)} w v_1^{-j/n} v_0^{j/n-1} |H_{j,t}^{(j)}|^p \\ &\approx (\epsilon f(t))^{-1} (v_1^{-j/n} v_0^{j/n-1})(t) \\ &\quad \times \int_t^{t+\epsilon f(t)} w |H_{j,t}^{(j)}|^p. \end{aligned}$$

Therefore if (2.5) holds we have

$$\begin{aligned} S(t) \quad = \quad & \mathrm{O}\Big\{(\epsilon f(t))^{-1}(v_1^{-j/n} v_0^{j/n-1})(t)\Big[\epsilon^{-jp} \int_{t-2\epsilon f(t)}^{t+2\epsilon f(t)} v_0 |H_{j,t}|^p \\ & +\epsilon^{(n-j)p} \int_{t-2\epsilon f(t)}^{t+2\epsilon f(t)} v_1 |H_{j,t}^{(n)}|^p\Big]\Big\}. \end{aligned}$$

However

$$
\begin{aligned}
&(\epsilon f(t))^{-1}(v_1^{-j/n}v_0(t)^{j/n-1})(t)\left(\int_{t-2\epsilon f(t)}^{t+2\epsilon f(t)} v_0|H_{j,t}|^p\right)\\
&\approx f(t)^{-jp}(\epsilon f(t))^{-1}\left(\int_{t-2\epsilon f(t)}^{t+2\epsilon f(t)} |H_{j,t}|^p\right)\\
&=\int_{-2}^{2}|H_j(u)|^p,
\end{aligned}
$$

and

$$
\begin{aligned}
&(\epsilon f(t))^{-1}(v_1^{-j/n}v_0^{j/n-1})(t)\left(\int_{t-2\epsilon f(t)}^{t+2\epsilon f(t)} v_1|H_{j,t}^{(n)}|^p\right)\\
&\approx f(t)^{-jp}(v_1/v_0)(\epsilon f(t))^{-1})\left(\int_{t-2\epsilon f(t)}^{t+2\epsilon f(t)} |H_{j,t}^{(n)}|^p\right)\\
&=\int_{-2}^{2}|H_j^{(n)}(u)|^p.
\end{aligned}
$$

Putting these two estimates together shows that $S(t)$ is uniformly bounded for $t\in I_a$ and $\epsilon\in(0,\epsilon_0]$ which is equivalent to (2.20) as was to be proved. □

Our next result extends an inequality of Anderson and Hinton [1, Theorem 3.1] from $L^2(I_a)$ to the L^p case.

Corollary 2.11. *Suppose* $v_0, v_1 \in L^p_{loc}(I_a)$, $v_1^{1/2}|v_0'|\le 2v_0^{3/2}$, *and* $|v_1'|\le (K/p)\sqrt{v_0v_1}$ *then the sum inequality*

$$\|v_1'y'\|_{p,I_a}\le K(\epsilon)\|v_0y\|_{p,I_a}+\epsilon\|v_1y''\|_{p,I_a} \tag{2.24}$$

holds for all $1<p<\infty$ *and* $\epsilon>0$.

Proof. Here the weights $(v_1')^p, v_0^p$, and v_1^p replace the weights w, v_0, and v_1. The conditions on v_1' and v_0' are equivalent to (2.19) for this choice of weights. $f(t)$ is given by $(v_1^p/v_0^p)^{1/2p}\equiv\sqrt{v_1/v_0}$. We have also that

$$S_1(t)\equiv S_2(t)=\epsilon^{-1}\sqrt{\frac{v_0}{v_1}}\int_t^{t+\epsilon\sqrt{v_1/v_0}}|v_1'|^p(v_1v_0)^{-p/2}\le (K/p)^p.$$

(2.24) follows by Theorem 2.9. □

Example 2.12. Suppose $w=v_0=v_1$ and $|v_1'|\le npv_1$. Then $f=1$ and $S_1(\infty)=S_2(\infty)=1$. By Theorem 2.9 we have the inequality

$$\int_{I_a} w|y^{(j)}|^p\le K_1\left(\epsilon^{-jp}\int_{I_a}w|y|^p+\epsilon^{(n-j)p}\int_{I_a}w|y^{(n)}|^p\right). \tag{2.25}$$

In this example unlike Examples 2.7 and 2.8 since the inequality holds for all $\epsilon>0$.

