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PERTURBATION ANALYSIS FOR THE MOORE–PENROSE METRIC GENERALIZED INVERSE OF BOUNDED LINEAR OPERATORS

FAPENG DU

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Abstract. Utilizing the gap between homogenous subsets which is introduced in this paper, the perturbations for the Moore–Penrose metric generalized inverses of bounded linear operators in Banach spaces are discussed. Under range–preseving, kernel–preseving and general case, respectively, we get some new results about error estimate of the perturbations for the Moore–Penrose metric generalized inverse of bounded linear operators.

1. Introduction

The expressions and perturbations of the generalized inverse have been widely studied in the last decades which have its genetic in context of the "ill–posed" linear problem. In 1997, Chen and Xue introduced the notation so–called stable perturbation of linear bounded operator in [3]. Using this notation, they established the perturbation analysis for the generalized inverse and for the operator equation $Tx = b$ on Banach space in [4]. Later the stable perturbation has been generalized to the Banach Algebra, Hilbert C*–module and closed operator on Banach space (resp. Hilbert space) (cf. [6, 17, 18]). However, linearly generalized inverse can not deal with the extremal solutions, the minimal norm solutions, and the best approximation solutions of an ill-posed linear operator equations in Banach spaces. In order to solve the best approximation problems for an ill–posed linear operator equation in Banach spaces, Nashed and Votruba

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introduced the concept of the (set–valued) metric generalized inverse of a linear operator in Banach spaces(cf. [11]). In 2003, H. Wang and Y. Wang introduced the Moore–Penrose metric generalized inverse for linear operator on Banach space in [16], which is a homogeneous and nonlinear operator.

In recent years, some papers on the perturbation of the Moore–Penrose metric generalized inverse have appeared. In [1], J.Cao and Y.Xue give the expression of $(T+\delta T)^M$ under the condition range–preserving and kernel–preserving and investigate the equivalent conditions for the Moore–Penrose metric generalized inverse of perturbed operator which have the simplest expression $T^M(I+\delta TT^M)^{-1}$. Meanwhile, the stability of some operator equations in Banach spaces is obtained. Some results on the perturbation of the Moore–Penrose metric generalized inverse similar to the linearly generalized inverse are obtained in [10] by H.Ma, et al. under the assumption that T^M is quasi–additive and metric projection $\pi_{N(T)}$ is linear and $R(\delta T)\subseteq R(T), N(T)\subseteq N(\delta T)$. Some other results about metric generalized inverse, please see [12, 14], etc.

It is well known the metric projection is a bounded homogeneous and nonlinear operator, and then the Moore–Penrose metric generalized inverse is different with the linearly generalized inverse. In this paper, utilizing the gap between homogeneous subsets, we investigate the perturbation of the Moore–Penrose metric generalized inverse again. Under the range–preserving, the kernel–preserving and general case, we present the upper bounds of $\|\overline{T}^M\|$ and $\|\overline{T}^M-T^M\|$, respectively.

2. Preliminaries

Throughout this paper X,Y will be Banach spaces. Let $B(X,Y)$ denote the set of all bounded linear operators from X to Y. For any $T\in B(X,Y)$, $D(T),R(T)$ and $N(T)$ denote the domain, the range and the kernel of T, respectively.

Let M be a subset in X. If $\lambda x\in M$ whenever $x\in M$ and $\lambda\in\mathbb{R}$, then we call M a homogeneous subset. A nonlinear operator $T:X\to Y$ is called a bounded homogeneous operator if T maps every bounded set in X into a bounded set in Y and $T(\lambda x)=\lambda Tx$ for all $\lambda\in\mathbb{R}$. Let $H(X,Y)$ denote the set of all bounded homogeneous operators from X to Y. Equipped with the usual linear operations on $H(X,Y)$ and norm on $T\in H(X,Y)$ defined as $\|T\|=\sup\limits_{\|x\|=1}\|Tx\|$, $H(X,Y)$ become a Banach space(cf. [13, 15]). Obviously, $B(X,Y)\subseteq H(X,Y)$.

Recall that a nonlinear operator T is called quasi–additive on a subspace $M\subset X$ if

$$T(x+z)=T(x)+T(z),\quad \forall x\in X,\quad \forall z\in M.$$

If a homogeneous operator $T\in H(X,X)$ is quasi–additive on $R(T)$, then we call T a quasi–linear operator.

Let $M\subset X$ be a subset of X, we define the distance of a point $x\in X$ to the set M as $dist(x,M)=\inf\limits_{\forall y\in M}\|x-y\|$. Then the (set–valued) metric projection P_M defined on X is a mapping from X to M:

$$P_M=\{z\in M\mid \|x-z\|=dist(x,M),\forall x\in X\}.$$

If $P_M \neq \emptyset$, then M is called proximinal set. If P_M is singleton, then M is said to be a Chebyshev set. In this case, we denote P_M by π_M. Moreover, π_M satisfies the following properties:

Proposition 2.1. [1, 10, 13] *Let $M \subset X$ be a subspace of X. Then*

(1) $\pi_M^2(x) = \pi_M(x)$, $\forall x \in X$, *i.e., π_M is idempotent.*
(2) $\|x - \pi_M(x)\| \leq \|x\|$ *and so that* $\|\pi_M(x)\| \leq 2\|x\|$, $\forall x \in X$.
(3) $\pi_M(\lambda x) = \lambda \pi_(x)$, $\forall x \in X, \forall \lambda \in \mathbb{R}$, *i.e., π_M is homogenous.*
(4) $\pi_M(x + z) = \pi_M(x) + \pi_M(z) = \pi_M(x) + z$ *for any* $z \in M$, *i.e., π_M is quasi–additive on M.*
(5) π_M *is a closed operator if M is a Chebyshev subspace.*

Lemma 2.2. [5, 13] *Let $M \subset X$ be a proximinal subspace. Then $\pi_M^{-1}(0)$ is a closed linear subspace if and only if M is Chebyshev and π_M is continuous and linear operator.*

Lemma 2.3. [9] *Let X be a reflexive Banach space. Then X is strictly convex if and only if every nonempty closed convex subset $M \subset X$ is a Chebyshev set.*

Let X^* be the dual space of X and $M^\perp = \{x^* \in X^* | \langle x, x^* \rangle = 0, x \in M\}$. Now, we recall the notation so called dual–mapping.

