Banach J. Math. Anal. 1 (2007), no. 1, 91–100

BANACH JOURNAL OF MATHEMATICAL ANALYSIS
ISSN: 1735-8787 (electronic)
http://www.math-analysis.org

STABILITY OF A JENSEN TYPE FUNCTIONAL EQUATION

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This paper is dedicated to Professor Themistocles M. Rassias.

Submitted by C. Park

ABSTRACT. In this paper we consider the general solution of a Jensen type functional equation. Moreover we prove the stability theorem of this equation in the spirit of Hyers, Ulam, Rassias and Găvruţa.

1. INTRODUCTION

In 1940, S.M. Ulam [20] raised a question concerning the stability of group homomorphisms:

> *Let f be a mapping from a group G_1 to a metric group G_2 with metric $d(\cdot,\cdot)$ such that*
> $$d(f(xy), f(x)f(y)) \le \varepsilon.$$
> *Then does there exist a group homomorphism $L : G_1 \to G_2$ and $\delta_\epsilon > 0$ such that*
> $$d(f(x), L(x)) \le \delta_\epsilon$$
> *for all $x \in G_1$?*

This problem was solved affirmatively by D.H. Hyers [3] under the assumption that G_2 is a Banach space.

Date: Received: 2 January 2007; Accepted: 17 October 2007.

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2000 *Mathematics Subject Classification.* Primary 39B82; Secondary 39B52.

Key words and phrases. Hyers–Ulam–Rassias stability, Jensen functional equation.

In 1978, Th.M. Rassias [12] firstly generalized Hyers's result to the unbounded Cauchy difference. The terminology *Hyers–Ulam–Rassias stability* originates from these historical backgrounds. Usually the functional equation

$$E_1(\phi) = E_2(\phi) \tag{1.1}$$

has the Hyers–Ulam–Rassias stability if for an approximate solution ϕ_s satisfying

$$d(E_1(\phi_s), E_2(\phi_s)) \le \psi(x)$$

for some fixed function $\psi(x)$ there exists a solution ϕ of equation (1.1) such that

$$d(\phi, \phi_s) \le \Psi(x)$$

for some fixed function $\Psi(x)$.

Also P. Găvruţa [2] obtained further generalization of the Hyers–Ulam–Rassias stability. Since then, stability problems concerning the various functional equations have been extensively investigated by numerous authors. We refer to [1, 4, 5, 7, 8, 10, 11, 13, 14, 15, 16, 17, 18] for more interesting results in connection with stability problems of functional equations.

T. Trif [19] solved a Jensen type functional equation

$$\begin{aligned} 3f\left(\frac{x+y+z}{3}\right) &+ f(x) + f(y) + f(z) \\ &= 2\left[f\left(\frac{x+y}{2}\right) + f\left(\frac{y+z}{2}\right) + f\left(\frac{z+x}{2}\right)\right] \end{aligned}$$

and investigated the Hyers–Ulam–Rassias stability of this equation.

In this paper we consider the following Jensen type functional equation

$$\begin{aligned} mf\left(\frac{x+y+z}{m}\right) &+ f(x) + f(y) + f(z) \\ &= n\left[f\left(\frac{x+y}{n}\right) + f\left(\frac{y+z}{n}\right) + f\left(\frac{z+x}{n}\right)\right], \end{aligned} \tag{1.2}$$

where m and n are nonnegative integers with $(m, n) \neq (1, 1)$. Moreover we prove the stability theorem concerning equation (1.2) in the spirit of Hyers, Ulam, Rassias and Găvruţa.

2. Solution of equation(1.2)

Now we consider the general solution of the Jensen type functional equation. Throughout this section X and Y will be real vector spaces.

