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A CHARACTERIZATION OF B-CONVEXITY AND J-CONVEXITY OF BANACH SPACES

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This paper is dedicated to Professor Themistocles M. Rassias.

Submitted by M. Abel

Abstract. In [K.-I. Mitani and K.-S. Saito, J. Math. Anal. Appl. 327 (2007), 898–907] we characterized the strict convexity, uniform convexity and uniform non-squareness of Banach spaces using ψ-direct sums of two Banach spaces, where ψ is a continuous convex function with some appropriate conditions on $[0, 1]$. In this paper, we characterize the B_n-convexity and J_n-convexity of Banach spaces using ψ-direct sums of n Banach spaces, where ψ is a continuous convex function with some appropriate conditions on a certain convex subset of $\mathbb{R}^n$.

1. Introduction and preliminaries

A norm $\|\cdot\|$ on $\mathbb{C}^n$ is said to be absolute if

$$\|(x_1, x_2, \cdots, x_n)\| = \|(|x_1|, |x_2|, \cdots, |x_n|)\|$$

for all $x_1, x_2, \cdots, x_n \in \mathbb{C}$, and normalized if

$$\|(1, 0, \cdots, 0)\| = \|(0, 1, 0, \cdots, 0)\| = \cdots = \|(0, \cdots, 0, 1)\| = 1.$$

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The ℓ_p-norms $\|\cdot\|_p$ are such examples:

$$\|(x_1,\cdots,x_n)\|_p = \begin{cases} \left(|x_1|^p+\cdots+|x_n|^p\right)^{1/p} & \text{if } 1\le p<\infty, \\ \max\left\{|x_1|,\cdots,|x_n|\right\} & \text{if } p=\infty. \end{cases}$$

Let AN_n be the family of all absolute normalized norms on $\mathbb{C}^n$. The second author, Kato and Takahashi in [9] showed that for every absolute normalized norm $\|\cdot\|$ on $\mathbb{C}^n$, there corresponds a unique continuous convex function on Δ_n with some appropriate conditions, where

$$\Delta_n = \left\{(t_1,\cdots,t_n)\in\mathbb{R}^n : t_j\ge 0\ (\forall j),\ \sum_{j=1}^n t_j = 1\right\}.$$

Indeed, for any $\|\cdot\|\in AN_n$, we define

$$\psi(t_1,\cdots,t_n) = \|(t_1,\cdots,t_n)\|, \quad (t_1,\cdots,t_n)\in\Delta_n. \tag{1.1}$$

Then ψ is a continuous convex function on Δ_n, and satisfies the following conditions:

$$\psi(1,0,\cdots,0)=\psi(0,1,0,\cdots,0)=\cdots=\psi(0,\cdots,0,1)=1 \tag{A_0}$$

$$\psi(t_1,\cdots,t_n)\ge(1-t_1)\psi\left(0,\frac{t_2}{1-t_1},\cdots,\frac{t_n}{1-t_1}\right), \quad \text{if } t_1\ne 1 \tag{A_1}$$

$$\psi(t_1,\cdots,t_n)\ge(1-t_2)\psi\left(\frac{t_1}{1-t_2},0,\frac{t_3}{1-t_2}\cdots,\frac{t_n}{1-t_2}\right), \quad \text{if } t_2\ne 1 \tag{A_2}$$

$$\vdots$$

$$\psi(t_1,\cdots,t_n)\ge(1-t_n)\psi\left(\frac{t_1}{1-t_n},\cdots,\frac{t_{n-1}}{1-t_n},0\right), \quad \text{if } t_n\ne 1. \tag{A_n}$$

Let Ψ_n be the set of all continuous convex functions ψ on Δ_n satisfying (A_0) – (A_n). Conversely, for any $\psi\in\Psi_n$, we define

$$\|(x_1,\cdots,x_n)\|_\psi = \begin{cases} \left(\sum_{j=1}^n|x_j|\right)\psi\left(\frac{|x_1|}{\sum_{j=1}^n|x_j|},\cdots,\frac{|x_n|}{\sum_{j=1}^n|x_j|}\right), \\ \qquad\qquad \text{if } (x_1,\cdots,x_n)\ne(0,\cdots,0), \\ 0, \qquad\qquad \text{if } (x_1,\cdots,x_n)=(0,\cdots,0). \end{cases}$$

