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RECENT DEVELOPMENTS OF THE CONDITIONAL STABILITY OF THE HOMOMORPHISM EQUATION

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Abstract. The issue of Ulam's type stability of an equation is understood in the following way: when a mapping which satisfies the equation approximately (in some sense), it is "close" to a solution of it. In this expository paper, we present a survey and a discussion of selected recent results concerning such stability of the equations of homomorphisms, focussing especially on some conditional versions of them.

1. Some history

This is an expository paper, but because of the demands of the journal, we had to restrict very significantly the number of references. We apologize to the authors of all the papers that are connected with the subjects considered here, but had to be omitted.

The property of additivity of a mapping is very important (not only in mathematics) and can be described by the following Cauchy functional equation

$$f(x+y) = f(x) + f(y), \tag{1.1}$$

where f is a mapping between semigroups endowed with binary operations denoted by $+$ (we abuse the notation and we use the same symbol for operations in two different structures). Every mapping satisfying equation (1.1) is said to be

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additive. The equation has drawn attention of researchers for a quite long time. Cauchy [32] proved in 1821 that every continuous solution $f\colon \mathbb{R} \to \mathbb{R}$ of (1.1) must be of the form $f(x) \equiv cx$ for some fixed $c \in \mathbb{R}$. In 1905, Hamel [60] used the so called Hamel basis to provide a description of its discontinuous solutions, and Ostrowski [83] proved that the functions constructed by Hamel cannot be Lebesgue measurable. Moreover, it is well known that if an additive function $f\colon \mathbb{R} \to \mathbb{R}$ is bounded on a set of positive inner Lebesgue measure, then it is linear (cf. [72]). There are numerous further important outcomes of similar type proved for this functional equation, also for functions with more general and/or abstract domains and ranges.

One of interesting questions is the following: *When is it true that a mapping satisfying equation* (1.1) *approximately (in some sense) must be close to an exact solution of* (1.1)*, i.e., to an additive function?*

Such a problem (in a more general form) was posed for the first time by Ulam (cf., e.g., [62, 115]) in 1940 during his talk before a Mathematical Colloquium at the University of Wisconsin. Namely, he asked about the following issue.

Given a group G_1, a metric group (G_2, d) and a positive number ε, does there exist a number $\delta > 0$ such that if a mapping $f\colon G_1 \to G_2$ satisfies the inequality

$$d(f(xy), f(x)f(y)) \le \delta, \qquad x, y \in G_1,$$

then there exists a homomorphism $T\colon G_1 \to G_2$ such that

$$d(f(x), T(x)) \le \varepsilon, \qquad x \in G_1\,?$$

The first partial answer to Ulam's problem was published by Hyers [62] in 1941 (in the context of Banach spaces) with $\delta = \varepsilon$ in the following form.

Suppose that E_1, E_2 are Banach spaces, $f\colon E_1 \to E_2$ is a mapping, $\varepsilon > 0$ and

$$\|f(x+y) - f(x) - f(y)\| \le \varepsilon, \qquad x, y \in E_1.$$

Then there is a unique additive mapping $T\colon E_1 \to E_2$, defined by

$$T(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}, \qquad x, y \in E_1,$$

such that

$$\|f(x) - T(x)\| \le \varepsilon, \qquad x \in E_1.$$

However, it seems that the question of Ulam was somehow anticipated in [88], where the subsequent outcome was presented.

For every real sequence $(a_n)_{n\in\mathbb{N}}$ with

$$\sup_{n,m\in\mathbb{N}} |a_{n+m} - a_n - a_m| \le 1,$$

there is a real number ω such that

$$\sup_{n\in\mathbb{N}} |a_n - \omega n| \le 1.$$

Moreover, $\omega = \lim_{n\to\infty} a_n/n$.

Ulam's problem has inspired numerous other authors (for details and further references see [16, 28, 27, 63, 67]); in particular Bourgin, who presented in [15] some remarks concerning approximately additive mappings and Aoki [5], who extended Hyers' theorem in the following way.