Definition 2.4. The set–valued mapping $F_X : X \to X^*$ defined as

$$F_X(x) = \{x^* \in X^* \mid \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}, \quad \forall x \in X.$$

is called the dual–mapping of X, where $\langle x, x^* \rangle = x^*(x)$.

Lemma 2.5. (Generalized Orthogonal Decomposition Theorem) [7, 13] *Let X be a Banach space and $M \subset X$ be a proximinal subspace. Then for any $x \in X$, we have*

(1) $x = x_1 + x_2$ *with* $x_1 \in M$ *and* $x_2 \in F_X^{-1}(M^\perp)$.
(2) *If $M \subset X$ is a Chebyshev subspace, then the decomposition in* (1) *is unique such that* $x = \pi_M(x) + x_2$. *In this case, we write* $X = M \dot{+} F_X^{-1}(M^\perp)$.

Where $F_X^{-1}(M^\perp) = \{x \in X \mid F_X(x) \cap M^\perp \neq \emptyset\}$.

Lemma 2.6. [13, Theorem 1.2.9] *Let X be a Banach spaces, $M \subset X$ be a subspace. Let $x \in X \backslash \overline{M}$, $x_0 \in M$. Then $x_0 \in P_M(x)$ iff $F_X(x - x_0) \cap M^\perp \neq \emptyset$.*

The following definition of the Moore–Penrose metric generalized inverse comes from [13, 16].

Definition 2.7. [13, 16] Let $T \in B(X, Y)$. Assume that $R(T)$ and $N(T)$ are Chebyshev subspaces. If there is a bounded homogeneous operator T^M such that

(1) $TT^M T = T$, (2) $T^M TT^M = T^M$,

(3) $TT^M = \pi_{R(T)}$, (4) $T^M T = I - \pi_{N(T)}$,

then T^M is called the Moore–Penrose metric generalized inverse of T.

By Definition 2.7 and Lemma 2.5, if T^M exists, then the space X, Y have the following unique decompositions:

$$X = N(T) \dot{+} F_X^{-1}(N(T)^\perp), \quad Y = R(T) \dot{+} F_Y^{-1}(R(T)^\perp).$$

From [13, Theorem 4.3.1], We know if $R(T)$ and $N(T)$ are Chebyshev subspaces, then T^M uniquely exists and

$$T^M(y) = (T|_{F_X^{-1}(N(T)^\perp)})^{-1}\pi_{R(T)}(y), \quad \forall y \in D(T^M) = Y.$$

The gap between homogenous subsets play an important role in the perturbation analysis of the Moore–Penrose metric generalized inverse in this paper. First, we look back the concept of the gap between subspaces and its properties. For convenience, we denote $\overline{M}$ the closure of the subspace (resp. homogeneous subset) $M \subset X$ in the context.

Definition 2.8. [8] Let M, N be the subspaces of Banach space X. Put

$$\delta(M, N) = \begin{cases} \sup\{dist(x, N) \,|\, x \in M, \; \|x\| = 1\}, & M \neq \{0\} \\ 0 & M = \{0\} \end{cases}.$$

We call $\hat{\delta} = \max\{\delta(M, N), \delta(N, M)\}$ the gap between M and N.

From Definition 2.8, we get a useful inequality as following:

$$dist(x, N) \leq \|x\|\delta(M, N), \quad \forall x \in M.$$

Proposition 2.9. [19, Proposition 1.3.2] *Let L, M, N be the subspaces of Banach space X. Then*

(1) $0 \leq \delta(M, N) \leq 1, 0 \leq \hat{\delta}(M, N) \leq 1$ *and* $\hat{\delta}(M, N) = \hat{\delta}(N, M)$.
(2) $\delta(\overline{M}, \overline{N}) = \delta(M, N), \hat{\delta}(\overline{M}, \overline{N}) = \hat{\delta}(M, N)$.
(3) $\delta(M, N) = 0$ *iff* $\overline{M} \subseteq \overline{N}$, $\hat{\delta}(M, N) = 0$ *iff* $\overline{M} = \overline{N}$.
(4) $\delta(L, N) \leq \delta(L, M) + [1 + \delta(L, M)]\delta(M, N)$.
(5) $\hat{\delta}(L, N) \leq 2[\hat{\delta}(L, M) + \hat{\delta}(M, N)]$.