Then $\|\cdot\|_\psi\in AN_n$ and satisfies (1.1) (cf. [9, Theorem 4.2]). Further, AN_n and Ψ_n are in a one-to-one correspondence. In particular, let ψ_p be the function corresponding to ℓ_p-norms on $\mathbb{C}^n$. According to Kato, the second author and Tamura in [6, 8], for any Banach spaces $X_1,X_2,\cdots,X_n$ and any $\psi\in\Psi_n$, we define the *ψ-direct sum* $(X_1\oplus X_2\oplus\cdots\oplus X_n)_\psi$ to be their direct sum equipped with the norm

$$\|(x_1,\cdots,x_n)\|_\psi := \|(\|x_1\|,\cdots,\|x_n\|)\|_\psi.$$

Let $B_X=\{x\in X:\|x\|\le 1\}$ be the closed unit ball of a Banach space X. A Banach space X is said to be *uniformly non-square* if there exists a $\delta>0$ such that $\|(x-y)/2\|>1-\delta$, $x,y\in B_X$ implies $\|(x+y)/2\|\le 1-\delta$ (cf. [5]). In [7], we characterized the uniform non-squareness of Banach spaces. That is, let

$\varphi, \psi \in \Psi_2$. Assume that $\varphi \neq \psi_\infty$ and ψ has a unique minimal point t_0 in Δ_2. Then a Banach space X is uniformly non-square if and only if

$$\|A_2 : (X \oplus X)_\psi \to (X \oplus X)_\varphi\| < \frac{\|(1,1)\|_\varphi}{\psi(t_0)}$$

holds, where A_2 is the Littlewood matrix, that is, $A_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$.

In this paper, we give some characterizations of the B-convexity (or uniformly non ℓ_1-ness) and the J-convexity (which is related to superreflexivity) for Banach spaces using ψ-direct sums $(X \oplus \cdots \oplus X)_\psi$. We note that these characterizations are an extension of some results in Takahashi and Kato [11].

We need some preliminaries. Recall the following properties concerning absolute norms on $\mathbb{C}^n$ (see [9]).

Lemma 1.1 ([9]). (i) *Let $\psi \in \Psi_n$. If $|x_i| \leq |y_i|$ for all i, then*

$$\|(x_1, x_2, \cdots, x_n)\|_\psi \leq \|(y_1, y_2, \cdots, y_n)\|_\psi.$$

(ii) *Let $\varphi, \psi \in \Psi_n$ with $\varphi \leq \psi$. Put $M = \max_{t \in \Delta_n} \psi(t)/\varphi(t)$. Then we have $\|\cdot\|_\varphi \leq \|\cdot\|_\psi \leq M\|\cdot\|_\varphi$.*

Let X be a Banach space. For any $\psi \in \Psi_n$, we define the space $\ell_\psi^n(X)$ by

$$\ell_\psi^n(X) = (\overbrace{X \oplus \cdots \oplus X}^{n})_\psi.$$

In particular, for the case $\psi = \psi_p$ $(1 \leq p \leq \infty)$, let the space $\ell_p^n(X)$ be $\ell_p^n(X) = \ell_{\psi_p}^n(X)$.

2. B-convexity

The notion of B-convexity was introduced by Beck [2] in order to obtain a strong law of large numbers for certain vector-valued random variables. A Banach space X is called *B_n-convex* (or uniformly non-ℓ_1^n) if there is a real number $\delta > 0$ such that for any $x_1, \cdots, x_n \in B_X$

$$\min_{\varepsilon_1, \cdots, \varepsilon_n = \pm 1} \left\| \sum_{j=1}^{n} \varepsilon_j x_j \right\| \leq n(1-\delta)$$

holds. If X is B_n-convex for some n, then X is called *B-convex*. For the fundamental properties of B-convexity, we refer to [1], [2], [3], [10], [11].