Let E_1 and E_2 be real normed spaces, E_2 be complete, $\varepsilon \geq 0$, $p \in [0,1)$ and $f\colon E_1 \to E_2$ be a mapping with

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon(\|x\|^p + \|y\|^p), \qquad x, y \in E_1. \tag{1.2}$$

Then there is a unique additive mapping $T\colon E_1 \to E_2$ such that

$$\|f(x) - T(x)\| \leq \frac{2\varepsilon}{2-2^p}\|x\|^p, \qquad x \in E_1. \tag{1.3}$$

Nearly thirty years later, Th.M. Rassias [90], independently, published a result resembling that of Aoki and concerning linear mappings. Namely, he proved the following.

Let E_1 and E_2 be real normed spaces with E_2 complete and $f\colon E_1 \to E_2$ be such that the mapping $t \mapsto f(tx)$ is continuous on $\mathbb{R}$ for each fixed $x \in E_1$. Assume that there exist $\varepsilon \geq 0$ and $p \in [0,1)$ with

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon(\|x\|^p + \|y\|^p)$$

for all $x, y \in E_1$. Then there exists a unique linear mapping $T\colon E_1 \to E_2$ such that

$$\|f(x) - T(x)\| \leq \frac{2\varepsilon}{2-2^p}\|x\|^p$$

for all $x \in E_1$.

It is easily seen that the latter outcome follows at once from that of Aoki. Namely, it is enough to notice that the continuity of the mapping $t \mapsto f(tx)$ and (1.3) imply that the mapping $t \mapsto T(tx)$ is bounded on a nontrivial real interval and, consequently, T must be linear, because it is additive (see, e.g., [2]). Actually, it is enough to assume that, for each fixed $x \in E_1$, the mapping $t \mapsto f(tx)$ is continuous at least at one point (or bounded, above or below, on a set with a positive inner Lebesgue measure).

It is well known that the reasonings given in [5, 90] also work in the case $p < 0$, if we assume that $\|0\|^p = \infty$. In 1990, during the 27th International Symposium on Functional Equations, Th.M. Rassias [91] asked the question whether his theorem can also be proved for $p \geq 1$. In 1991, Gajda [49] gave an affirmative solution to this question for $p > 1$ by following the same approach as in Rassias' paper [90]. He actually used the Hyers method, but with T defined by

$$T(x) := \lim_{n\to\infty} 2^n f(2^{-n}x).$$

It also was proved by Gajda [49] (cf., Th.M. Rassias and Šemrl [92]) that one cannot prove a theorem of that type when $p = 1$. In 1994, Găvruta [52] provided a generalization of the Aoki-Rassias result; he replaced the bound $\varepsilon(\|x\|^p + \|y\|^p)$ in (1.2) by a general control function $\varphi(x, y)$ and considered the inequality

$$\|f(x+y) - f(x) - f(y)\| \leq \varphi(x, y). \tag{1.4}$$

Namely, he proved the following.

Theorem 1.1. *Suppose $(G,+)$ is an abelian group, E is a Banach space and the so-called admissible control function $\varphi\colon G\times G\to[0,\infty)$ satisfies*

$$\widetilde{\varphi}(x,y):=\frac{1}{2}\sum_{n=0}^{\infty}2^{-n}\varphi(2^n x,2^n y)<\infty, \qquad x,y\in G.$$

If $f\colon G\to E$ fulfils (1.4) for all $x,y\in G$, then there exists a unique additive mapping $T\colon G\to E$ such that $\|f(x)-T(x)\|\le\widetilde{\varphi}(x,x)$ for all $x\in G$.

All the results presented so far suggest the following formal definition.

Definition 1.2. Let $(A,+)$ and $(S,+)$ be semigroups, d be a metric in S, $\mathcal{E}\subset\mathcal{C}\subset\mathbb{R}_+^{A^2}$ be nonempty, and $\mathcal{T}$ be an operator mapping $\mathcal{C}$ into $\mathbb{R}_+^{A}$. We say that the Cauchy equation (1.1) is $(\mathcal{E},\mathcal{T})$−stable provided for every $\varepsilon\in\mathcal{E}$ and $\varphi_0\in S^A$ with

$$d\big(\varphi_0(x+y),\varphi_0(x)+\varphi_0(y)\big)\le\varepsilon(x,y), \qquad x,y\in A, \tag{1.5}$$

there exists a solution $\varphi\in S^A$ of equation (1.1) such that

$$d\big(\varphi(x),\varphi_0(x)\big)\le\mathcal{T}\varepsilon(x), \qquad x\in A. \tag{1.6}$$

Roughly speaking, $(\mathcal{E},\mathcal{T})$−stability of equation (1.1) means that every approximate (in the sense of (1.5)) solution of (1.1) is always close (in the sense of (1.6)) to an exact solution of (1.1).