Lemma 2.1. *Let m and n be nonnegative integers. A function $f : X \to Y$ satisfies equation* (1.2) *and $f(-x) = -f(x)$ for all $x, y, z \in X$ if and only if f is additive.*

Proof. (Sufficiency) This is obvious.
(Necessity) Putting $y = z = 0$ in (1.2) we have

$$mf\left(\frac{x}{m}\right) + f(x) = 2nf\left(\frac{x}{n}\right). \tag{2.1}$$

Letting $z = -y$ in (1.2) we get

$$mf\left(\frac{x}{m}\right) + f(x) = nf\left(\frac{x+y}{n}\right) + nf\left(\frac{x-y}{n}\right). \tag{2.2}$$

It follows from (2.1) and (2.2) that we obtain

$$2f(x) = f(x+y) + f(x-y). \tag{2.3}$$

Putting $y = x$ in (2.3) yields

$$2f(x) = f(2x). \tag{2.4}$$

Replacing $x = \frac{x+y}{2}, y = \frac{x-y}{2}$ in (2.3) and using (2.4) we have

$$f(x+y) = f(x) + f(y)$$

for all $x, y \in X$. Therefore f is additive. □

Lemma 2.2. *Let m and n be nonnegative integers with $(m,n) \neq (1,1)$. A function $f : X \to Y$ satisfies equation (1.2), $f(-x) = f(x)$ and $f(0) = 0$ for all $x, y, z \in X$ if and only if $f(x) = 0$ for all $x \in X$.*

Proof. (Sufficiency) This is obvious.
(Necessity) Putting $y = -x, z = 0$ in (1.2) we have

$$f(x) = nf\left(\frac{x}{n}\right). \tag{2.5}$$

Letting $y = z = 0$ in (1.2) and using (2.5) we get

$$f(x) = mf\left(\frac{x}{m}\right). \tag{2.6}$$

Thus (1.2) is converted into

$$f(x+y+z) + f(x) + f(y) + f(z) = f(x+y) + f(y+z) + f(z+x). \tag{2.7}$$

Putting $z = -x$ in (2.7) yields

$$2f(x) + 2f(y) = f(x+y) + f(x-y). \tag{2.8}$$

In view of (2.8) we have

$$f\left(\frac{x}{k}\right) = \frac{1}{k^2}f(x) \tag{2.9}$$

for any rational number k. It follows from (2.5), (2.6) and (2.9) that

$$f(x) = mf\left(\frac{x}{m}\right) = \frac{1}{m}f(x),$$
$$f(x) = nf\left(\frac{x}{n}\right) = \frac{1}{n}f(x)$$

for all $x \in X$. Since $(m,n) \neq (1,1)$, we get $f(x) = 0$ for all $x \in X$. □

Theorem 2.3. *Let m and n be positive integers with $(m,n) \neq (1,1)$. A function $f : X \to Y$ satisfies equation (1.2) for all $x, y, z \in X$ if and only if there exist an additive function $A : X \to Y$ and an element $C \in Y$ such that*

$$f(x) = A(x) + C$$

for all $x \in X$. In particular if $m + 3 \neq 3n$, then $C = 0$.

Proof. (Necessity) This is obvious.
(Sufficiency) Let $A(x) := \frac{1}{2}[f(x) - f(-x)]$ and $B(x) := \frac{1}{2}[f(x) + f(-x)] - f(0)$ for all $x \in X$. Then we have $A(0) = 0, A(-x) = -A(x), B(0) = 0, B(-x) = B(x)$,

$$\begin{aligned} &mA\left(\frac{x+y+z}{m}\right) + A(x) + A(y) + A(z) \\ &\qquad = nA\left(\frac{x+y}{n}\right) + nA\left(\frac{y+z}{n}\right) + nA\left(\frac{z+x}{n}\right) \end{aligned}$$

and

$$\begin{aligned} &mB\left(\frac{x+y+z}{m}\right) + B(x) + B(y) + B(z) \\ &\qquad = nB\left(\frac{x+y}{n}\right) + nB\left(\frac{y+z}{n}\right) + nB\left(\frac{z+x}{n}\right) \end{aligned}$$