Remark 2.13. If a sum inequality of the form (2.5) holds for arbitrary $\epsilon > 0$ we can minimize the right-hand side of the inequality as a function of ϵ provided $j \neq 0$ and $\int_{I_a} v_1 |y^{(n)}|^p \neq 0$. This procedure applied to (2.25) in the previous example will yield the product inequality

$$\int_{I_a} w|y^{(j)}|^p \leq K_2 \left(\int_{I_a} w|y|^p \right)^{\frac{n-j}{n}} \left(\int_{I_a} w|y^{(n)}|^p \right)^{\frac{j}{n}}.$$

Note that v_1 can be taken as $\mathrm{e}^{\pm bt}$ where $0 < b \leq np$.

Remark 2.14. So far we have supposed because of the applications we have in mind that each of the terms in our weighted sum or product inequalities have a common L^p norm. However, versions of these inequalities exist when the three norms are different. One can have, for instance, an inequality of the form

$$\int_{I_a} w|y^{(j)}|^p \leq K_1 \left\{ \epsilon^{-\mu_1} \left[\int_{I_a} v_0 |y|^s \right]^{p/s} + \epsilon^{\mu_2} \left[\int_{I_a} v_1 |y^{(m)}|^t \right]^{p/t} \right)$$

where $1 \leq p, s, t < \infty$,

$$\mu_1 = p(j + s^{-1} - p^{-1})$$
$$\mu_2 = p(n - j - t^{-1} + p^{-1}),$$

and n, j, p, s, t satisfy various relationships. Also generalizations exist in $\mathbb{R}^n$, $n > 1$. For information on these more general cases see [5], [7], [6], [9], and [11]. There are additionally other approaches to weighted norm inequalities similar to (2.5). See for example Wojteczek-Laszczak [26] and Kwong and Zettl [22].

3. Some Applications to Relative Boundedness and Limit-point conditions for differential operators in L^p spaces

In [11] we gave applications of sum and product inequalities to various spectral theoretic problems involving Sturm-Liouville operators in $L^2(I_a)$. In this section we look at applications to operators determined by expressions of Sturm-Liouville form but defined in L^p spaces. We first require some preliminary definitions and abstract results. In what follows $\|(\cdot)\|$ will denote the norm in an arbitrary Banach space.

Definition 3.1. Suppose A and T are operators from a Banach space X to a Banach space Y. Then A is said to be T bounded if the domain of T is contained in the domain of A and the inequality

$$\|A(x)\| \leq K(\|x\| + \|T(x)\|) \tag{3.1}$$

holds for all x in the domain of T. Furthermore A is said to have T bound 0 if A is T bounded and the inequality (3.1) has the form

$$\|A(x)\| \leq K(\epsilon)\|x\| + \epsilon \|T(x)\|)$$

for all $\epsilon \in (0, \epsilon_0)$ for some $\epsilon_0 \in (0, \infty)$.

Lemma 3.2. *Suppose A, B, C, and L are operators from a Banach space X to a Banach space Y. Suppose that A is L-bounded and B, C are L-bounded with L bound 0. Then A is $L + B + C$ bounded.*

Proof. By the triangle inequality

$$\|x\| + \|L(x)\| \le \|x\| + \|(L + B + C)(x)\| + \|B(x)\| + \|C(x)\|. \tag{3.2}$$

¿From the hypotheses on B and C we also have the estimates

$$\|B(x)\| \le K_1(\epsilon/2)\|x\| + (\epsilon/2)\|L(x)\| \tag{3.3}$$

$$\|C(x)\| \le K_2(\epsilon/2)\|x\| + (\epsilon/2)\|L(x)\|. \tag{3.4}$$

Substituting (3.3) and (3.4) into (3.2 gives that

$$(1 - \epsilon)(\|x\| + \|L(x)\|) \le (1 + K_1(\epsilon/2) + K_2(\epsilon/2))\|x\| + \|(L + B + C)(x)\|. \tag{3.5}$$