If $\mathcal{C}=\mathcal{E}$ consists only of all the constant functions in $\mathbb{R}_+^{A^2}$ and $\mathcal{T}$ takes values only in the family of constant functions in $\mathbb{R}_+^{A}$, then (sometimes under some additional conditions) the $(\mathcal{E},\mathcal{T})$−stability is quite often called the Hyers-Ulam stability.

Clearly, according to Definition 1.2, we can say that Theorem 1.1 states that the Cauchy equation is $(\mathcal{E},\mathcal{T})$−stable with

$$\mathcal{E}:=\Big\{\varphi\in E^G:\sum_{n=0}^{\infty}2^{-n}\varphi(2^n x,2^n y)<\infty, \quad x,y\in G\Big\},$$

$$\mathcal{T}\varphi(x):=\frac{1}{2}\sum_{n=0}^{\infty}2^{-n}\varphi(2^n x,2^n x), \qquad \varphi\in\mathcal{E}, x\in G.$$

Jun et al. [66] obtained some stability results for the so-called pexiderized Cauchy equation $f(x+y)=g(x)+h(y)$.

Another significant version of stability is the conditional stability (sometimes called also *stability on the restricted domain* or *stability of conditional equations*), in which, for example, inequality (1.4) is assumed to hold for (x,y) belonging to a subset D of X^2, where X is the domain of f.

Modifying slightly the proof of Theorem 1.1 (e.g., arguing analogously as in the proof of Theorem 3.4), we can obtain the following version of it on a restricted domain (with $D=\{(x,y)\in W^2: x+y\in W\}$).

Theorem 1.3. *Suppose $(G,+)$ is an abelian semigroup, $W \subset G$ is nonempty, $2W \subset W$, E is a Banach space and $\varphi\colon G\times G \to [0,\infty)$ satisfies*

$$\widetilde{\varphi}(x,y) := \frac{1}{2}\sum_{n=0}^{\infty} 2^{-n}\varphi(2^n x, 2^n y) < \infty, \qquad x,y \in W, x+y \in W. \tag{1.7}$$

If $f\colon G \to E$ fulfils (1.4) for all $x,y \in W$ with $x+y \in W$, then there exists a unique mapping $T\colon W \to E$ such that $T(x+y) = T(x)+T(y)$ for all $x,y \in W$ with $x+y \in W$ and $\|f(x)-T(x)\| \le \widetilde{\varphi}(x,x)$ for $x \in W$.

We say that Cauchy functional equation is satisfied on a sphere if for all x and y from a normed space X

$$\|x\| = \|y\| \quad \Longrightarrow \quad f(x+y) = f(x)+f(y). \tag{1.8}$$

In this case we have $D = \{(x,y) \in X^2 : \|x\| = \|y\|\}$. Alsina and Garcia-Roig in [3] considered this equation in case where $f\colon X \to Y$ is a continuous mapping from a real inner product space X with $\dim X \ge 2$ into a real topological space Y. Szabó in [112] proved that if X is a real normed space with $\dim X \ge 3$, and $(Y,+)$ is an abelian group, then $f\colon X \to Y$ satisfies conditional Cauchy equation (1.8) if and only if f is additive.