We consider the *Rademacher matrices* R_n; that is,

$$R_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad R_{n+1} = \left(\begin{array}{c|c} 1 & \\ \vdots & R_n \\ 1 & \\ \hline -1 & \\ \vdots & R_n \\ -1 & \end{array}\right) \quad (n = 1, 2, \cdots).$$

Proposition 2.1. *Let $\varphi \in \Psi_{2^n}$ and $\psi \in \Psi_n$ where $n \geq 2$. Then for any Banach space X*

$$\|R_n : \ell_\psi^n(X) \to \ell_\varphi^{2^n}(X)\| \leq \frac{\|(1, \cdots, 1)\|_\varphi}{\min\limits_{t \in \Delta_n} \psi(t)}. \tag{2.1}$$

Proof. Put $R_n = (a_{ij})$. From Lemma 1.1 we have for any $x_1, \cdots, x_n \in X$,

$$\begin{aligned}
\left\| \left\{ \sum_{j=1}^{n} a_{ij} x_j \right\}_{i=1}^{2^n} \right\|_\varphi &= \left\| \left\{ \left\| \sum_{j=1}^{n} a_{ij} x_j \right\| \right\}_{i=1}^{2^n} \right\|_\varphi \\
&\leq \left\| \left\{ \sum_{j=1}^{n} \|x_j\| \right\}_{i=1}^{2^n} \right\|_\varphi \\
&= \left(\sum_{j=1}^{n} \|x_j\| \right) \|(1, \cdots, 1)\|_\varphi \\
&= \|(x_1, \cdots, x_n)\|_1 \|(1, \cdots, 1)\|_\varphi \\
&\leq \max_{t \in \Delta_n} \frac{\psi_1(t)}{\psi(t)} \|(x_1, \cdots, x_n)\|_\psi \|(1, \cdots, 1)\|_\varphi \\
&= \frac{\|(1, \cdots, 1)\|_\varphi}{\min\limits_{t \in \Delta_n} \psi(t)} \|(x_1, \cdots, x_n)\|_\psi,
\end{aligned}$$

which implies (2.1). □

The main theorem is the following.

Theorem 2.2. *Let $\varphi \in \Psi_{2^n}$ and $\psi \in \Psi_n$ where $n \geq 2$. Assume that ψ has a unique minimal point $t_0 = \{t_j\}_{j=1}^n \in \Delta_n$ with $t_j > 0$ for all j and φ has the condition:*

$$\|(1, \cdots, 1, \overset{(i)}{0}, 1, \cdots, 1)\|_\varphi < \|(1, \cdots, 1)\|_\varphi \tag{2.2}$$

for every i. Then a Banach space X is B_n-convex if and only if

$$\|R_n : \ell_\psi^n(X) \to \ell_\varphi^{2^n}(X)\| < \frac{\|(1, \cdots, 1)\|_\varphi}{\psi(t_0)}$$

holds.

We reformulate Theorem 2.2 as follows.

Theorem 2.3. *Let $\varphi \in \Psi_{2^n}$ and $\psi \in \Psi_n$ where $n \geq 2$. Assume that ψ has a unique minimal point $t_0 = \{t_j\}_{j=1}^n \in \Delta_n$ with $t_j > 0$ for all j and φ has the condition:*

$$\|(1, \cdots, 1, \overset{(i)}{0}, 1, \cdots, 1)\|_\varphi < \|(1, \cdots, 1)\|_\varphi \tag{2.3}$$

for every i. Then for a Banach space X the following are equivalent:
(i) *X is B_n-convex.*

(ii) *There exists a real number* δ $(0 < \delta < 1)$ *such that for all* $x_1, \cdots, x_n \in X$,

$$\left\| \left\{ \sum_{j=1}^{n} a_{ij} t_j x_j \right\}_{i=1}^{2^n} \right\|_{\varphi} \leq (1-\delta) \frac{\|(1, \cdots, 1)\|_{\varphi}}{\psi(t_0)} \|\{t_j x_j\}_{j=1}^{n}\|_{\psi}$$

holds, where $R_n = (a_{ij})$.

We shall show Theorem 2.3. To do it, we need the following lemma about norm convexity.