But since A is L bounded $\|A(x)\| \le K(\|x\| + \|L(x)\|)$. Combining this with (3.5) yields that

$$\|A(x)\| \le K(1 - \epsilon)^{-1}[(1 + K_1(\epsilon/2) + K_2(\epsilon/2))\|x\| + \|(L + B + C)(x)\|].$$

□

Lemma 3.3. *Let A, B, C and L be operators from a Banach space X to a Banach space Y. Suppose the inequalities*

$$\|A(y)\| \le K(\epsilon)\|B(y)\| + \epsilon\|L(y)\|) \tag{3.6}$$

$$\|B(y)\| \le K(\epsilon)\|C(y)\| + \epsilon\|L(y)\| \tag{3.7}$$

$$\|C(y)\| \le K(\|y\| + \|T(y)\|) \tag{3.8}$$

where $T(y) = (A + B + L)(y)$. Then A is T bounded with relative bound 0.

Proof. By (3.7) and the triangle inequality

$$\|B(y)\| \le K(\epsilon)\|C(y)\| + \epsilon\|(L + B)(y)\| + \epsilon\|B(y)\|.$$

Hence,

$$\|B(y)\| \le K(\epsilon)(1 - \epsilon)^{-1}\|C(y)\| + \epsilon(1 - \epsilon)^{-1}\|(L + B)(y)\|.$$

Substituting this into (3.6) after noting again that $\|L(y)\| \le \|(L + B)(y)\| + \|B(y)\|$ gives the inequality

$$\|A(y)\| \le K(\epsilon)(1 + \epsilon(1 - \epsilon)^{-1})\|C(y)\| + \epsilon^2(1 - \epsilon)^{-1}\|(L + B)(y)\|. \tag{3.9}$$

Finally,

$$\begin{aligned} \|(L + B)(y)\| &\le \|T(y) + \|C(y)\| \\ &\le K\|y\| + (K + 1)\|T(y)\|. \end{aligned}$$

Substitution into (3.9) now gives the desired conclusion. □

Given a Banach space X with dual X^*, $[x, x^*]$ signifies the complex conjugate $\overline{x^*(x)}$ for $x \in X$ and $x^* \in X^*$. If T is an operator on X we consider the set of pairs $G(T^*) := (z, z') \in X \times X^*$ such that

$$[T(y), z] = [y, z']. \tag{3.10}$$

The density of $\mathcal{D}(T)$ implies that (3.10) determines an operator T^* called the *adjoint* of T such that $T^*(z) = z'$. If $T : X^* \to X^*$ has a domain $\mathcal{D}(T)$ which is *total* over X (i.e., $[x, x^*] = 0$ for a fixed $x \in X$ and all $x^* \in \mathcal{D}(T) \Rightarrow x = 0$) the set $G(^*T)$ of pairs $(y', y) \in X \times X$ satisfying

$$[y', z] = [y, T(z)]$$

also determines an operator which we denote by $^*T$ and call the adjoint of T in X.[1] Both T^* and $^*T$ are closed and $[T(y), z] = [y, T^*(z)]$ or $[^*T(y), z] = [y, T(z)]$ for all $y \in \mathcal{D}(T)$ or $\mathcal{D}(^*T) \subset X$ and $z \in \mathcal{D}(T^*)$ or $\mathcal{D}(T) \subset X^*$. In the particular case $X = L^p(I_a)$ and $X^* = L^{p'}(I_a)$ where $1 \le p \le \infty$ the pairing $[(\cdot), (\cdot)]$ on $L^p(J) \times L^{p'}(J)$ for some interval J is given by $[y, z]_J := \int_J y\bar{z}$. Consider now the differential expression $M[y] := -(ry')' + qy$. Assume that $r > 0$, $r \in C^1(I_a)$, and $q \in C(I_a)$. Define

$$\{y, z\}(t) := r(t)(y(t)\bar{z}'(t) - y'(t)\bar{z}(t)).$$

for $y, z \in AC_{\mathrm{loc}}(I_a)$ and the following operators and domains in $L^p(I_a)$.