Now we generalize the definition of the gap between subspaces to the gap between homogenous subsets. It is naturally to define the gap between homogenous subsets as:

Definition 2.10. Let M, N be the homogenous subsets. Set

$$\eta(M, N) = \begin{cases} \sup\{dist(x, N) \,|\, x \in M, \; \|x\| = 1\}, & M \neq \{0\} \\ 0 & M = \{0\} \end{cases}.$$

Define the gap between homogenous subsets M and N as

$$\hat{\eta}(M, N) = \max\{\eta(M, N), \eta(N, M)\}.$$

Since M and N are homogeneous subsets, we have $x_0 = \dfrac{x}{\|x\|} \in M, \forall x \in M$ with $\|x_0\| = 1$ and so that

$$dist(x, N) \leq \|x\|\eta(M, N), \quad \forall x \in M.$$

From the definition 2.10, we get the following properties:

Proposition 2.11. *Let M, N be the homogenous subsets. Then*

(1) $0 \leq \eta(M,N) \leq 1$ *if* $0 \in N$ *and* $0 \leq \hat{\eta}(M,N) \leq 1$ *if* $0 \in M$ *and* $0 \in N$.
(2) $0 \leq \eta(M,N) \leq 2$, $0 \leq \hat{\eta}(M,N) \leq 2$ *and* $\hat{\eta}(M,N) = \hat{\eta}(N,M)$.
(3) $\eta(\overline{M}, \overline{N}) = \eta(M,N), \hat{\eta}(\overline{M}, \overline{N}) = \hat{\eta}(M,N)$.
(4) $\eta(M,N) = 0$ *iff* $\overline{M} \subseteq \overline{N}$, $\hat{\eta}(M,N) = 0$ *iff* $\overline{M} = \overline{N}$.

Proof. (1) is obvious.

(2) For any $z \in N$, we have $z_0 = \dfrac{z}{\|z\|} \in N$ with $\|z_0\| = 1$ since N is a homogenous subset. Thus, for any $x \in M$ with $\|x\| = 1$, we have

$$0 \leq dist(x,N) \leq \|x - z_0\| \leq \|x\| + \|z_0\| \leq 2.$$

So, the results follows.

(3) Similar to the proof of [19, Proposition 1.3.2 (2)].

(4) For any $x \in \overline{M}$, $\dfrac{x}{\|x\|} \in \overline{M}$ since M is a homogenous subset. Hence, $\eta(M,N) = 0$ implies that $dist(\dfrac{x}{\|x\|}, N) = 0$ for any $x \in \overline{M}$ by (2). Thus, $\dfrac{x}{\|x\|} \in \overline{N}$ and consequently $\overline{M} \subseteq \overline{N}$ for N is a homogenous subset.

Conversely, if $\overline{M} \subseteq \overline{N}$, then $dist(\dfrac{x}{\|x\|}, N) = 0$ for any $x \in \overline{M}$ and so that $\eta(M,N) = 0$.

Since $\hat{\eta}(M,N) = 0$ iff $\eta(M,N) = \eta(N,M) = 0$, we have $\hat{\eta}(M,N) = 0$ iff $\overline{M} = \overline{N}$ by the above argument. □

In the following, we investigate the properties of the gap between homogenous Chebyshev subsets.

Proposition 2.12. *Let $M \subset X$ be a Chebyshev subspace. Then*

$$\eta(F_X^{-1}(M^{\perp}), M) = 1.$$

Proof. Since M is a Chebyshev subspace, we have $N(\pi_M) = F_X^{-1}(M^{\perp})$ by Lemma 2.5. Thus, for any $x \in F_X^{-1}(M^{\perp})$, $\pi_M(\lambda x) = 0$ for π_M is a homogeneous operator. This show $\lambda x \in F_X^{-1}(M^{\perp})$, i.e., $F_X^{-1}(M^{\perp})$ is a homogenous subset.

For any $x \in F_X^{-1}(M^{\perp})$ with $\|x\| = 1$, by Lemma 2.5, $\pi_M(x) = 0$. Thus,

$$dist(x, M) = \|x - \pi_M(x)\| = \|x\| = 1.$$

This illustrate $\eta(F_X^{-1}(M^{\perp}), M) = 1$. □

Proposition 2.13. *Let X be a Banach space, M, N are homogenous Chebyshev subsets of X. Then*

(1) $\eta(M,N) < 1$ *implies that* $M \cap F_X^{-1}(N^{\perp}) = \{0\}$.
(2) $\dfrac{1}{2}\|(I - \pi_N)\pi_M\| \leq \eta(M,N) \leq \|(I - \pi_N)\pi_M\| \leq 2\|\pi_M - \pi_N\|$ *if* $0 \in M$.

Proof. (1). If $M \cap F_X^{-1}(N^{\perp}) \neq \{0\}$, then there is a $x \in M \cap F_X^{-1}(N^{\perp})$ such that $\|x\| = 1$ since M and $F_X^{-1}(N^{\perp})$ are homogenous subsets. Thus, $\pi_N(x) = 0$. Consequently, $dist(x,N) = \|x - \pi_N(x)\| = \|x\| = 1$. This shows $\eta(M,N) = 1$, which contradicts to the assume that $\eta(M,N) < 1$.

(2). For any $x \in X$,

$$\|(I-\pi_N)\pi_M(x)\| = dist(\pi_M(x), N) \leq \|\pi_M(x)\|\eta(M,N) \leq 2\|x\|\eta(M,N)$$

implies $\frac{1}{2}\|(I-\pi_N)\pi_M\| \leq \eta(M,N)$.

On the other hand, for any $z \in M$ with $\|z\| = 1$, $z = \pi_M(z)$ and so that

$$\begin{aligned} dist(z,N) = \|z - \pi_N(z)\| &= \|\pi_M(z) - \pi_N(\pi_M(z))\| \\ &= \|(I-\pi_N)\pi_M(z)\| \\ &\leq \|(I-\pi_N)\pi_M\|. \end{aligned}$$

This indicates $\eta(M,N) \leq \|(I-\pi_N)\pi_M\| = \|(\pi_M - \pi_N)\pi_M\| \leq 2\|\pi_M - \pi_N\|$. □

Let T be a linear operator. The reduced modulus $\gamma(T)$ of T is defined as

$$\gamma(T) = \inf\{\|Tx\| \mid dist(x, N(T)) = 1, \forall x \in D(T)\}$$

Obviously, $\gamma(T) dist(x, N(T)) \leq \|Tx\|$.