Lemma 2.4. *Let* X *be a Banach space. Then the following are equivalent:*
(i) *There exists a real number* δ $(0 < \delta < 1)$ *such that for all* $x_1, \cdots, x_n \in B_X$,

$$\min_{1 \leq i \leq 2^n} \left\| \sum_{j=1}^{n} a_{ij} x_j \right\| \leq n(1-\delta).$$

(ii) *For any (resp. some)* $\{t_j\}_{j=1}^{n} \in \Delta_n$ *with* $t_j > 0$ *for all* $j = 1, \cdots, n$, *there exists a real number* δ_0 $(0 < \delta_0 < 1)$ *such that for all* $x_1, \cdots, x_n \in B_X$,

$$\min_{1 \leq i \leq 2^n} \left\| \sum_{j=1}^{n} a_{ij} t_j x_j \right\| \leq 1 - \delta_0.$$

Proof. (i)⇒(ii): Assume that the assertion (i) holds. We take $\{t_j\}_{j=1}^{n} \in \Delta_n$ with $t_j > 0$ for all j, and fix $x_1, \cdots, x_n \in B_X$. Then there exists a number i_0 satisfying

$$\left\| \sum_{j=1}^{n} a_{i_0 j} x_j \right\| \leq n(1-\delta).$$

We choose a number k such that $t_k = \min\{t_1, \cdots, t_n\}$. Note that $t_k > 0$. Then we have

$$\begin{aligned} \left\| \sum_{j=1}^{n} a_{i_0 j} t_j x_j \right\| &= \left\| \sum_{j=1}^{n} a_{i_0 j} t_k x_j + \sum_{j=1}^{n} a_{i_0 j} (t_j - t_k) x_j \right\| \\ &\leq t_k \left\| \sum_{j=1}^{n} a_{i_0 j} x_j \right\| + \sum_{j=1}^{n} (t_j - t_k) \\ &\leq n t_k (1-\delta) + 1 - n t_k = 1 - n t_k \delta. \end{aligned}$$

Put $\delta_0 = n t_k \delta$. Then the assertion (ii) holds.

(ii)⇒(i): Assume that the assertion (ii) holds. We take $\{t_j\}_{j=1}^{n} \in \Delta_n$ with $t_j > 0$ for all j, and fix $x_1, \cdots, x_n \in B_X$. Then there exists a number i_0 such that

$$\left\| \sum_{j=1}^{n} a_{i_0 j} t_j x_j \right\| \leq 1 - \delta_0.$$

We choose a number ℓ such that $t_\ell = \max\{t_1, \cdots, t_n\}$. Then

$$\left\|\sum_{j=1}^n a_{i_0 j} x_j\right\| = \left\|\sum_{j=1}^n a_{i_0 j} \frac{t_j}{t_\ell} x_j + \sum_{j=1}^n \left(1 - \frac{t_j}{t_\ell}\right) a_{i_0 j} x_j\right\|$$
$$\leq \frac{1}{t_\ell}\left\|\sum_{j=1}^n a_{i_0 j} t_j x_j\right\| + \sum_{j=1}^n \left(1 - \frac{t_j}{t_\ell}\right)$$
$$\leq \frac{1}{t_\ell}(1-\delta_0) + n - \frac{1}{t_\ell} = n\left(1 - \frac{1}{n t_\ell}\delta_0\right).$$

Put $\delta = \frac{1}{n t_\ell}\delta_0$. Then the assertion (i) holds. This completes the proof. □

Proof of Theorem 2.3. We put

$$K = \frac{\|(1, \cdots, 1)\|_\varphi}{\psi(t_0)}.$$

(i)⇒(ii): Assume that the assertion (ii) fails to hold. Then, for each positive number ℓ, there exist $x_{\ell 1}, \cdots, x_{\ell n} \in X$ such that

$$K\left(1 - \frac{1}{\ell}\right)\left\|\{t_j x_{\ell j}\}_{j=1}^n\right\|_\psi < \left\|\left\{\sum_{j=1}^n a_{ij} t_j x_{\ell j}\right\}_{i=1}^{2^n}\right\|_\varphi. \tag{2.4}$$

Since $(x_{\ell 1}, \cdots, x_{\ell n}) \neq (0, \cdots, 0)$ for all ℓ, we may assume

$$\max\left\{\|x_{\ell 1}\|, \cdots, \|x_{\ell n}\|\right\} = 1$$

for all ℓ. So we can take sequences $\{\ell(k)\}_{k=1}^\infty, \{\alpha_j\}_{j=1}^n$ and $\{\beta_i\}_{i=1}^{2^n}$ such that for all i, j,

$$\|x_{\ell(k)j}\| \to \alpha_j \ (k \to \infty)$$

and

$$\left\|\sum_{j=1}^n a_{ij} t_j x_{\ell(k)j}\right\| \to \beta_i \ (k \to \infty). \tag{2.5}$$