Definition 3.4. For $p \in [1, \infty]$ let let $T'_{0,p}$, T_p, $T_{0,p}$, and be the operators with domain and range in $L^p(I_a)$ determined by M on

$$\begin{aligned}
\mathcal{D}'_{0,p} &:= \{y \in C_0^\infty(I_a)\},\\
\mathcal{D}_p &:= \{y \in L^p(I_a) : y' \in AC_{\mathrm{loc}}(I_a); M[y] \in L^p(I_a)\},\\
\mathcal{D}_{0,p} &:= \{y \in \mathcal{D}_p : y(a) = y'(a) = 0; \lim_{t\to\infty} \{y, z\}(t) = 0, \forall z \in \mathcal{D}_{p'}\}.
\end{aligned}$$

We call $T'_{0,p}$, $T_{0,p}$ respectively the "preminimal" and "minimal" operators, and T_p the "maximal" operator determined by M.

Theorem B. *If $1 \le p < \infty$ the operators $T'_{0,p}$, $T_{0,p}$, and T_p have the following properties:*

(i) *$T_{0,p}$ and T_p are closed densely defined operators.*
(ii) $[T_p(y), z]_{[a,t]} = \{y, z\}(t) - \{y, z\}(a) + [y, T_{p'}(z)]_{[a,t]}$.
(iii) *$T_p^* = T_{0,p'}$ and $T_{0,p}^* = T_{p'}$.*
(iv) *$T'_{0,p}$ is closable and $\overline{T'_{0,p}} = T_{0,p}$.*

Moreover, $T_{0,\infty}$ and T_∞ are closed, the closure of $T'_{0,\infty}$ is $T_{0,\infty}$, $^*T_{0,\infty} = T_1$, and $^*T_\infty = T_{0,1}$.

Proofs of (i)–(iii), the last statement, as well as more general results may be found in one of [18, Chapter VI], [25], or [13]). The L^2 theory is thoroughly treated in Naimark [23, §17]. It is almost certain that by extending the procedure of Naimark given in the L^2 case that q, r and $r^{-p'/p}$ need only be locally integrable for Theorem B to hold. However, the verification of this is technically complicated and will be omitted here.

Definition 3.5. The operators $T_{0,p}$ or T_p are separated if on $\mathcal{D}_{0,p}$ or $\mathcal{D}_p$ $(ry')' \in L^p(I_a)$

[1] Goldberg calls this operator the preconjugate of T. Its properties are studied in [18, VI.I].

Remark 3.6. By the triangle inequality the separation of $T_{0,p}$ or T_p is equivalent to $qy \in L^p(I_a)$ on the domains of these operators. The closed graph theorem in turn implies that separation is equivalent to the existence of an inequality of the form

$$\|(ry')'\|_{p,I_a} + \|qy\|_{p,I_a} \le K\{\|y\|_{p,I_a} + \|M[y]\|_{p,I_a}\}.$$

Necessary and sufficient conditions for separation in L^p when $M[y] = -y'' + qy$ have been given by Chernyavskaya and Shuster [14]. However their conditions can be difficult to verify. For $p = 2$ various sufficient conditions may be found in [12], [10], [2], [16]. The simplest condition guaranteeing separation for all p is to require that q be essentially bounded.

Limit-point Results in L^p spaces.

Definition 3.7. We say that T_p is p-limit-point (p-LP) at ∞ if $\lim_{t\to\infty}\{y,z\}(t) = 0$ for all $y \in \mathcal{D}_p$ and $z \in \mathcal{D}_{p'}$

Theorem 3.8. *Consider the following conditions:*

(i) *T_p is p-LP.*
(ii) *There do not exist linearly independent solutions $y_p \in L^p(I_a)$ and $z_{p'} \in L^{p'}(I_a)$ of $M[y] = 0$.*
(iii) *$dim\,\mathcal{D}_p/\mathcal{D}_{0,p} = 2$.*

Then (i) and (iii) are equivalent conditions and (i) $\Rightarrow$ (ii).

