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INNER SPECTRAL RADIUS OF POSITIVE OPERATOR MATRICES

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ABSTRACT. In this paper we give more results about inner radius spectrum of operators on Hilbert spaces with several examples. Also, we established an inequality for inner radius spectrum of a positive operator matrix and its minimum moduli block matrix.

1. INTRODUCTION AND PRELIMINARIES

Let H be a complex Hilbert space and $B(H)$ denote the algebra of all bounded linear operators on H. For operator $a \in B(H)$, let $m(a)$, $\sigma(a)$, $W(a)$, $r(a)$, $w(a)$, $i(a)$, and $w_i(a)$ denote the minimum moduli, spectrum, numerical range, spectral radius, numerical radius, inner spectral radius, and inner numerical radius of a, respectively.

$$\begin{gathered}
m(a) = \inf\{\|ax\|,\ x \in H,\ \text{and}\ \|x\| = 1\ \},\\
\sigma(a) = \{\lambda \in \mathbb{C} \,:\, a - \lambda I \text{ is not invertible }\},\\
W(a) = \{\langle x, ax\rangle,\ \|x\| = 1\ \},\\
r(a) = \sup\{|\lambda| \,:\, \lambda \in \sigma(A)\ \},\\
w(a) = \sup\{|\langle x, ax\rangle|,\ \|x\| = 1\ \},\\
i(a) = \inf\{|\lambda| \,:\, \lambda \in \sigma(a)\ \},\\
w_i(a) = \inf\{|\langle x, ax\rangle|,\ \|x\| = 1\ \}.
\end{gathered}$$

Also, by definitions of $m(a)$ and $i(a)$ it is clear that; if A is not invertible in $B(H)$, then $m(a) = i(a) = 0$, and if a is invertible, then $m(a) = \|a^{-1}\|^{-1}$ and

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$i(a) = r(a^{-1})^{-1}$, where $r(a)$ is the spectral radius of A. It is well known that for every $a \in B(H)$, we have

$$i(a) \geq m(a) \tag{1.1}$$

In addition to the inequality (1.1), the most important properties of the inner spectral radius are the inner spectral radius formula

$$i(a) = \lim_{n\to\infty} (m(a^n))^{\frac{1}{n}},$$

if and only if

$$\lim_{n\to\infty} (m(a^n))^{\frac{1}{n}} \leq \lim_{n\to\infty} (m((a^*)^n))^{\frac{1}{n}}.$$

Also, a special of the spectral mapping theorem, which assert that

$$i(a^n) = (i(a))^n, \text{ for every positive integer } n,$$

and, if a is normal, then

$$i(a) = m(a) = w_i(a).$$

For the proof of above inequalities and additional properties of inner spectral radius, the reader is referred to [6].

It follows easily from the Theorem 1.3.4 [7] that if $a,\ b \in B(H)$ are such that $ab = ba$, then

$$r(a+b) \leq r(a) + r(b),$$

and

$$r(ab) \leq r(a)r(b).$$

The following examples show that the inner spectral radius is neither subadditive nor submultiplicative.

Example 1.1. Suppose that

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}.$$

Then $i(A) = i(B) = 0$, but $i(A+B) = \sqrt{2} - 1$. In this example $AB \neq BA$.

Example 1.2. Suppose that

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \qquad B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then $i(A) = 2$ and $i(B) = 1$, but $i(A+B) = 1 \leq i(A) + i(B)$. In this example $AB = BA$.

Example 1.3. Suppose that

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

Then A and B are positive as operators on $\mathbb{C}^2$ with $BA = AB$, and $i(A+B) = 3 \geq 1 + 0 = i(A) + i(B)$.

As a result of Theorem 1.3.4 in [7] we have the following corollary.

Corollary 1.4. *If $a, b \in B(H)$ are positive operators such that $ab = ba$, then*

$$r(a+b) \geq r(a) + r(b)\,.$$

Example 1.5. Let

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

Then $AB \neq BA$ and $i(AB) = \sqrt{2} - 1 \leq \frac{\sqrt{5}-1}{2} = i(B)\; i(A)\,.$

Example 1.6. Let

$$C = \begin{pmatrix} 3 & 2 \\ 0 & 2 \end{pmatrix}, \qquad D = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

Then $CD \neq DC$ and $i(CD) = 2\sqrt{3} \geq 2\sqrt{2} = i(C)\; i(D)$.