Further, we say that a function f is orthogonally additive if for any x,y from the domain of f,

$$x \perp y \quad \Longrightarrow \quad f(x+y) = f(x)+f(y). \tag{1.9}$$

Initially, we consider the orthogonality relation $\perp$ as the one defined in an inner product space. However, one can define various notions of orthogonality, in arbitrary normed spaces, playing a fundamental role in geometry of normed spaces. Many mathematicians have introduced different types of orthogonality for normed linear spaces, cf. [4]. Suppose $(X,\|.\|)$ is a real normed linear space whose dimension is at least two. In 1934, Roberts [95] introduced the first orthogonality relation: $x \in X$ is said to be orthogonal in the sense of Roberts to $y \in X$ ($x \perp_R y$) if $\|x+ty\| = \|x-ty\|$ for all $t \in \mathbb{R}$. Later, in 1935, Birkhoff [14] introduced one of the most important types of orthogonality: x is said to be Birkhoff orthogonal to y ($x \perp_B y$) if $\|x+ty\| \ge \|x\|$ for all $t \in \mathbb{R}$. In 1945, James [64] introduced the so-called James (isosceles) orthogonality: x is said to be isosceles orthogonal to y ($x \perp_I y$) if $\|x+y\| = \|x-y\|$. Trivially, the implications $\perp_R \Longrightarrow \perp_I$ and $\perp_R \Longrightarrow \perp_B$ hold while each of the reciprocals of these implications holds only in real-valued inner product spaces.

In [59], Gudder and Strawther gave the first axiomatic definition of an abstract orthogonality relation in linear spaces. In what follows, we give a definition proposed by Rätz in [94] (for some historical background and a number of references see Sikorska [108]). We say that a pair $(X,\perp)$ is an orthogonality space if X is a linear space with dimension at least 2 and $\perp$ is a binary relation defined on X and fulfilling four properties:

- $x \perp 0$ and $0 \perp x$ for all $x \in X$;
- if $x,y \in X \setminus \{0\}$ and $x \perp y$, then x and y are linearly independent;
- if $x,y \in X$ and $x \perp y$, then for all $\alpha,\beta \in \mathbb{R}$ we have $\alpha x \perp \beta y$;

- for every 2-dimensional subspace P of X and for every $x \in P$, $\lambda \in [0, \infty)$, there exists $y \in P$ such that $x \perp y$ and $x + y \perp \lambda x - y$.

A normed space with Birkhoff orthogonality is a typical example of an orthogonality space, and on the other hand James orthogonality, since it is not homogeneous, is not an example of a binary relation in such a space.

In [112], Szabó studied also a conditional Cauchy equation of the form

$$\|x + y\| = \|x - y\| \implies f(x + y) = f(x) + f(y). \tag{1.10}$$

It is worth underlying that equation (1.10) is nothing else but equation of orthogonal additivity (1.9) with the James orthogonality relation.

In [57], Ger and Sikorska studied (1.8) and (1.10) in more general structures than it was done in [3] and [112]. They also replaced a norm by an abstract function γ, satisfying given conditions. Namely, the following conditional equations were investigated:

$$\gamma(x) = \gamma(y) \implies f(x + y) = f(x) + f(y), \tag{1.11}$$

$$\gamma(x + y) = \gamma(x - y) \implies f(x + y) = f(x) + f(y). \tag{1.12}$$

All just mentioned examples are conditional Cauchy equations of the form

$$(x, y) \in D \implies f(x + y) = f(x) + f(y), \tag{1.13}$$

where D is a subset of $X \times X$ with some given properties.

Concerning the structure of the target space, we may consider another form of a conditional homomorphism, namely a conditional exponential equation:

$$(x, y) \in D \implies f(x + y) = f(x)f(y).$$

In [20], Brzdęk solved the orthogonally exponential equation in an orthogonality space ($D = \{(x, y) : x \perp y\}$), and in [21] he solved this equation in an normed space with James orthogonality ($D = \{(x, y) : \|x + y\| = \|x - y\|\}$).

Stability problem for the orthogonal additivity appeared first in the paper by Ger and Sikorska [56], being a starting point for a number of stability results obtained for various functional equations defined for orthogonal vectors. Results from [56] were improved and generalized by Fechner and Sikorska in [43] (for some more general investigations, see also [107]). In fact, they showed that if f is a mapping from an orthogonality space X into a real Banach space Y, $\varepsilon > 0$ is given and for all $x, y \in X$ with $x \perp y$ we have $f(x + y) = f(x) + f(y)$, then there exists exactly one orthogonally additive mapping $g\colon X \to Y$ such that for all $x \in X$ there is $\|f(x) - g(x)\| \leq 5\varepsilon$.