Proof. Suppose that (i) is true and that there were linearly independent solutions $y_p \in L^p(I_a)$ and $z_{p'} \in L^{p'}(I_a)$. Let $t \in I_a$ By (ii) of Theorem B $\{y,z\}(t) = \{y,z\}(a)$ and the fact that T_p is p-LP we have

$$0 = \lim_{t\to\infty}\{y_p, z_{p'}\}(t) = \{y_p, z_{p'}\}(a) = r^{-1}(a)\{y_p, z_{p'}\}(a).$$

But this is impossible since $r^{-1}\{y,z\}$ is just the Wronskian of of the solutions $y_p, z_{p'}$ and its zero value at a or t contradicts their assumed linear independence. Turning now to (iii), let ϕ_1 and ϕ_2 be $C_0^\infty(I_a)$ functions such that $\phi_1(a) = 1$, $\phi_1'(a) = 0$ and $\phi_2(a) = 0$, $\phi_2'(a) = 1$. Since $\phi_1, \phi_2 \in \mathcal{D}_p$ and are linearly independent, we see that $\dim \mathcal{D}_p/\mathcal{D}_{0,p} \ge 2$. Suppose there exists $u \in \mathcal{D}_p$ such that $\{\phi_1, \phi_2, u\}$ is linearly independent mod $\mathcal{D}_{0,p}$. Let h be a linear combination of these functions such that $h(a) = h'(a) = 0$. Let G_p and $G_{0,p}$ respectively be $\mathcal{D}_p$ or $\mathcal{D}_{0,p}$ endowed with the graph norm. Now the dual G_p^* of G_p can be identified with the space of pairs $\xi = (z, z^*)$ in $L^{p'}(I_a) \times L^{p'}(I_a)$ such that $\xi(y)$, $y \in G_p$, is given by

$$\int_{I_a} y\bar{z} + \int_{I_a} M[y]\bar{z}^*.$$

Since $h \notin G_{0,p}$ and $G_{0,p}$ is closed in G_p there exists an element $\xi \in G_p^*$ such that $\xi(h) = 1$ and $\xi(y) = 0$ for all $y \in G_{0,p}$, implying that $\xi = (-M[z^*], z^*)$ for some $z \in \mathcal{D}_p$. It follows that

$$\begin{aligned} 1 = \xi(h) &= \int_{I_a} h(-\overline{M[z]} + \int_{I_a} M[h]\bar{z} \\ &= \{h, z\}(\infty), \end{aligned}$$

contradicting (i). We conclude that $\dim \mathcal{D}/\mathcal{D}_0 = 2$. In the above argument we have shown that (i)$\Rightarrow$ (ii) and that (i)$\Rightarrow$ (iii). It is clear that (iii) $\Rightarrow$ (i). For if $\mathcal{D}_p$ is a two dimensional extension of $\mathcal{D}_{0,p}$ then $\operatorname{span}\{\phi_1, \phi_2, \mathcal{D}_{0,p}\} = \mathcal{D}_p$. Since ϕ_1 and ϕ_2 have compact support and by the definition of $\mathcal{D}_{0,p}$, (i) will hold. □

Remark 3.9. For $p \neq 2$ Theorem 3.8 represents an extension of the well-known limit point concept for differential operators in $L^2(I_a)$. In particular (ii) is a generalization of the fact that if M is 2-LP at ∞ then M has at most one L^2 solution. In the limiting case $p = 1$, $p' = \infty$, if there is a solution y in $\mathcal{D}_1$ in $L(I_a)$, (ii) says that that there cannot be another solution independent of y which is bounded. As in the Hilbert space case we have also shown that T_p is p-limit-point if and only if $\mathcal{D}_p/\mathcal{D}_{0,p} = 2$.