However, If $a,\ b \in B(H)$ are such that $ab = ba$, then by Theorem 1.3.4 in [7] we have

$$i(ab) \geq i(a) i(b).$$

2. Inner spectral radius of block-minimum moduli matrices

In this section we try to introduce block-minimum moduli matrices associated with an operator matrix $A = (a_{ij})_{n\times n}$. Also we will give an open problem about minimum moduli and inner spectral radius of the block-minimum moduli matrix associated with an operator matrix. Let H_i, $i = 1, \cdots, n$, be complex Hilbert spaces with inner product $\langle \cdot\ , \cdot \rangle$. As usual $B(H_i, H_j)$ is the Banach space of all bounded linear operators from H_i to H_j with operator norm topology. For $H = \bigoplus_{i=1}^{n} H_i$, and $A \in B(H)$, the operator A can be represented as an $n \times n$ matrix, that is $A = (a_{ij})_{n\times n}$ with $a_{ij} \in B(H_j, H_i)$. The block-norm and block-minimum moduli matrices associated with an operator matrix $A = (a_{ij})_{n\times n}$ are defined respectively by $\tilde{A} = (\|a_{ij}\|)_{n\times n}$ and $\hat{A} = \left(m(a_{ij})\right)_{n\times n}$. Note that, $\tilde{A}$ and $\hat{A}$ are nonnegative matrices. Recall that an $n \times n$ complex matrix $T = (t_{ij})_{n\times n}$ is said to be a nonnegative matrix if each entry t_{ij} is a nonnegative number. An operator $a \in B(H)$ is called positive and denoted by $a \geq 0$ if $\langle ax, x\rangle \geq 0$ for all $x \in H$.

In recent years, a number of researchers have considered questions concerning the spectral radius of an operator matrix A and its block-norm $\tilde{A}$ (see for example,[1, 2, 4]).Jin-Chuan Hou and Hong-Ke Du proved the following theorem [2]. In what follows, $M_n(B(H))$ shall denote the algebra of all $n \times n$ matrices with entries in $B(H)$.

Theorem 2.1. *Let $A = (A_{ij})_{n\times n} \in M_n(B(H))$ and $\tilde{A} = (\|A_{ij}\|)_{n\times n}$ be its block-norm matrix. Then*

(1) $\|A\| \leq \|\tilde{A}\|$.
(2) $w(A) \leq w(\tilde{A})$.
(3) $r(A) \leq r(\tilde{A})$.

As an application of this theorem in [4] F. Kittaneh proved the following theorems.

Theorem 2.2. *If $a_1, a_2, b_1, b_2 \in B(H)$, then*

$$r(a_1b_1 + a_2b_2) \leq \frac{1}{2}(\|b_1a_1\| + \|b_2a_2\|) + \sqrt{(\|b_1a_1\| + \|b_2a_2\|)^2 + 4\|b_1a_2\|\|b_2a_1\|}\,.$$

For any vector $x = (x_1, \cdots, x_n)^T, x_i \in H$, we write $|x| = (\|x_i\|, \cdots, \|x_n\|)^T$. Then $|x|$ is a unit vector in the Hilbert space $\mathbb{C}^n$ if x is a unit vector in H. The proof of Theorem 2.1 is based on the norm monotonicity of nonnegative matrices and the following equations.

$$\sup_{\|x\|=1} \langle \tilde{A}|x|, |x|\,\rangle \;\leq\; w(\tilde{A}), \qquad \sup_{\|x\|=1} \langle \tilde{A}^*\tilde{A}|x|, |x|\,\rangle \;=\; \|\tilde{A}\|\,.$$

Proposition 2.3. *The minimum moduli has monotone property that is;*

(1) *If $A,\ B \in B(H)$ such that $0 \leq A \leq B$, then $m(A) \leq m(B)$.*
(2) *If A and B are two positive semidefinite matrices such that $B - A$ is also positive semidefinite, then $m(A) \leq m(B)$.*

Proof. If $m(A) = 0$, then the result is clear. Let $m(A) \neq 0$. Then A is invertible. By Proposition 4.2.8 [3] we get B is invertible and $\|B^{-1}\| \leq \|A^{-1}\|$. Thus

$$m(A) \;=\; \|A^{-1}\|^{-1} \;\leq\; \|B^{-1}\|^{-1} \;=\; m(B).$$

□

The following example shows that we can not replace positive semidefinte with nonnegative in statement (2) in the above proposition.