Lemma 2.14. *Let X, Y be Banach spaces and $T \in B(X,Y)$ with $R(T), N(T)$ are Chebyshev subspaces. Then*

$$\frac{1}{\|T^M\|} \leq \gamma(T) \leq \frac{\|TT^M\|}{\|T^M\|}$$

Proof. Since $R(T), N(T)$ are Chebyshev subspaces, T^M exists. Thus, we have

$$dist(x, N(T)) = \|x - \pi_{N(T)}x\| = \|T^M Tx\| \leq \|T^M\|\|Tx\|$$

and so that $\gamma(T) \geq \frac{1}{\|T^M\|}$.

Noting that $dist(x, N(T)) = \|T^M Tx\|$, we have

$$\gamma(T)\|T^M Tx\| = \gamma(T) dist(x, N(T)) \leq \|Tx\|.$$

Hence, for any $y \in Y$,

$$\gamma(T)\|T^M y\| \leq \|TT^M y\|$$

and consequently, $\gamma(T) \leq \frac{\|TT^M\|}{\|T^M\|}$. □

From [19, Lemma 1.3.5], we know

$$\gamma(T)\delta(N(\overline{T}), N(T)) \leq \|\delta T\| \quad \text{and} \quad \gamma(T)\delta(R(T), R(\overline{T})) \leq \|\delta T\|.$$

Associated with Lemma 2.14, we have the following proposition:

Proposition 2.15. *Let X, Y be Banach spaces and $T \in B(X,Y), \overline{T} = T + \delta T \in B(X,Y)$ with $R(T), N(T)$ are Chebyshev subspaces. Then*

$$\delta(R(T), R(\overline{T})) \leq \|\delta T\|\|T^M\| \quad \textit{and} \quad \delta(N(\overline{T}), N(T)) \leq \|\delta T\|\|T^M\|.$$

The linear outer generalized inverse with prescribed range and kernel $A^{(2)}_{T,S}$ is a kind of important generalized inverse. Most linear generalized inverse such as Moore–Penrose inverse, Drazin inverse, group inverse, etc. can be written as $A^{(2)}_{T,S}$ when we chose the suitable subspaces T and S. In [2], J.Cao and Y.Xue first define and characterize the homogeneous (resp. quasi–linear) operator out generalized inverse with prescribed range and kernel $A^{(2,H)}_{T,S}(resp. A^{(2,h)}_{T,S})$, where T and S are homogeneous subsets.

Definition 2.16. [2] Let $A \in B(X,Y)$. T and S are homogeneous subsets of X and Y, respectively. The operator $B \in H(Y,X)$ such that the following equations:

$$BAB = B, \quad R(B) = T, \quad N(B) = S$$

is called the homogeneous outer generalized inverse of A with prescribed range T and kernel S. Denoted by $A^{(2,H)}_{T,S}$. In addition, if B is quasi–additive on AT, then B is called the quasi-linear outer generalized inverse of A with prescribed range T and kernel S. We denoted it by $A^{(2,h)}_{T,S}$.

Lemma 2.17. [2] *Let $A \in B(X,Y)$. T and S are homogeneous subsets of X and Y, respectively. Then $A^{(2,h)}_{T,S}$ exists if and only if $Y = AT \dot{+} S$ and $N(A) \cap T = \{0\}$ and T is closed linear subspace. In addition, if $A^{(2,h)}_{T,S}$ exists, then it is unique.*

Proposition 2.18. *Let $A \in B(X,Y)$ be a linear operator. $N(A), R(A)$ are Chebyshev subspaces. Assume that $\pi_{N(A)}$ is a continuous linear operator. Then*

$$A^M = A^{(2,h)}_{F_X^{-1}(N(A)^\perp), F_Y^{-1}(R(A)^\perp)}.$$

Proof. Since $\pi_{N(A)}$ is continuous linear operator, by Lemma 2.2, $F_X^{-1}(N(A)^\perp)$ is a closed linear subspace.

Since $N(A), R(A)$ are Chebyshev subspaces, by Lemma 2.5,

$$N(A) \cap F_X^{-1}(N(A)^\perp) = \{0\}, \quad \text{and} \quad AF_X^{-1}(N(A)^\perp) \dot{+} F_Y^{-1}(R(A)^\perp) = Y.$$

So, $A^{(2,h)}_{F_X^{-1}(N(A)^\perp), F_Y^{-1}(R(A)^\perp)}$ exists by Lemma 2.17.

For any $y \in Y$, set $y_0 = AA^{(2,h)}_{F_X^{-1}(N(A)^\perp), F_Y^{-1}(R(A)^\perp)} y$. Since $A^{(2,h)}_{F_X^{-1}(N(A)^\perp), F_Y^{-1}(R(A)^\perp)}$ is quasi–additive on $AF_X^{-1}(N(A)^\perp)$, we have

$$y - y_0 \in N(A^{(2,h)}_{F_X^{-1}(N(A)^\perp), F_Y^{-1}(R(A)^\perp)}) = F_Y^{-1}(R(A)^\perp).$$

This shows $F_X(y - y_0) \cap R(A)^\perp \neq \emptyset$. By Lemma 2.6, we have $y_0 \in P_{R(A)}(y)$. Since $R(A)$ is a Chebyshev subspace, we have $y_0 = \pi_{R(A)}(y)$, that is,

$$AA^{(2,h)}_{F_X^{-1}(N(A)^\perp), F_Y^{-1}(R(A)^\perp)} y = \pi_{R(A)}(y).$$

For any $x \in X$, set $x_0 = A^{(2,h)}_{F_X^{-1}(N(A)^\perp), F_Y^{-1}(R(A)^\perp)} Ax$

$$x - (x - x_0) = A^{(2,h)}_{F_X^{-1}(N(A)^\perp), F_Y^{-1}(R(A)^\perp)} Ax \in F_X^{-1}(N(A)^\perp).$$