Remark 3.10. If any one of $T_{0,p}$, T_p, $T_{0,p'}$ and $T_{p'}$ has closed range then more can be said. Specifically all the other minimal and maximal operators also have closed range. In particular, the minimal operators are one-to-one and have bounded inverses while the maximal operators are surjective and

$$\dim(\mathcal{D}_p/\mathcal{D}_{0,p}) = \dim \mathcal{N}(T_p) + \dim \mathcal{N}(T_{p'}).$$

For the proof in a considerably more general setting see Goldberg [18] Theorems VI.2.7 and VI.2.11. Furthermore, since the dimensions of both null spaces do not exceed 2 and since as we have seen $\mathcal{D}_p$ is at least a two dimensional extension of $\mathcal{D}_{0,p}$ we have that

$$2 \leq \dim(\mathcal{D}_p/\mathcal{D}_{0,p}) \leq 4.$$

Brown and Cook [13, Corollary 2.9] have shown that $T_{0,p}$ defined on I_a has a bounded inverse and thus closed range for $1 \leq p \leq \infty$ if both $\int_{I_a} r^{-1} < \infty$ and $\int_{I_a} q^{-1} < \infty$

Remark 3.11. If $q \geq 0$ then M is disconjugate and by by Corollary 6.4 and Theorem 6.4 of Hartman [20] there is a fundamental set of of positive linearly independent solutions y_1 and y_2 of $M[y] = 0$, called respectively the principal and nonprincipal solutions, such that $y_1' \leq 0$ and $y' > 0$ on I_a. Additionally, $\lim_{t\to\infty} y_1/y_2 = 0$. It follows at once in our setting that $\dim \mathcal{N}(T_p) = \dim \mathcal{N}(T_{p'}) \leq 1$. In [3] it is furthermore shown that if $r = 1$, $q \geq 0$, and there exists $b \in (0, \infty)$ such that

$$\inf_{x\in I_a, x-b>a} \int_{x-b}^{x+b} q > 0, \tag{3.11}$$

then

(i) T_p has closed range.
(ii) $\dim(\mathcal{D}_p/\mathcal{D}_{0,p}) = 2$ and $\dim(R(T_p)/R(T_{0,p})) = 1$,
(iii) $\dim \mathcal{N}(T_p) = 1$,
(iv) If y_1 denotes the principal or "small" solution of $M[y] = 0$ then $y_1 \in L^p(I_a)$ for all $p \in [1, \infty]$.

The condition (3.11) was shown by Chernyavskya and Shuster [15] to be necessary and sufficient for T_p defined on $\mathbb{R}$ to have a bounded inverse. In the case when $q \geq k > 0$ one can show that $M[y] = 0$ has exponentially growing and exponentially decaying solutions. Read [24] has extended this by showing that the same is true if

$$\liminf_{x\to\infty} \int_x^{x+a} q^{1/2}\,dt > 0$$

for some $a > 0$.

Relative Boundedness for perturbations of Differential Operators in L^p spaces.

The following two results are generalizations to L^p of relative boundedness criteria given by Anderson and Hinton [1] in the L^2 setting.

Theorem 3.12. *Let $A_j : L^p \to L^p$ be given by $\int_{I_a} a_j y^{(j)}$, $j = 0, 1$, on $\mathcal{D}_p$ where the a_j are locally integrable functions. Let $T_{0,p}$ be the minimal operator in $L^p(I_a)$ determined by $M[y] = -(ry')' + qy$ and Assume that $|r'| \le (K/p)\sqrt{r}$ and that*

$$\sup_{t\in I_a} \frac{1}{\sqrt{r}} \int_t^{t+\epsilon\sqrt{r}} |q|^p < \infty. \tag{3.12}$$

Then the A_j are $T_{0,p}$-bounded if and only if

$$\frac{1}{\sqrt{r}} \int_t^{t+\epsilon\sqrt{r}} |a_j|^p r^{-jp/2} < \infty, \quad j = 0, 1, \tag{3.13}$$

is bounded on I_a. If T_p is p-LP at ∞ then the A_j are also T_p bounded.