for all $x, y, z \in X$. It follows from Lemmas 2.1 and 2.2 that A is additive and $B \equiv 0$. Letting $C := f(0)$ we get

$$f(x) = A(x) + C$$

for all $x \in X$. $\square$

Remark 2.4. If $(m, n) = (1, 1)$, then equation (1.2) is

$$f(x+y+z) + f(x) + f(y) + f(z) = f(x+y) + f(y+z) + f(z+x). \tag{2.10}$$

This equation was solved by Pl. Kannappan. In fact, he proved that a function f on a real vector space is a solution of equation (2.10) if and only if there exist a symmetric biadditive function B and an additive function A such that $f(x) = B(x, x) + A(x)$ for all $x \in X$ (see [9]). Moreover, S.-M. Jung [6] investigated the Hyers–Ulam–Rassias stability of equation (2.10) on restricted domains and applied the result to the study of an interesting asymptotic behavior of the quadratic functions.

3. Hyers–Ulam–Rassias stability of equation (1.2)

Now we are going to prove the stability theorem for Jensen type functional equation. Throughout this section X and Y will be a normed vector space and a Banach space, respectively. Let n be positive integer with $n \neq 1$ and let $\phi : X^3 \to [0, \infty)$ be a mapping satisfying one of the conditions (a), (b) and one of the conditions (c), (d):

$$(a)\ \ \Phi_1(x,y,z) := \sum_{k=0}^{\infty} \frac{1}{2^k}\phi(2^k x, 2^k y, 2^k z) < \infty,$$

$$(b)\ \ \Phi_2(x,y,z) := \sum_{k=1}^{\infty} 2^k\phi\left(\frac{x}{2^k}, \frac{y}{2^k}, \frac{z}{2^k}\right) < \infty,$$

$$(c)\ \ \Phi_3(x,y,z) := \sum_{k=1}^{\infty} \frac{1}{n^k}\phi(n^k x, n^k y, n^k z) < \infty,$$

$$(d)\ \ \Phi_4(x,y,z) := \sum_{k=0}^{\infty} n^k\phi\left(\frac{x}{n^k}, \frac{y}{n^k}, \frac{z}{n^k}\right) < \infty$$

for all $x, y, z \in X$.

For convenience, we define the operator T by

$$\begin{aligned}(Tf)(x,y,z) := mf\left(\frac{x+y+z}{m}\right) + f(x) + f(y) + f(z)\\ -nf\left(\frac{x+y}{n}\right) - nf\left(\frac{y+z}{n}\right) - nf\left(\frac{z+x}{n}\right).\end{aligned}$$

Lemma 3.1. *Let m and n be positive integers and let $\phi : X^3 \to [0,\infty)$ be a mapping satisfying one of the conditions (a), (b). Suppose that a function $f : X \to Y$ satisfies $f(-x) = -f(x)$ and*

$$\|(Tf)(x,y,z)\| \le \phi(x,y,z) \tag{3.1}$$

for all $x, y, z \in X$. Then there exist a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \Psi_o(x),$$

where

$$\Psi_o(x) := \begin{cases} \dfrac{1}{2n}[\Phi_1(nx,0,0) + \Phi_1(nx,nx,-nx)] & \text{if } \phi \text{ satisfies (a)},\\ \dfrac{1}{2n}[\Phi_2(nx,0,0) + \Phi_2(nx,nx,-nx)] & \text{if } \phi \text{ satisfies (b)}\end{cases}$$

for all $x \in X$.