Note that $0 \leq \alpha_j \leq 1$ for all j, and $0 \leq \beta_i \leq 1$ for all i. By (2.4) and Lemma 1.1, we have

$$K\left(1 - \frac{1}{\ell(k)}\right)\left\|\{t_j \|x_{\ell(k)j}\|\}_{j=1}^n\right\|_\psi < \left\|\left\{\left\|\sum_{j=1}^n a_{ij} t_j x_{\ell(k)j}\right\|\right\}_{i=1}^{2^n}\right\|_\varphi$$
$$\leq \left\|\left\{\sum_{j=1}^n t_j \|x_{\ell(k)j}\|\right\}_{i=1}^{2^n}\right\|_\varphi \tag{2.6}$$
$$= \sum_{j=1}^n t_j \|x_{\ell(k)j}\| \|(1, \cdots, 1)\|_\varphi.$$

As $k \to \infty$, we obtain

$$K\|\{t_j \alpha_j\}_{j=1}^n\|_\psi \leq \sum_{j=1}^n t_j \alpha_j \|(1, \cdots, 1)\|_\varphi,$$

and so

$$\psi\left(\frac{t_1\alpha_1}{\sum_{j=1}^n t_j\alpha_j}, \cdots, \frac{t_n\alpha_n}{\sum_{j=1}^n t_j\alpha_j}\right) \le \psi(t_0).$$

Since $\psi(t) > \psi(t_0)$ for all $t \in \Delta_n$ with $t \neq t_0$ by the assumption, we have

$$t_0 = (t_1, \cdots, t_n) = \left(\frac{t_1\alpha_1}{\sum_{j=1}^n t_j\alpha_j}, \cdots, \frac{t_n\alpha_n}{\sum_{j=1}^n t_j\alpha_j}\right),$$

which implies

$$\alpha_1 = \alpha_2 = \cdots = \alpha_n = \sum_{j=1}^n t_j\alpha_j,$$

by $t_j > 0$ for all j. Since $\|x_{\ell(k)j}\| \to \alpha_j$ for all j and

$$\max\{\|x_{\ell(k)1}\|, \cdots, \|x_{\ell(k)n}\|\} = 1,$$

we have $\max\{\alpha_1, \cdots, \alpha_n\} = 1$, and so $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 1$. We also have by (2.6),

$$\left\| \left\{ \left\| \sum_{j=1}^n a_{ij} t_j x_{\ell(k)j} \right\| \right\}_{i=1}^{2^n} \right\|_\varphi \to \|(1, \cdots, 1)\|_\varphi.$$

Hence we have by (2.5),

$$\|(\beta_1, \beta_2, \cdots, \beta_{2^n})\|_\varphi = \|(1, 1, \cdots, 1)\|_\varphi.$$

If $\beta_1 < 1$, then we have by the assumption

$$\begin{aligned}\|(\beta_1, \beta_2, \cdots, \beta_n)\|_\varphi &\le \|(\beta_1, 1, 1, \cdots, 1)\|_\varphi \\ &\le (1-\beta_1)\|(0, 1, 1, \cdots, 1)\|_\varphi + \beta_1\|(1, 1, \cdots, 1)\|_\varphi \\ &< (1-\beta_1)\|(1, 1, \cdots, 1)\|_\varphi + \beta_1\|(1, 1, \cdots, 1)\|_\varphi \\ &= \|(1, 1, \cdots, 1)\|_\varphi,\end{aligned}$$

which is a contradiction. Hence $\beta_1 = 1$. We similarly have $\beta_2 = \beta_3 = \cdots = \beta_n = 1$. Namely, we obtain

$$\left\| \sum_{j=1}^n a_{ij} t_j x_{\ell(k)j} \right\| \to 1 \quad (k \to \infty)$$

for all i. Therefore it follows from Lemma 2.4 that (i) fails to hold.