So, $F_Y(x-(x-x_0))\cap N(A)^\perp \neq \emptyset$. By Lemma 2.6, we have $x-x_0=\pi_{N(A)}x$ and so that

$$A^{(2,h)}_{F_X^{-1}(N(A)^\perp),F_Y^{-1}(R(A)^\perp)}Ax = x-\pi_{N(A)}x.$$

It is easy to verify

$$AA^{(2,h)}_{F_X^{-1}(N(A)^\perp),F_Y^{-1}(R(A)^\perp)}Ax = Ax,$$

and

$$A^{(2,h)}_{F_X^{-1}(N(A)^\perp),F_Y^{-1}(R(A)^\perp)}AA^{(2,h)}_{F_X^{-1}(N(A)^\perp),F_Y^{-1}(R(A)^\perp)}y = A^{(2,h)}_{F_X^{-1}(N(A)^\perp),F_Y^{-1}(R(A)^\perp)}y.$$

Consequently, $A^M = A^{(2,h)}_{F_X^{-1}(N(A)^\perp),F_Y^{-1}(R(A)^\perp)}$. □

Proposition 2.19. *Let $A\in B(X,Y)$ with $R(A), N(A)$ are Chebyshev subspaces. Then $A^M=(I-\pi_{N(A)})A^-\pi_{R(A)}$ for any $A^-\in B(Y,X)$ which satisfy $AA^-A=A$.*

Proof. Since $R(A), N(A)$ are Chebyshev subspaces, we have A^M exists and

$$AA^M=\pi_{R(A)},\quad A^MA=I_X-\pi_{N(A)}.$$

Set $B=(I-\pi_{N(A)})A^-\pi_{R(A)}$. Obviously, B is a bounded homogeneous operator and

$$B=(I-\pi_{N(A)})A^-\pi_{R(A)}=A^MAA^-AA^M=A^M.$$

□

3. The perturbation analysis of the Moore–Penrose metric generalized inverse

In this section, we consider the perturbation for the Moore–Penrose metric generalized inverse of bounded linear operators on Banach space. In virtue of the gap between homogenous subsets, we obtain some new results.

Theorem 3.1. *Let X,Y be reflexive strictly convex Banach spaces. $T\in B(X,Y)$ with $R(T)$ closed and $\overline{T}=T+\delta T\in B(X,Y)$. Assume that $R(T)=R(\overline{T})$ and T^M is quasi–additive on $R(T)$. If*

$$\eta(R(\overline{T}^M),R(T^M))<\frac{1-\|T^M\|\|\delta T\|}{1+\|T\|\|T^M\|}$$

then

(1)

$$\frac{\|\overline{T}^M-T^M\|}{\|T^M\|}\leq\frac{\|T^M\|\|\delta T\|+(1+\|T\|\|T^M\|)\eta(R(\overline{T}^M),R(T^M))}{1-\|T^M\|\|\delta T\|-(1+\|T\|\|T^M\|)\eta(R(\overline{T}^M),R(T^M))},$$

(2)

$$\|\overline{T}^M\|\leq\frac{\|T^M\|}{1-\|T^M\|\|\delta T\|-(1+\|T\|\|T^M\|)\eta(R(\overline{T}^M),R(T^M))}.$$

Proof. Since X, Y are reflexive strictly convex Banach spaces, $R(T) = R(\overline{T})$, $N(T), N(\overline{T})$ are Chebyshev subspaces. Thus, $T^M, \overline{T}^M$ exist and $R(T^M), R(\overline{T}^M)$ are homogeneous subsets.

Since $R(T) = R(\overline{T})$ closed, we have

$$D(T^M) = D(\overline{T}^M) = R(T) \dot{+} F_Y^{-1}(R(T)^\perp) = Y.$$

Let $W = \overline{T}^M - T^M$. For any $\xi \in D(T^M) = D(\overline{T}^M) = R(T) \dot{+} F_Y^{-1}(R(T)^\perp) = Y$, there exist $u \in R(\overline{T}) = R(T), u'$ such that $\xi = u + u'$. Thus, $u = \overline{T}\overline{T}^M y$ for some $y \in D(\overline{T}^M)$. Since

$$dist(\overline{T}^M y, R(T^M)) \leq \|\overline{T}^M y\| \eta(R(\overline{T}^M), R(T^M)),$$

for any $\epsilon > 0$, there is $y' \in D(T^M)$ such that

$$\|\overline{T}^M y - T^M y'\| \leq dist(\overline{T}^M y, R(T^M)) + \epsilon \leq \|\overline{T}^M y\| \eta(R(\overline{T}^M), R(T^M)) + \epsilon.$$

Let $v = TT^M y' \in R(T)$, then

$$\begin{aligned}\|\overline{T}\overline{T}^M y - TT^M y'\| &= \|T\overline{T}^M y - TT^M y' + \delta T\overline{T}^M y\| \\ &\leq \|T\overline{T}^M y - TT^M y'\| + \|\delta T\overline{T}^M y\| \\ &\leq \|T\|\|\overline{T}^M y\| \eta(R(\overline{T}^M), R(T^M)) + \epsilon\|T\| + \|\delta T\|\|\overline{T}^M y\|.\end{aligned}$$

Noting that $N(T^M) = F_Y^{-1}(R(T)^\perp) = F_Y^{-1}(R(\overline{T})^\perp) = N(\overline{T}^M)$ and T^M is quasi–additive on $R(T)$, we have

$$\begin{aligned}&\|W\xi\| = \|\overline{T}^M \xi - T^M \xi\| = \|\overline{T}^M u - T^M u\| \\ &\leq \|\overline{T}^M u - T^M v\| + \|T^M u - T^M v\| \\ &\leq \|\overline{T}^M y - T^M y'\| + \|T^M\|\|\overline{T}\overline{T}^M y - TT^M y'\| \\ &\leq \|\overline{T}^M y\|\{\|T^M\|\|\delta T\| + (1 + \|T\|\|T^M\|)\eta(R(\overline{T}^M), R(T^M))\} + \epsilon(1 + \|T\|\|T^M\|).\end{aligned}$$