Example 2.4. Suppose that

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

It is clear that $B - A$ is a nonnegative matrix but $m(A) = i(A) = w_i(A) = 1$ and $m(B) = i(B) = w_i(B) = 0$.

This example also shows that the monotonicity property does not hold for spectral inner radius and numerical inner radius.

Proposition 2.5. *Let $A = (a_{ij}I)_{n\times n} \in M_n(B(H))$, where I is the identity element in $B(H)$ and $\tilde{A} = (a_{ij})_{n\times n} \in M_n(\mathbb{R})$ be a nonnegative matrix. Then*

(1) *$w_i(A) \leq w_i(\tilde{A})$*
(2) *$m(A) \leq m(\tilde{A})$.*
(3) *$i(A) \leq i(\tilde{A})$.*

Proof. **(1).** Let e_0 be a unit vector in the Hilbert space H. For every unit vector $X = (x_1, \cdots, x_n)^T \in \mathbb{C}^n$ we write $y = (x_1e_0, \cdots, x_ne_0) \in H^n$. We have

$$\langle \tilde{A}X, X\rangle \;=\; \sum_{i,j=1}^{n} a_{ij}x_j\bar{x}_i \;=\; \sum_{i,j=1}^{n} \langle a_{ij}Iy_j, y_i\rangle \;=\; \langle Ay, y\rangle\,.$$

Thus,

$$w_i(\tilde{A}) = \inf_{\|X\|=1} |\langle \tilde{A}X, X\rangle| = \inf_{\|y\|=1} |\langle Ay, y\rangle| \geq w_i(A)\,.$$

(2). Let X and y be as in part (1).

$$\begin{aligned} m(\tilde{A})^2 &= \inf_{\|x\|=1} \|\tilde{A}X\|^2 = \inf_{\|X\|=1} \langle \tilde{A}^*\tilde{A}X, X\rangle \\ &= \inf_{\|x\|=1} |\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} a_{kj}x_j a_{ki}\bar{x}_i| \\ &= \inf_{\|x\|=1} |\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} \langle a_{kj}y_j, a_{ki}y_i\rangle| \\ &= \inf_{\|y\|=1} \langle A^*Ay, y\rangle = \inf_{\|y\|=1} \|Ay\|^2 \\ &\geq m(A)^2. \end{aligned}$$

So $m(A) \leq m(\tilde{A})$.

(3). Notice that for operators matrices $A = (a_{ij}I)_{n\times n}$, $B = (b_{ij}I)_{n\times n}$ in $M_n(H)$ and nonnegative matrices $\tilde{A} = (a_{ij})_{n\times n}$, $\tilde{B} = (b_{ij})_{n\times n}$ in $M_n(\mathbb{R})$ we have

$$m(\widetilde{AB}) = m(\tilde{A}\tilde{B})\,.$$

Using induction, we have

$$m(A^n) = m(\widetilde{A^n}) = m((\tilde{A})^n),$$

for every positive integer n. Also,

$$i(A) \leq \lim_{n\to\infty} (m(A^n))^{\frac{1}{n}} = \lim_{n\to\infty} \left(m(\widetilde{A^n})\right)^{\frac{1}{n}} = i(\tilde{A}).$$

(The first inequality above is result of Equation (3.1) in [6] and the last equality is the Fact 3.1 in [6] about properties of the inner spectral radius.) □

The following examples show that in general neither $m(A) \leq m(\tilde{A})$ $(i(A) \leq i(\tilde{A}))$ nor $m(A) \geq m(\tilde{A})$ $(i(A) \geq i(\tilde{A}))$.