Skof [110] studied the stability of the Cauchy equation on an interval, Kominek [69] investigated the equation on an N-dimensional cube in the space $\mathbb{R}^N$, and Sikorska [100, 101] dealt with such stability postulated for orthogonal vectors in a ball centered at the origin. A somewhat more abstract approach to the conditional stability is presented in [116, 117].

The stability of the linear mappings in Banach modules was studied by Park [85]. Moslehian [77] investigated the stability of orthogonal Cauchy equation of Pexider type in the framework of Banach modules over unital Banach algebras. Further, Moslehian and Eshaghi Gordji [39] utilized the notion of module extension to reduce the problem of stability of derivations to that of ring homomorphisms in the context of Banach bimodules over Banach algebras.

Another nice version of stability is the so-called *superstability*. An equation $E(f) = 0$ is said to be superstable if the boundedness of $E(f)$ implies that either f is bounded or f is a solution of the equation. The most famous result on the superstability of the exponential Cauchy functional equation is the following [11]:

Suppose that X is a vector space over the rationals and $f\colon X \to \mathbb{C}$ is a function satisfying $|f(x+y) - f(x)f(y)| \le \delta$ for some $\delta \ge 0$ and all $x, y \in X$. Then either $|f(x)| \le \max\{4, 4\delta\}$ for all $x \in X$, or f is exponential on X.

A very particular case of superstability is hyperstability, considered recently for various equations, e.g., in [10, 23, 24, 25, 87] (for further references see [28]). We discuss that issue in the further parts of the paper in a more detailed way.

During the last decades several stability problems of various functional equations such as the quadratic functional equation

$$f(x + y) + f(x - y) = 2f(x) + 2f(y)$$

(cf., e.g., [78, 76]) the Jensen functional equation

$$2f((x + y)/2) = f(x) + f(y)$$

(cf., e.g., [69, 68]) and their generalizations have been investigated in the spirit of Hyers–Ulam–Aoki–Rassias results (cf., e.g., [16, 28, 27, 36, 63, 67]). They have several applications in Information Theory, Physics, Economy and Social and Behaviour Sciences (see [86]).

In most stability theorems for functional equations, the completeness of the target space of the unknown functions contained in the equation is assumed and essential. Now, we may ask the question whether the stability of a functional equation implies this completeness of the target space. During the 25$^{\text{th}}$ International Symposium on Functional Equations in 1988, this problem was first considered by Schwaiger [97], who proved that if X is a normed space then the stability of Cauchy functional equation (1.1) for functions $f\colon \mathbb{Z} \to X$ implies the completeness of X. Forti and Schwaiger proved in [47] that an analogous statement is valid if the domain of f is an abelian group containing an element of infinite order. In [48], authors prove that a normed space X is complete if there exists a functional equation of the type

$$\sum_{i=1}^{n} a_i f(\varphi_i(x_1, \ldots, x_k)) = 0 \qquad (x_1, \ldots, x_k \in D)$$

with given real numbers $a_1, \ldots, a_n$, given mappings $\varphi_1 \ldots, \varphi_n\colon D^k \to D$ and unknown function $f\colon D \to X$, which has a Hyers–Ulam stability property on an infinite subset D of the integers, see also [81].

2. The main methods of proof

We can actually distinguish four main methods in the investigations of stability of functional equations; the other methods can be considered to be their modifications (cf., [54]). The first method is the *direct method* in which one uses an iteration process producing the so-called *Hyers type sequences* [62]. Another method is based on *sandwich theorems*, which are generalizations of the Hahn–Banach separation theorems (see, e.g., [84]). The foundation of the third method are *fixed point techniques* (see [28, 31, 35]) and the fourth technique uses the *invariant means* (see [113, 114]).

2.1. Direct Method.

The most famous method which has widely been applied to prove the stability of functional equations is the direct method based on an iteration process. We show how to use it in the proof of the next theorem (to obtain the Hyers result on a restricted domain).