Since

$$\|\overline{T}^M y\| = \|\overline{T}^M u\| = \|W\xi + T^M \xi\| \leq \|W\xi\| + \|T^M\|\|\xi\|,$$

it follows that

$$\begin{aligned}\|W\xi\| \leq (\|W\xi\| + \|T^M\|\|\xi\|)\{\|T^M\|\|\delta T\| \\ + (1 + \|T\|\|T^M\|)\eta(R(\overline{T}^M), R(T^M))\} + \epsilon(1 + \|T\|\|T^M\|).\end{aligned}$$

Therefore, by letting $\epsilon \to 0^+$,

$$\|W\xi\| \leq \frac{\|T^M\|\|\delta T\| + (1 + \|T\|\|T^M\|)\eta(R(\overline{T}^M), R(T^M))}{1 - \|T^M\|\|\delta T\| - (1 + \|T\|\|T^M\|)\eta(R(\overline{T}^M), R(T^M))}\|T^M\|\|\xi\|.$$

Consequently,

$$\|\overline{T}^M - T^M\| \leq \frac{\|T^M\|\|\delta T\| + (1 + \|T\|\|T^M\|)\eta(R(\overline{T}^M), R(T^M))}{1 - \|T^M\|\|\delta T\| - (1 + \|T\|\|T^M\|)\eta(R(\overline{T}^M), R(T^M))}\|T^M\|.$$

Furthermore,

$$\begin{aligned}\|\overline{T}^M\| &= \|W + T^M\| \le \|W\| + \|T^M\| \\ &\le \frac{\|T^M\|\|\delta T\| + (1+\|T\|\|T^M\|)\eta(R(\overline{T}^M), R(T^M))}{1-\|T^M\|\|\delta T\| - (1+\|T\|\|T^M\|)\eta(R(\overline{T}^M), R(T^M))}\|T^M\| + \|T^M\| \\ &= \frac{\|T^M\|}{1-\|T^M\|\|\delta T\| - (1+\|T\|\|T^M\|)\eta(R(\overline{T}^M), R(T^M))}.\end{aligned}$$

□

Theorem 3.2. *Let X, Y be reflexive strictly convex Banach spaces. $T \in B(X, Y)$ with $R(T)$ closed and $\overline{T} = T + \delta T \in B(X, Y)$. Assume that $N(T) = N(\overline{T})$ and T^M is quasi-additive on $R(T)$ and $R(\delta T)$. If*

$$\eta(N(\overline{T}^M), N(T^M)) < \frac{1-\|T^M\|\|\delta T\|}{\|\overline{T}\|\|T^M\|}$$

then

(1)

$$\frac{\|\overline{T}^M - T^M\|}{\|T^M\|} \le \frac{\left\{1+\|T^M\|\|\overline{T}\|\right\}\eta(N(\overline{T}^M), N(T^M)) + \|T^M\|\|\delta T\|}{1-\|T^M\|\|\delta T\| - \|T^M\|\|\overline{T}\|\eta(N(\overline{T}^M), N(T^M))}.$$

(2)

$$\|\overline{T}^M\| \le \frac{\|T^M\| + \|T^M\|\eta(N(\overline{T}^M), N(T^M))}{1-\|T^M\|\|\delta T\| - \|T^M\|\|\overline{T}\|\eta(N(\overline{T}^M), N(T^M))}.$$

Proof. Let $W = \overline{T}^M - T^M$ and $B' = I - \overline{T}\overline{T}^M$, $B = I - TT^M$. Since $N(T) = N(\overline{T})$, we have

$$R(\overline{T}^M) = F_X^{-1}(N(\overline{T})^\perp) = F_X^{-1}(N(T)^\perp) = R(T^M).$$

Since T^M is quasi-additive on $R(T)$ and $R(\delta T)$, it follows that

$$\begin{aligned}W = \overline{T}^M - T^M &= T^M T\overline{T}^M - T^M TT^M \\ &= T^M\overline{T}\overline{T}^M - T^M\delta T\overline{T}^M - T^M TT^M \\ &= T^M(B - B') - T^M\delta T\overline{T}^M.\end{aligned}$$

Since $B'\xi \in N(\overline{T}^M)$ for any $\xi \in Y$, we have

$$dist(B'\xi, N(T^M)) \le \|B'\xi\|\eta(N(\overline{T}^M), N(T^M)).$$

Thus, for any $\epsilon > 0$, there is a $u \in Y$ such that

$$\|B'\xi - Bu\| \le \|B'\xi\|\eta(N(\overline{T}^M), N(T^M)) + \epsilon$$

and so that

$$\|T^M(B'\xi - Bu)\| \le \|T^M\|\|B'\xi\|\eta(N(\overline{T}^M), N(T^M)) + \|T^M\|\epsilon.$$

Noting that $T^M B = 0$, we have

$$\begin{aligned}\|W\xi\| &= \|T^M(B-B')\xi - T^M\delta T\overline{T}^M\xi\| \\ &\le \|T^M B'\xi\| + \|T^M\delta T\overline{T}^M\xi\| \\ &= \|T^M(B'\xi - Bu)\| + \|T^M\delta T\overline{T}^M\xi\| \\ &\le \|T^M\|\|B'\xi\|\eta(N(\overline{T}^M), N(T^M)) + \|T^M\|\epsilon + \|T^M\|\|\delta T\|\|\overline{T}^M\xi\|.\end{aligned}$$