Example 2.6. Let $a = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, and $b = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Consider the 2×2 operator matrix

$$A = \begin{pmatrix} a & b \\ b & a \end{pmatrix}.$$

We have $m(A) = i(A) = \sqrt{3} - 1$, but $m(\tilde{A}) = i(\tilde{A}) = 0$. Thus, $m(A) = i(A) \geq m(\tilde{A}) = i(\tilde{A})$.

Example 2.7. Let $a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, and $b = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Consider the 2×2 operator matrix

$$A = \begin{pmatrix} a & b \\ b & a \end{pmatrix}.$$

Since A is a positive operators, we have $m(A) = i(A) = 0 \leq m(\tilde{A}) = i(\tilde{A}) = 1$.

As we see in Example 2.6, $m(\hat{A}) = i(\hat{A}) = 1 \geq m(A) = i(A) = \sqrt{3} - 1$ and in Example 2.7, $m(\hat{A}) = i(\hat{A}) = 0 = m(A) = i(A)$. Therefore in the above examples we have $m(\hat{A}) \geq m(A)$ and $i(\hat{A}) \geq i(A)$. But for arbitrary operator matrix A still we do not know what is the relation between $m(A)$ and $m(\hat{A})$ ($i(A)$ and $i(\hat{A})$)?

At the end of this section we give an inequality for positive operator matrices. Let $A = (a_{ij})_{n\times n}$ be a positive operator matrix. Then a_{ii} is positive for every $i = 1, \cdots, n$ and $a_{ij} = a_{ji}^*$ when $i \neq j$.

Theorem 2.8. *If $A = (a_{ij})_{n\times n}$ is a positive operator matrix, then*

$$m(A) \leq \min_{1\leq i\leq n} \{m(a_{ii})\} .$$

Proof. Let x be unit vectors in H. Then, consider unit vector $X_i = (x_1, \cdots, x_n)^T \in H^n$, where $x_i = x$ and $x_j = 0$ for $j \neq i$.

$$\langle AX_i \ , \ X_i\rangle = \langle a_{ii}x, x\rangle .$$

Since A and a_{ii} are positive operators for every $1 \leq i \leq n$,

$$m(a_{ii}) = \inf_{\|x\|=1} \langle a_{ii}x, x\rangle = \inf_{\|X_i\|=1} \langle AX_i \ , \ X_i\rangle \geq \inf_{\|X\|=1} \langle AX \ , \ X\rangle = m(A) .$$

Thus,

$$i(A) = w_i(A) = m(A) \leq \min\left(m(a_{ii}) = w_i(a_{ii}) = i(a_{ii}), m(a_{ii}) = w_i(a_{ii}) = i(a_{ii})\right) .$$

$\square$

Remark 2.9. In Example 2.6; A is not positive because one of its eigenvalues is -1 and we have $m(A) = 1 \geq \min(m(a) = 0 \ , \ m(b) = 1)$. Therefore, Theorem 2.8 does not hold for arbitrary operator matrix. If $A = (a_{ij}) \geq 0$, with $a_{ij} = 0$ whenever $i \neq j$, then $m(A) = \min_{1\leq i\leq n}\{m(a_{ii})\}$. If $A = (a_{ij})_{n\times n}$ is a positive semidefinite matrix with $a_{ii} \neq 0$ and $a_{ij} = \sqrt{a_{ii}a_{jj}}$ for every $1 \leq i, j \leq n$, then $m(A) = 0$ which is strictly less than $\min_{1\leq i\leq n}\{m(a_{ii})\}$.

3. Positive operator matrices on C*-Algebras

In this section we show that the last results hold for a positive operator matrix on an arbitrary C*-algebra by GNS construction defined as for scalar matrices. If $\mathcal{A}$ is an algebra, then $M_n(\mathcal{A})$ denotes the algebra of all $n \times n$ matrices with entries in $\mathcal{A}$. The operations on $M_n(\mathcal{A})$ are define just as for scalar matrices. If $\mathcal{A}$ is a $*$-algebra, so is $M_n(\mathcal{A})$, where the involution is given by $(a_{ij})^*_{n\times n} = (a^*_{ij})_{n\times n}$. If $\varphi : \mathcal{A} \to \mathcal{B}$ is a $*$-homomorphism between $*$-algebras, then

$$\varphi : M_n(\mathcal{A}) :\to \ M_n(\mathcal{B}), \ (a_{ij})_{n\times n} \ \to \ (\varphi(a_{ij}))_{n\times n}$$

is $*$-homomorphism and also denoted by φ.