Theorem 2.1. *Let $(X,+)$ be a commutative semigroup, $X_0 \subset X$ be a nonempty set such that $2X_0 \subset X_0$, and let Y be a Banach space. Suppose that $f\colon X_0 \to Y$ is a mapping such that*

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon \tag{2.1}$$

for some $\varepsilon > 0$ and for all $x, y \in X_0$ with $x+y \in X_0$. Then there is a unique mapping $T\colon X_0 \to Y$ that is additive on X_0 (i.e., $f(x+y) = f(x) + f(y)$ for $x, y \in X_0$ with $x+y \in X_0$) and such that

$$\sup_{x\in X_0} \|f(x) - T(x)\| \leq \varepsilon.$$

Proof. Putting $y = x$ in (2.1) we have

$$\|f(2x) - 2f(x)\| \leq \varepsilon, \qquad x \in X_0.$$

Using induction, one can show that

$$\|2^{-n}f(2^n x) - 2^{-m}f(2^m x)\| \leq \frac{1}{2}\sum_{k=m}^{n-1} 2^{-k}\varepsilon, \qquad x \in X_0, \tag{2.2}$$

for all nonnegative integers m, n with $n > m$. Hence $\{2^{-n}f(2^n x)\}$ is a Cauchy sequence in Y for each $x \in X_0$. Due to the completeness of Y we conclude that this sequence is convergent. Set $T(x) := \lim_{n\to\infty} 2^{-n}f(2^n x)$ for $x \in X_0$.

Replacing x by $2^n x$ and y by $2^n y$ in (2.1), we obtain

$$\left\|\frac{f(2^n(x+y))}{2^n} - \frac{f(2^n x)}{2^n} - \frac{f(2^n y)}{2^n}\right\| \leq \frac{\varepsilon}{2^n}$$

for all $x, y \in X_0$ with $x+y \in X_0$, whence letting $n \to \infty$ we get that T is additive on X_0. In addition, setting $m = 0$ in (2.2) we arrive at

$$\|f(x) - 2^{-n}f(2^n x)\| \leq \frac{1}{2}\sum_{k=0}^{n-1} 2^{-k}\varepsilon,$$

which with n tending to infinity yields

$$\|f(x) - T(x)\| \leq \varepsilon, \qquad x \in X_0.$$

If $T' : X_0 \to Y$ is also a mapping that is additive on X_0 and fulfils the inequality

$$\sup_{x \in X_0} \|f(x) - T'(x)\| \leq \varepsilon,$$

then

$$\begin{aligned} \|T(x) - T'(x)\| &= 2^{-n}(\|T(2^n x) - T'(2^n x)\| \\ &\leq 2^{-n}(\|T(2^n x) - f(2^n x)\| + \|f(2^n x) - T'(2^n x)\|) \leq 2^{-n+1}\varepsilon \end{aligned}$$

for every $x \in X_0$ and $n \in \mathbb{N}$ (positive integers). Tending with n to infinity, we get $T = T'$. □

2.2. **Sandwich Technique.** The following lemma [82] provides a necessary and sufficient condition for the existence of an additive separation function.

Lemma 2.2. *Let S be an abelian semigroup and $p, q \colon S \to [-\infty, \infty)$ be functions. Then there is an additive function $T \colon S \to [-\infty, \infty)$ such that $p(x) \leq T(x) \leq q(x)$ if and only if the following condition holds:*

($\mathcal{A}$) $p(x_1) + \cdots + p(x_m) \leq q(y_1) + \cdots + q(y_n)$ *for all $m, n \in \mathbb{N}$ and for all $x_1, \cdots, x_m, y_1, \cdots, y_n \in S$ with $x_1 + \cdots + x_m = y_1 + \cdots + y_n$.*

We are now ready to show an application of the above lemma (cf. [84]) in obtaining a stability result that is somewhat different from that of Hyers.