(ii)⇒(i): Assume that the assertion (ii) holds. For any $x_1, \cdots, x_n \in B_X$, we have

$$\begin{aligned}\min_{1\le i\le 2^n} \left\| \sum_{j=1}^n a_{ij} t_j x_j \right\| \|(1, \cdots, 1)\|_\varphi &\le \left\| \left\{ \left\| \sum_{j=1}^n a_{ij} t_j x_j \right\| \right\}_{i=1}^{2^n} \right\|_\varphi \\ &\le K(1-\delta)\|\{t_j\|x_j\|\}_{j=1}^n\|_\psi \\ &\le K(1-\delta)\|\{t_j\}_{j=1}^n\|_\psi \\ &= \|(1, \cdots, 1)\|_\varphi(1-\delta)\end{aligned}$$

and so

$$\min_{1\le i\le 2^n}\left\|\sum_{j=1}^{n} a_{ij}t_jx_j\right\|\le 1-\delta.$$

Thus it follows from Lemma 2.4 that the assertion (i) holds. This completes the proof. □

In Theorem 2.2, we suppose that φ is strictly convex on Δ_n. Then φ satisfies (2.2) for every i. Therefore we have

Corollary 2.5. *Let $\varphi\in\Psi_{2^n}$ and $\psi\in\Psi_n$ where $n\ge 2$. Assume that ψ has a unique minimal point $t_0=\{t_j\}_{j=1}^n\in\Delta_n$ with $t_j>0$ for all j and φ is strictly convex on Δ_n. Then a Banach space X is B_n-convex if and only if*

$$\|R_n:\ell_\psi^n(X)\to\ell_\varphi^{2^n}(X)\|<\frac{\|(1,\cdots,1)\|_\varphi}{\psi(t_0)}$$

holds.

We remark that Theorem 2.2 is an extension of the following result in Takahashi and Kato [11].

Corollary 2.6 ([11])**.** *Let $1<r\le\infty$ and $1\le s<\infty$. Then a Banach space X is B_n-convex if and only if $\|R_n:\ell_r^n(X)\to\ell_s^{2^n}(X)\|<2^{n/s}n^{1/r'}$ holds, where $1/r+1/r'=1$.*

Proof. Note that $\psi_r(t_1,\cdots,t_n)>\psi_r(\frac{1}{n},\cdots,\frac{1}{n})$ for all $t=(t_1,\cdots,t_n)\in\Delta_n$ with $t\ne(\frac{1}{n},\cdots,\frac{1}{n})$, and ψ_s satisfies (2.2), for all i. Therefore we can apply Theorem 2.2 to $\psi=\psi_r$ and $\varphi=\psi_s$, and

$$\frac{\|(1,\cdots,1)\|_{\psi_s}}{\psi_r(\frac{1}{n},\cdots,\frac{1}{n})}=\frac{(1^s+\cdots+1^s)^{1/s}}{((\frac{1}{n})^r+\cdots+(\frac{1}{n})^r)^{1/r}}=2^{n/s}n^{1/r'}.$$

□

Further we consider Theorem 2.2 for the case $n=2$.

Corollary 2.7 ([7])**.** *Let $\psi\in\Psi_2$ and $\varphi\in\Psi_2$. Assume that ψ has a unique minimal point $t_0\in\Delta_2$ and $\varphi\ne\psi_\infty$. Then a Banach space X is uniformly non-square if and only if*

$$\|A_2:\ell_\psi^2(X)\to\ell_\varphi^2(X)\|<\frac{\|(1,1)\|_\varphi}{\psi(t_0)}$$

holds.

Proof. If $\varphi\in\Psi_2$ satisfies $\varphi\ne\psi_\infty$, then by the convexity of φ, we have $\varphi(\frac{1}{2},\frac{1}{2})>\frac{1}{2}$, which implies $\|(0,1)\|_\varphi<\|(1,1)\|_\varphi$ and $\|(1,0)\|_\varphi<\|(1,1)\|_\varphi$. Also, if $\psi\in\Psi_2$ has a unique minimal point $t_0=(t_1,t_2)\in\Delta_2$, then it is obvious that $t_1>0$ and $t_2>0$. Thus we obtain this corollary. □