But $\|B'\xi\| \le \|\xi\| + \|\overline{T}\|\|\overline{T}^M\xi\|$ and $\|\overline{T}^M\xi\| \le \|W\xi\| + \|T^M\xi\|$. Thus,

$$\begin{aligned}\|W\xi\| &\le \|T^M\|\|B'\xi\|\eta(N(\overline{T}^M), N(T^M)) + \|T^M\|\epsilon + \|T^M\|\|\delta T\|\|\overline{T}^M\xi\| \\ &\le \|T^M\|\eta(N(\overline{T}^M), N(T^M))\|\xi\| \\ &+ \|\overline{T}^M\xi\|\|T^M\|\{\|\overline{T}\|\eta(N(\overline{T}^M), N(T^M)) + \|\delta T\|\} + \|T^M\|\epsilon \\ &\le \|T^M\|\eta(N(\overline{T}^M), N(T^M))\|\xi\| \\ &+ \{\|T^M\xi\| + \|W\xi\|\}\|T^M\|\{\|\overline{T}\|\eta(N(\overline{T}^M), N(T^M)) + \|\delta T\|\} + \|T^M\|\epsilon.\end{aligned}$$

By $\epsilon \to 0^+$, we have

$$\|W\xi\| \le \frac{\eta(N(\overline{T}^M), N(T^M))\|\xi\| + \|T^M\xi\|\{\|\overline{T}\|\eta(N(\overline{T}^M), N(T^M)) + \|\delta T\|\}}{1 - \|T^M\|\|\delta T\| - \|T^M\|\|\overline{T}\|\eta(N(\overline{T}^M), N(T^M))}\|T^M\|.$$

Consequently,

$$\|\overline{T}^M - T^M\| \le \frac{\Big\{1 + \|T^M\|\|\overline{T}\|\Big\}\eta(N(\overline{T}^M), N(T^M)) + \|T^M\|\|\delta T\|}{1 - \|T^M\|\|\delta T\| - \|T^M\|\|\overline{T}\|\eta(N(\overline{T}^M), N(T^M))}\|T^M\|.$$

Furthermore,

$$\|\overline{T}^M\| \le \frac{\|T^M\| + \|T^M\|\eta(N(\overline{T}^M), N(T^M))}{1 - \|T^M\|\|\delta T\| - \|T^M\|\|\overline{T}\|\eta(N(\overline{T}^M), N(T^M))}.$$

□

In order to estimate the upper bounds of $\|\overline{T}^M\|$ and $\|\overline{T}^M - T^M\|$ for the general case, we need to construct an operator B such that $R(B) = R(T)$ and $N(B) = N(\overline{T})$.

Theorem 3.3. *Let X, Y be reflexive strictly convex Banach spaces. Let $T \in B(X,Y)$, $\overline{T} = T + \delta T \in B(X,Y)$ with $R(T), R(\overline{T})$ closed and T^M is quasi-additive on $R(T)$ and $R(\delta T)$. Set $\delta_1 = \eta(R(\overline{T}^M), R(T^M))$, $\delta_2 = \eta(N(\overline{T}^M), N(T^M))$ and $\kappa = \|T\|\|T^M\|$. Assume that $R(\overline{T}) \cap N(T^M) = \{0\}$ and $\|T^M\|\|\delta T\| \le \dfrac{\kappa}{3(1+\kappa)}$. If*

$$\delta_1 < \frac{1}{(1+\kappa)^2} \quad \textit{and} \quad \delta_2 < \frac{\|T\| - 3(1+\kappa)\|\delta T\|}{\|\overline{T}\|(1+\kappa)},$$

then

$$\|\overline{T}^M\| \le \frac{1+\delta_2}{1 - \rho\{\|\delta T\| - \|\overline{T}\|\delta_2\}}\rho$$

and

$$\|\overline{T}^M - T^M\| \le \Big\{\frac{\delta_2 + \rho\Big\{\|\overline{T}\|\delta_2 + \|\delta T\|\Big\}}{1-\rho\Big\{\|\overline{T}\|\delta_2 + \|\delta T\|\Big\}} + 2\|T^M\|\|\delta T\| + (1+\kappa)\delta_1\Big\}\rho.$$

Here,

$$\rho = \frac{\|T^M\|}{1 - 2\|T^M\|\|\delta T\| - (1+\kappa)\delta_1}.$$

Proof. Let $B = \pi_{R(T)}\overline{T}$. Since T^M is quasi-additive on $R(\delta T)$, it follows that $\pi_{R(T)}\delta T$ is linear operator and $B = T + \pi_{R(T)}\delta T \in B(X, Y)$. Clearly, $R(B) = R(T)$. For any $x \in N(B)$, $TT^M\overline{T}x = \pi_{R(T)}\overline{T}x = 0$ indicates $\overline{T}x \in R(\overline{T}) \cap N(T^M) = \{0\}$. Therefore $N(B) \subseteq N(\overline{T})$ and consequently $N(B) = N(\overline{T})$. Since $R(T)$ and $N(\overline{T})$ are Chebyshev subspaces, B^M exists.

By Defintion 2.7,we have

$$BB^M = \pi_{R(B)} = \pi_{R(T)} = TT^M$$

and

$$B^M B = I - \pi_{N(B)} = I - \pi_{N(\overline{T})} = \overline{T}^M\overline{T}.$$

Consequently, $R(B^M) = R(\overline{T}^M)$ and $N(B^M) = N(T^M)$.

Put $\delta_1 = \eta(R(\overline{T}^M), R(T^M)) = \eta(R(B^M), R(T^M))$.