Proof. Putting $y = z = 0$ in (3.1) we have

$$\left\|mf\left(\frac{x}{m}\right) + f(x) - 2nf\left(\frac{x}{n}\right)\right\| \le \phi(x,0,0). \tag{3.2}$$

Letting $y = x$ and $z = -x$ in (3.1) we get

$$\left\|mf\left(\frac{x}{m}\right) + f(x) - nf\left(\frac{2x}{n}\right)\right\| \le \phi(x,x,-x). \tag{3.3}$$

Adding (3.2) to (3.3) we obtain

$$\left\|2nf\left(\frac{x}{n}\right) - nf\left(\frac{2x}{n}\right)\right\| \le \phi(x,0,0) + \phi(x,x,-x). \tag{3.4}$$

Assume that ϕ satisfies the condition (a). Replacing x by nx and dividing by $2n$ in (3.4) we have

$$\left\|f(x) - \frac{f(2x)}{2}\right\| \le \frac{1}{2n}[\phi(nx,0,0) + \phi(nx,nx,-nx)].$$

Making use of induction argument we have

$$\left\|f(x) - \frac{f(2^k x)}{2^k}\right\| \le \frac{1}{2n}[\Phi_1(nx,0,0) + \Phi_1(nx,nx,-nx)] \tag{3.5}$$

for all $k \in \mathbb{N}$ and $x \in X$. Replacing x by $2^l x$ and dividing by 2^l yields

$$\left\|\frac{f(2^l x)}{2^l} - \frac{f(2^{k+l}x)}{2^{k+l}}\right\| \le \frac{1}{2n \cdot 2^l}[\Phi_1(2^l nx,0,0) + \Phi_1(2^l nx, 2^l nx, -2^l nx)]$$

for all $k, l \in \mathbb{N}$ and $x \in X$. It follows from the condition (a) that $\{\frac{f(2^k x)}{2^k}\}$ is a Cauchy sequence which converges uniformly. Thus we can define a function $A : X \to Y$ by

$$A(x) := \lim_{k\to\infty} \frac{f(2^k x)}{2^k}.$$

Now replacing x, y, z by $2^k x, 2^k y, 2^k z$ in (3.1), respectively and then dividing by 2^k we have

$$\|(Tf)(2^k x, 2^k y, 2^k z)\| \le \frac{1}{2^k}\phi(2^k x, 2^k y, 2^k z)$$

for all $k \in \mathbb{N}$ and $x, y, z \in X$. Letting $k \to \infty$ we get

$$(TA)(x,y,z) = 0$$

for all $x, y, z \in X$. Also we obtain $A(-x) = -A(x)$ for all $x \in X$ by virtue of the assumption of f. According to Lemma 2.1, A is additive. Taking the limit as $k \to \infty$ in (3.5) we get

$$\|f(x) - A(x)\| \le \frac{1}{2n}[\Phi_1(nx,0,0) + \Phi_1(nx,nx,-nx)] \tag{3.6}$$

for all $x \in X$.

Finally we prove the uniqueness. Suppose that A' is another additive mapping satisfying (3.6). Then we have

$$\begin{aligned} k\|A(x) - A'(x)\| &= \|A(kx) - A'(kx)\| \\ &\le \frac{1}{n}[\Phi_1(knx,0,0) + \Phi_1(knx,knx,-knx)] \end{aligned}$$

for all $k \in \mathbb{N}$ and $x \in X$. Taking the limit as $k \to \infty$, we see from the condition (a) that $A(x) = A'(x)$ for all $x \in X$.

On the other hand, assume that ϕ satisfies the condition (b). Replacing x by $\frac{nx}{2}$ in (3.4) and then dividing by n we have

$$\left\|f(x) - 2f\left(\frac{x}{2}\right)\right\| \le \frac{1}{n}\left[\phi\left(\frac{nx}{2},0,0\right) + \phi\left(\frac{nx}{2},\frac{nx}{2},-\frac{nx}{2}\right)\right].$$

Making use of induction argument we get

$$\left\|f(x) - 2^k f\left(\frac{x}{2^k}\right)\right\| \le \frac{1}{2n}[\Phi_2(nx,0,0) + \Phi_2(nx,nx,-nx)].$$