3. J-convexity

A finite sequence of signs $\varepsilon_1, \cdots, \varepsilon_n$ will be called *admissible* if all $+$ signs are before all $-$ signs. A Banach space X is called J_n*-convex* if there exists some $\delta > 0$ such that for every $x_1, x_2, \cdots, x_n \in B_X$, there is an admissible choice of signs $\varepsilon_1, \cdots, \varepsilon_n$ such that

$$\Big\| \sum_{j=1}^{n} \varepsilon_j x_j \Big\| \leq n(1-\delta)$$

holds. If X is J_n-convex for some n, then X is called J*-convex*. Note that X is uniformly non-square if and only if it is J_2-convex. It is well-known that X is J-convex if and only if it is super-reflexive (see [1]). For the fundamental properties of J-convexity, we refer to [1], [5], [11] and so on.

The $n \times n$ matrices A_n (called *admissible matrices*) are defined by

$$A_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \; A_{n+1} = \left(\begin{array}{c|ccc} 1 & & & \\ \vdots & & A_n & \\ 1 & & & \\ \hline 1 & -1 & \cdots & -1 \end{array}\right) \quad (n = 2, 3, \cdots).$$

As in Theorem 2.2, we can characterize J_n-convexity of Banach spaces using ψ-direct sums.

Theorem 3.1. *Let $\psi \in \Psi_n$ and $\varphi \in \Psi_n$ where $n \geq 2$. Assume that ψ has a unique minimal point $t_0 = \{t_j\} \in \Delta_n$ with $t_j > 0$ for all j and φ has the condition:*

$$\|(1, \cdots, 1, \overset{(i)}{0}, 1, \cdots, 1)\|_\varphi < \|(1, \cdots, 1)\|_\varphi$$

for every i. Then a Banach space X is J_n-convex if and only if

$$\|A_n : \ell^n_\psi(X) \to \ell^n_\varphi(X)\| < \frac{\|(1, \cdots, 1)\|_\varphi}{\psi(t_0)}$$

holds.

In particular, we have

Corollary 3.2 ([11])**.** *Let $1 < r \leq \infty$ and $1 \leq s < \infty$. Then a Banach space X is J_n-convex if and only if $\|A_n : \ell^n_r(X) \to \ell^n_s(X)\| < n^{1/s+1/r'}$ holds, where $1/r + 1/r' = 1$.*

References

1. B. Beauzamy, *Introduction to Banach Spaces and Their Geometry*, 2nd ed., North-Holland, Amsterdam-New York-Oxford, 1985.
2. A. Beck, *A convexity condition in Banach spaces and the strong law of large numbers*, Proc. Amer. Math. Soc. **13** (1962), 329–334.
3. D. P. Giesy and R. C. James, *Uniformly non-$\ell^{(1)}$ and B-convex Banach spaces*, Studia Math. **48** (1973), 61–69.
4. D.H. Hyers, G. Isac and Th.M. Rassias, *Topics in nonlinear analysis and applications*. World Scientific Publishing Co., Inc., River Edge, NJ, 1997.
5. R. C. James, *Uniformly non-square Banach spaces*, Ann. of Math. **80** (1964), 542–550.

6. M. Kato, K.-S. Saito and T. Tamura, *On ψ-direct sums of Banach spaces and convexity*, J. Austral. Math. Soc. **75** (2003), 413–422.
7. K.-I. Mitani and K.-S. Saito, *A note on geometrical properties of Banach spaces using ψ-direct sums*, J. Math. Anal. Appl. **327** (2007), 898–907.
8. K.-S. Saito and M. Kato, *Uniform convexity of ψ-direct sums of Banach spaces*, J. Math. Anal. Appl. **277** (2003), 1–11.
9. K.-S. Saito, M. Kato and Y. Takahashi, *Absolute norms on $\mathbb{C}^n$*, J. Math. Anal. Appl. **252** (2000), 879–905.
10. M. A. Smith and B. Turett, *Rotundity in Lebesgue-Bochner function spaces*, Trans. Amer. Math. Soc. **257** (1980), 105–118.
11. Y. Takahashi and M. Kato, *Geometry of Banach spaces and norms of ± 1 matrices*, RIMS Kōkyūroku **1039** (1998), 165–169.

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