Since $\|T^M\|\|\delta T\| \le \dfrac{\kappa}{3(1+\kappa)} < \dfrac{\kappa}{2(1+\kappa)}$ and

$$\delta_1 < \frac{1}{(1+\kappa)^2} \le \frac{1 - 2\|T^M\|\|\delta T\|}{1+\kappa} \le \frac{1 - \|T^M\|\|\pi_{R(T)}\delta T\|}{1+\kappa},$$

by Theorem 3.1, we have

$$\begin{aligned}\|B^M - T^M\| &\le \frac{\|T^M\|\|\pi_{R(T)}\delta T\| + (1+\kappa)\delta_1}{1 - \|T^M\|\|\pi_{R(T)}\delta T\| - (1+\kappa)\delta_1}\|T^M\| \\ &\le \frac{2\|T^M\|\|\delta T\| + (1+\kappa)\delta_1}{1 - 2\|T^M\|\|\delta T\| - (1+\kappa)\delta_1}\|T^M\|\end{aligned}$$

and

$$\begin{aligned}\|B^M\| &\le \frac{\|T^M\|}{1 - \|T^M\|\|\pi_{R(T)}\delta T\| - (1+\kappa)\delta_1} \\ &\le \frac{\|T^M\|}{1 - 2\|T^M\|\|\delta T\| - (1+\kappa)\delta_1}.\end{aligned}$$

Let $\delta_2 = \eta(N(\overline{T}^M), N(T^M)) = \eta(N(\overline{T}^M), N(B^M))$.

Noting that $\overline{T} = B + (I - \pi_{R(T)})\delta T$ and

$$\delta_2 < \frac{\|T\| - 3(1+\kappa)\|\delta T\|}{\|\overline{T}\|(1+\kappa)} = \frac{1 - 3\|T^M\|\|\delta T\| - \dfrac{1}{1+\kappa}}{\|\overline{T}\|\|T^M\|}$$
$$< \frac{1 - 3\|T^M\|\|\delta T\| - (1+\kappa)\delta_1}{\|\overline{T}\|\|T^M\|}$$
$$< \frac{1 - \|B^M\|\|\delta T\|}{\|\overline{T}\|\|B^M\|}$$
$$\leq \frac{1 - \|B^M\|\|(I - \pi_{R(T)})\delta T\|}{\|\overline{T}\|\|B^M\|},$$

by Theorem 3.2,

$$\|\overline{T}^M - B^M\| \leq \frac{\Big\{1 + \|B^M\|\|\overline{T}\|\Big\}\delta_2 + \|B^M\|\|(I - \pi_{R(T)})\delta T\|}{1 - \|B^M\|\|(I - \pi_{R(T)})\delta T\| - \|B^M\|\|\overline{T}\|\delta_2}\|B^M\|$$
$$\leq \frac{\Big\{1 + \|B^M\|\|\overline{T}\|\Big\}\delta_2 + \|B^M\|\|\delta T\|}{1 - \|B^M\|\|\delta T\| - \|B^M\|\|\overline{T}\|\delta_2}\|B^M\|,$$

and

$$\|\overline{T}^M\| \leq \frac{\|B^M\| + \|B^M\|\delta_2}{1 - \|B^M\|\|(I - \pi_{R(T)})\delta T\| - \|B^M\|\|\overline{T}\|\delta_2}$$
$$\leq \frac{\|B^M\| + \|B^M\|\delta_2}{1 - \|B^M\|\|\delta T\| - \|B^M\|\|\overline{T}\|\delta_2}.$$

For convenience, we set $\rho = \dfrac{\|T^M\|}{1 - 2\|T^M\|\|\delta T\| - (1+\kappa)\delta_1}$. Thus, $\|B^M\| \leq \rho$ and

$$\|\overline{T}^M - T^M\| = \|\overline{T}^M - B^M + B^M - T^M\|$$
$$\leq \|\overline{T}^M - B^M\| + \|B^M - T^M\|$$
$$\leq \frac{\Big\{1 + \|B^M\|\|\overline{T}\|\Big\}\delta_2 + \|B^M\|\|\delta T\|}{1 - \|B^M\|\|\delta T\| - \|B^M\|\|\overline{T}\|\delta_2}\|B^M\| + \frac{2\|T^M\|\|\delta T\| + (1+\kappa)\delta_1}{1 - 2\|T^M\|\|\delta T\| - (1+\kappa)\delta_1}\|T^M\|$$
$$\leq \frac{\delta_2 + \|B^M\|\Big\{\|\overline{T}\|\delta_2 + \|\delta T\|\Big\}}{1 - \|B^M\|\Big\{\|\overline{T}\|\delta_2 + \|\delta T\|\Big\}}\|B^M\| + \Big\{2\|T^M\|\|\delta T\| + (1+\kappa)\delta_1\Big\}\rho$$
$$\leq \left\{\frac{\delta_2 + \rho\Big\{\|\overline{T}\|\delta_2 + \|\delta T\|\Big\}}{1 - \rho\Big\{\|\overline{T}\|\delta_2 + \|\delta T\|\Big\}} + 2\|T^M\|\|\delta T\| + (1+\kappa)\delta_1\right\}\rho.$$

Furthermore,

$$\|\overline{T}^M\| \leq \frac{\|B^M\| + \|B^M\|\delta_2}{1 - \|B^M\|\|\delta T\| - \|B^M\|\|\overline{T}\|\delta_2} \leq \frac{1+\delta_2}{1 - \rho\{\|\delta T\| - \|\overline{T}\|\delta_2\}}\rho.$$

□

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Department of Mathematics, Southeast University, Nanjing, 211100, Jiangsu Province, P. R. China;
Department of Mathematics, School of mathematical & Physical Sciences, Xuzhou Institute of Technology, Xuzhou, 221008, Jiangsu Province, P. R. China.
E-mail address: jsdfp@163.com