By the similar method as that of the case (a), we can define a function $A : X \to Y$ by

$$A(x) := \lim_{k\to\infty} 2^k f\left(\frac{x}{2^k}\right)$$

and also easily have that A is a unique additive mapping such that

$$\|f(x) - A(x)\| \leq \frac{1}{2n}[\Phi_2(nx, 0, 0) + \Phi_2(nx, nx, -nx)]$$

for all $x \in X$. □

Lemma 3.2. *Let m and n be positive integers with $n \neq 1$ and let $\phi : X^3 \to [0, \infty)$ be a mapping satisfying one of the conditions (c), (d). Suppose that a function $f : X \to Y$ satisfies $f(-x) = f(x)$, $f(0) = 0$ and*

$$\|(Tf)(x, y, z)\| \leq \phi(x, y, z) \tag{3.7}$$

for all $x, y, z \in X$. Then

$$\|f(x)\| \leq \Psi_e(x),$$

where

$$\Psi_e(x) := \begin{cases} \frac{1}{2}\Phi_3(x, -x, 0) & \text{if } \phi \text{ satisfies (c)},\\ \frac{1}{2}\Phi_4(x, -x, 0) & \text{if } \phi \text{ satisfies (d)} \end{cases}$$

for all $x \in X$.

Proof. Putting $y = -x$ and $z = 0$ in (3.7) and dividing by 2 yields

$$\left\|f(x) - nf\left(\frac{x}{n}\right)\right\| \leq \frac{1}{2}\phi(x, -x, 0). \tag{3.8}$$

Replacing x by nx in (3.8) and dividing by n we have

$$\left\|f(x) - \frac{f(nx)}{n}\right\| \leq \frac{1}{2n}\phi(nx, -nx, 0).$$

Assume that ϕ satisfies the condition (c). Making use of induction argument we get

$$\left\|f(x) - \frac{f(n^k x)}{n^k}\right\| \leq \frac{1}{2}\Phi_3(x, -x, 0) \tag{3.9}$$

for all $k \in \mathbb{N}$ and $x \in X$. Replacing x by $n^l x$ and dividing by n^l yields

$$\left\|\frac{f(n^l x)}{n^l} - \frac{f(n^{l+k} x)}{n^{l+k}}\right\| \leq \frac{1}{2n^l}\Phi_3(n^l x, -n^l x, 0)$$

for all $k, l \in \mathbb{N}$ and $x \in X$. By virtue of the condition (c), we can see that $\{\frac{f(n^k x)}{n^k}\}$ is a Cauchy sequence. Thus we define a function $B : X \to Y$ by

$$B(x) := \lim_{k\to\infty} \frac{f(n^k x)}{n^k}.$$

Now replacing x, y, z by $n^k x, n^k y, n^k z$ in (3.7), respectively and then dividing by n^k we have

$$\|(Tf)(n^k x, n^k y, n^k z)\| \le \frac{1}{n^k}\phi(n^k x, n^k y, n^k z)$$

for all $k \in \mathbb{N}$ and $x, y, z \in X$. Letting $k \to \infty$ we get

$$(TB)(x, y, z) = 0$$

for all $x, y, z \in X$. Also we obtain $B(-x) = B(x), B(0) = 0$ for all $x \in X$ by virtue of the assumption of f. According to Lemma 3.2, $B(x) = 0$ for all $x \in X$. Taking the limit in (3.9) as $k \to \infty$ we obtain

$$\|f(x)\| \le \frac{1}{2}\Phi_3(x, -x, 0)$$

for all $x \in X$.