Proof. This is an application of Theorem 2.9 and Lemma 3.2. Define the operators $L, B, C : L^p(I_a) \to L^p(I_a)$ on $C_0^\infty(I_a)$ by $L(y) = ry''$, $B(y) = r'y'$, and $C(y) = qy$ where $y \in \mathcal{D}_{0,p}$. Then $T'_{0,r} = L + B + C$. Let $f = \sqrt{r}$, and take $v_0 = 1$, The condition on r' is just the first condition in (2.19) with r^p replacing v_1. By Theorem 2.9 the A_j are L-bounded or equivalently the sum inequalities

$$\|A_j(y)\|_{p,I_a} \le K(\|y\|_{p,I_a} + \|ry''\|_{p,I_a}), \quad j = 0, 1,$$

hold if and only if (3.13) is true. By Corollary 2.11 (with $v_0 = 1$) B is L-bounded with relative bound 0. Finally, another application of Theorem 2.9 using (3.12) gives that C is L-bounded with relative bound 0. The fact that the A_j are $T'_{0,p}$ bounded now now follows from Lemma 3.2. A closure argument shows that the A_j are $T_{0,p}$ bounded. Finally, if T_p is p-LP at ∞, then T_p is a two dimensional extension of $T_{0,p}$ via the functions ϕ_1 and ϕ_2. Hence, if $y \in \mathcal{D}_p$, $y = y_0 + z$ where $z = c_1\phi_1 + c_2\phi_2$. Since A_j is T_0 bounded and by an elementary inequality we have

$$\|A_j(y_0)\|_{p,I_a}^p \le K^p 2^{p-1}\{\|y_0\|_{p,I_a}^p + \|T(y_0)\|_{p,I_a}^p\}.$$

The same inequality is true for z since the p-th root of each side defines two norms on $Z := \operatorname{span}\{\phi_1, \phi_2\}$ and the mapping $j : Z \to Z$ given by $j(z) = z$ is continuous with respect to these norms since Z is finite (2!) dimensional. Hence,

$$\begin{aligned}
\|A_j(y_0 + z)\|_{p,I_a}^p &\le 2^{p-1}[\|A_j(y_0)\|_{p,I_a}^p + A_j(z)\|_{p,I_a}^p \\
&\le K_1[\|y_0\|_{p,I_a}^p + \|z\|_{p,I_a}^p + \|T(y_0)\|_{p,I_a}^p + \|T(z)\|_{p,I_a}^p] \\
&\le K_1[\|y_0 + z\|_{p,I_a}^p + \|T(y_0 + z)\|_{p,I_a}^p] \\
&= K_1\{\|y\|_{p,I_a}^p + \|T(y)\|_{p,I_a}^p\}.
\end{aligned}$$

□

Theorem 3.13. *Suppose that $T_{0,p}$ is separated Additionally suppose that*

$$|r'| \leq (K/p)\sqrt{r|q|}$$
$$|q'| \leq 2|q|^{3/2}r^{-1/2}.$$

Let A_j, $j = 0, 1$, be as in Theorem 3.12. Then A_j is $T_{0,p}$-bounded with T_p-bound 0 if and only if

$$S(t) = \sqrt{\frac{|q|}{r}} \int_t^{t+\epsilon\sqrt{r/q}} |a_j|^p r^{-jp/2} |q|^{p(j-2)/2} \tag{3.14}$$

is bounded on I_a.

Proof. As before we begin with the C_0^∞ functions. Let $C(y) = qy$, $B(y) = r'y'$ and $L(y) = ry''$. By the hypothesis of separation (3.8) holds. By Theorem 2.9 where $f(t) = (r^p/q^p)^{1/2p} = \sqrt{r/q}$ (3.6) holds if and only if (3.14) is true. By Corollary 2.11 (3.7) is true. The conclusion that A_j is $T'_{0,p}$ bounded follows from Lemma 3.3. □