A representation of a C*-algebra $\mathcal{A}$ is a pair (H, φ) where H is a Hilbert space and $\varphi : \mathcal{A} \to \ B(H)$ is a $*$-homomorphism. We say (H, φ) is faithful if φ is injective. Recall that if $\varphi : \mathcal{A} \to \mathcal{B}$ is an injective $*$-homomorphism between C*-algebras $\mathcal{A}$ and $\mathcal{B}$, then φ is necessarily isometric [7, Theorem 3.2.7]. Let (H, φ) be the universal representation (in GNS construction). Then, for element

$a \in \mathcal{A}$ we define $W(a) = \{\langle \varphi(a)x, y\rangle,\ x, y \in H\}$. By the GNS representation of a C*-algebra $\mathcal{A}$ in [7] the following theorem and inequalities were established.

Theorem 3.1. *If $\mathcal{A}$ is a C*-algebra, then there is a unique norm on $M_n(\mathcal{A})$ making it a C*-algebra.*

If $\mathcal{A}$ is a C*-algebra and $A \in M_n(\mathcal{A})$, then

$$\|a_{ij}\| \leq \|A\| \leq \sum_{k,l=1}^{n} \|a_{kl}\|,\ i, j = 1, \cdots, n.$$

As far as C*-algebras are concerned, the results to this point lead to the following theorem about positive operator matrices on C*-algebras.

Theorem 3.2. *Let $A = (a_{ij})_{n\times n} \in M_n(\mathcal{A})$ be an operator matrix and $\tilde{A} = (\|a_{ij}\|)_{n\times n}$ its block-norm matrix, where $\mathcal{A}$ is a C*-algebra. Then*
(1) $\|A\| \leq \|\tilde{A}\|$.
(2) $r(A) \leq r(\tilde{A})$.

Theorem 3.3. *If $a_1, a_2, b_1, b_2 \in \mathcal{A}$, then*

$$r(a_1b_1 + a_2b_2) \leq \frac{1}{2}(\|b_1a_1\| + \|b_2a_2\|) + \sqrt{(\|b_1a_1\| + \|b_2a_2\|)^2 + 4\|b_1a_2\|\|b_2a_1\|}\,.$$

Theorem 3.4. *If $A = (a_{ij})_{n\times n}$ is a positive operator matrix on the C*-algebra $\mathcal{A}$, then*

$$m(A) \leq \min_{1\leq i\leq n}\{m(a_{ii})\}\,.$$

These inequalities follow from corresponding inequalities in $M_n(B(H))$ and the unique norm on $M_n(\mathcal{A})$ is the norm induced by the norm defined above on the corresponding $M_n(B(H))$.

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References

1. J.C. Hou and M.C. Gao, *Positive matrices of operators* (Chinese), J. Systems Sci. Math. Sci., **14** (1994), no. 3, 252–267.
2. J.C. Hou and H.K. Du, *Norm inequalities of positive operator matrices*, Integral Equations Operator Theory, **22** (1995), no. 3, 281–294.
3. R.V. Kadison and J.R. Ringrose, *Fundamentals of the Theory of Operator Algebras, Volume I*, Academic Press, New York, 1986.
4. F. Kittaneh, *Spectral radius inequalities for Hilbert space operators*, Proc. Amer. Math. Soc., **134** (2006), no. 2, 385–390.
5. M. Mathias, *The Hadamard operator norm of a circulant and applications, SIAM J. Matrix Anal.*, **14** (1993), 1152-1167.
6. S.M. Manjegani, *A formula for the inner spectral radius*, Int. J. Math. Math. Sci., **2004** (2004), no. 61-64, 3285–3290.

7. G.J. Murphy, *C^*-Algebra and Operator Theory*, Academic Press, San Diego, 1990.

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