On the other hand, assume that ϕ satisfies the condition (d). It follows from (3.8) and induction argument that

$$\left\| f(x) - n^k f\left(\frac{x}{n}\right)\right\| \le \frac{1}{2}\Phi_4(x, -x, 0).$$

By the similar method as that of the case (c), we easily have that

$$\|f(x)\| \le \frac{1}{2}\Phi_4(x, -x, 0)$$

for all $x \in X$. □

Theorem 3.3. *Let m and n be positive integers with $n \ne 1$ and let $\phi : X^3 \to [0, \infty)$ be a mapping satisfying one of the conditions (a), (b) and one of the conditions (c), (d). Suppose that the function $f : X \to Y$ satisfies*

$$\|(Tf)(x, y, z)\| \le \phi(x, y, z)$$

for all $x, y, z \in X$. Then there exist a unique additive mapping $A : X \to Y$ and a unique element $C \in Y$ such that

$$\|f(x) - A(x) - C\| \le \frac{1}{2}[\Psi_o(x) + \Psi_o(-x) + \Psi_e(x) + \Psi_e(-x)] + \Psi_e(0),$$

where Ψ_o and Ψ_e are defined as in Lemma 3.1 and Lemma 3.2, respectively.

Proof. Let $f_o(x) := \frac{1}{2}[f(x) - f(-x)]$. Then we have $f_o(-x) = -f_o(x)$ and

$$\|(Tf_o)(x, y, z)\| \le \frac{1}{2}[\phi(x, y, z) + \phi(-x, -y, -z)].$$

According to Lemma 3.1, there exists a unique additive mapping A such that

$$\|f_o(x) - A(x)\| \le \frac{1}{2}[\Psi_o(x) + \Psi_o(-x)].$$

On the other hand, let $f_e(x) := \frac{1}{2}[f(x) + f(-x)] - f(0)$. Then we have $f_e(-x) = f_e(x), f_e(0) = 0$ and

$$\|(Tf_e)(x, y, z)\| \le \frac{1}{2}[\phi(x, y, z) + \phi(-x, -y, -z)] + \phi(0, 0, 0).$$

By virtue of Lemma 3.2, we have

$$\|f_e(x)\| \le \frac{1}{2}[\Psi_e(x) + \Psi_e(-x)] + \Psi_e(0).$$

Let $C = f(0)$. Since $f(x) = f_o(x) + f_e(x) + f(0)$ for all $x \in X$, it follows that

$$\begin{aligned}\|f(x) - A(x) - C\| &\le \|f_o(x) - A(x)\| + \|f_e(x)\| \\ &\le \frac{1}{2}[\Psi_o(x) + \Psi_o(-x) + \Psi_e(x) + \Psi_e(-x)] + \Psi_e(0)\end{aligned}$$

for all $x \in X$. This completes the proof. □

As a consequence of Theorem 3.3 we have the following corollaries.

Corollary 3.4. *Let m and n be positive integers with $n \neq 1$ and let $p \neq 1$. Suppose that the function $f : X \to Y$ satisfies*

$$\|(Tf)(x, y, z)\| \le \epsilon(\|x\|^p + \|y\|^p + \|z\|^p)$$

for all $x, y, z \in X$. Then there exist a unique additive mapping $A : X \to Y$ and a unique element $C \in Y$ such that

$$\|f(x) - A(x) - C\| \le \epsilon \left(\frac{4}{n|2 - 2^p|} + \frac{1}{|n - n^p|} \right) n^p \|x\|^p$$

for all $x \in X$ and for all $x \in X \backslash \{0\}$ if $p < 0$.

Corollary 3.5. *Let m and n be positive integers with $n \neq 1$. Suppose that the function $f : X \to Y$ satisfies*

$$\|(Tf)(x, y, z)\| \le \epsilon$$

for all $x, y, z \in X$. Then there exist a unique additive mapping $A : X \to Y$ and a unique element $C \in Y$ such that

$$\|f(x) - A(x) - C\| \le \frac{5n - 4}{2n(n - 1)}\epsilon$$

for all $x \in X$.

Acknowledgements: This work was supported by Korea Research Foundation Grant (KRF-2004-015-C00032) and the Special Grant of Sogang University in 2005.

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