References

1. T. G. Anderson and D. B. Hinton, *Relative boundedness and compactness for second order differential operators*, J. Ineq. and Appl. **1** (4) (1997), 375–400.
2. R. C. Brown, *Separation and discongugacy*, JPAM **4** (3), Article 56, 2002 (Electronic).
3. ______, *A limit-point criterion for a class of Sturm-Liouville operators defined in L^p spaces*, Proc. Am. Math. Soc **132** (2004), 2273–2280.
4. R. C. Brown and D. B. Hinton, *Sufficient conditions for weighted inequalities of sum form*, J. Math. Anal. Appl. **112** (1985), 563–578.
5. ______, *Sufficient conditions for weighted Gabushin inequalities*, Casopis Pěst. Mat. **111** (1986), 113–122.
6. ______, *Weighted interpolation inequalities of sum and product form in R^n*, J. London Math Soc. (3) **56** (1988), 261–280.
7. ______, *Weighted interpolation inequalities and embeddings in R^n*, Canad. J. Math. **47** (1990), 959–980.
8. ______, *A Weighted Hardy's inequality and nonoscillatory differential equations*, Quaes. Mathematicae **115** (1992), 197–212.
9. ______, *An interpolation inequality and applications*, Inequalities and Applications (R. P. Agarwal, ed.), World Scientific, Singapore-New Jersey-London-Hong Kong, 1994, 87–101.
10. ______, *Two separation criteria for second order ordinary or partial differential operators*, Math. Bohem. **124** (1999), 273–292.
11. ______, *Some One Variable Weighted Norm Inequalities and Their Applications to Sturm-Liouville and other Differential Operators*, to appear in Inequalities and Applications, (C. Bandle, A. Gilányi, L. Losonczi, Z. Pales, M. Plum, eds.), Vol. 157, International Series of Numerical Mathematics, Birkhäuser, Boston, Basel, Stuttgart, 2008.
12. R. C. Brown, D. B. Hinton, and M. F. Shaw, *Some separation criteria and inequalities associated with linear second order differential operators*, in Function Spaces and Aplications (D. E. Edmunds, et al., eds.), Narosa Publishing House, New Delhi, 2000, 7–35.
13. R. C. Brown and J. Cook, *Continuous invertibility of minimal Sturm-Liouville operators in Lebesgue spaces*, Proc. Roy. Soc. Edinburgh. A **136** (01) (2006), 53–70.
14. N. Chernyavskaya and L. Schuster, *Weight summability of solutions of the Sturm-Liouville equation*, J. Diff. Eqs. **151** (1999), 456–473.

15. ———, *A criterion for correct solvability of the Sturm-Liouville equation in the space $L_p(R)$*, Proc. Amer. Math. Soc. **130**(4) (2001), 1043–1054.
16. W. N. Everitt and M. Giertz, *Some properties of the domains of certain differential operators*, Proc. London Math. Soc. (3) **23** (1971), 301–324.
17. ———, M. Giertz, and J. Weidmann, *Some remarks on a separation and limit-point criterion of second-order ordinary differential expressions*, Math. Ann. **200** (1973), 335–346.
18. S. Goldberg, *Unbounded Linear Operators Theory and Applications*, McGraw-Hill Series in Higher Mathematics (E. H. Spanier, ed.), McGraw-Hill Book Company, New York, St. Louis, San Francisco, Toronto, London, Sydney, 1966.
19. M. De Guzman, "Differentiation of Integrals in $\mathbb{R}^n$" (Lecture Notes in Mathematics 481), Springer-Verlag, Berlin, 1975.
20. P. Hartman, *Ordinary Differential Equations*, Second Edition, Birkhäuser, Boston, Basel, Stuttgart, 1982.
21. E. Hewitt and K. Stromberg, *Real and Abstract Analysis*, Springer-Verlag, New York Heidelberg, Berlin, 1969.
22. M. K. Kwong and A. Zettl, *Norm Inequalities for Derivatives and Differences*, Lecture Notes in Mathematics 1536, Springer-Verlag, Berlin, 1992.
23. M. A. Naimark, *Linear Differential Operators, Part II*, Frederick Ungar, New York, 1968.
24. T. T. Read, *Exponential solutions of $y'' + (r - q)y = 0$ and the least eigenvalues of Hill's equation* Proc. Amer. Math. Soc. **50** (1975), 273–280.
25. G. Rota, *Extension theory of differential operators I*, Comm. in Pure and Applied Math. **13** (1958), 23–65.
26. K. Wojteczek-Laszczak, *On quadratic integral inequalities of the second order*, J. Math. Anal. Appl. **342** (2) (2008), 1356–1352.

Department of Mathematics, University of Alabama-Tuscaloosa, AL 35487-0350, USA
E-mail address: dicbrown@bama.ua.edu