Theorem 2.3. *Let S be an abelian semigroup and $p, q \colon S \to [-\infty, \infty)$ be functions such that ($\mathcal{A}$) is valid. Assume that a function $f \colon S \to \mathbb{R}$ satisfies*

$$p(y) \leq f(x + y) - f(x) \leq q(y) \tag{2.3}$$

for all $x, y \in S$. Then there exists an additive function $T \colon S \to [-\infty, \infty)$ such that

$$p(x) \leq T(x) \leq q(x), \qquad x \in S.$$

Proof. Let $m, n \in \mathbb{N}$ and $x_1, \cdots, x_m, y_1, \cdots, y_n \in S$ be such that $z := x_1 + \cdots + x_m = y_1 + \cdots + y_n$. Let $s \in S$ be an arbitrary fixed element. By the left hand side inequality of (2.3), we have

$$\begin{aligned} &p(x_1) \leq f(s + x_1) - f(s) \\ &p(x_2) \leq f(s + x_1 + x_2) - f(s + x_1) \\ &\quad \cdots \\ &p(x_m) \leq f(s + x_1 + \cdots + x_m) - f(s + x_1 + \cdots + x_{m-1}). \end{aligned}$$

Adding up these inequalities we get

$$p(x_1) + \cdots + p(x_m) \leq f(s + z) - f(s).$$

Similarly, applying the right hand side inequality of (2.3), we obtain

$$f(s + z) - f(s) \leq q(x_1) + \cdots + q(x_n).$$

Thus $p(x_1) + \cdots + p(x_m) \le q(x_1) + \cdots + q(x_n)$. By Lemma 2.2, we conclude the existence of the required additive separation function. □

Corollary 2.4. *Suppose that $(S, +)$ is an abelian semigroup, $\varepsilon_1, \varepsilon_2 \ge 0$ and $f\colon S \to \mathbb{R}$ is a mapping satisfying $-\varepsilon_2 \le f(x+y) - f(x) - f(y) \le \varepsilon_1$ for all $x, y \in S$. Then there is an additive mapping $T\colon S \to \mathbb{R}$ such that $-\varepsilon_2 \le T(x) - f(x) \le \varepsilon_1$ for all $x \in S$.*

Proof. Put $p(x) = f(x) - \varepsilon_2$ and $q(x) = f(x) + \varepsilon_1$ in Theorem 2.3. □

2.3. **Fixed Point Method.** Now we present the third method used in investigations of the stability of functional equations. For a more detailed survey on various versions of it we refer to [27]. Here we present only an application of the following well known fixed point theorem named as the fixed point alternative (see [37]).

Theorem 2.5. *Let $(\mathcal{M}, d)$ be a complete generalized metric space and $J\colon \mathcal{M} \to \mathcal{M}$ be a strictly contractive mapping with the Lipschitz constant L. Then, for each given element $x \in \mathcal{M}$, either*

(A1) $d(J^n x, J^{n+1}x) = \infty$ *for all $n \ge 0$,*

or

(A2) *there exists a natural number n_0 such that:*
- (A20) $d(J^n x, J^{n+1}x) < \infty$ *for all $n \ge n_0$;*
- (A21) *the sequence $\{J^n x\}$ is convergent to a fixed point y^* of J;*
- (A22) *y^* is the unique fixed point of J in the set $\{y \in \mathcal{M} : d(J^{n_0}x, y) < \infty\}$;*
- (A23) $d(y, y^*) \le \frac{1}{1-L} d(y, Jy)$ *for all $y \in U$.*

Now we present an alternative proof of Theorem 2.1, based on that fixed point alternative.

Proof. Write $\mathcal{M} := \{g : X_0 \to Y : g(0) = 0\}$ and let the generalized metric d on $\mathcal{M}$ be defined by

$$d(g, h) = \inf\{c \in (0, \infty) : \|g(x) - h(x)\| \le c\varepsilon \text{ for } x \in X_0\}.$$

It is easy to see that $(\mathcal{M}, d)$ is complete. The mapping $J\colon \mathcal{M} \to \mathcal{M}$ given by $(Jg)(x) := \frac{1}{2} g(2x)$ for $x \in X_0$ is a strictly contractive mapping.

Take $c > 0$ and $g, h \in \mathcal{M}$ with $d(g, h) \le c$. Then $\|g(x) - h(x)\| \le c\varepsilon$ for $x \in X_0$, whence

$$\left\| \frac{1}{2} g(2x) - \frac{1}{2} h(2x) \right\| \le \frac{1}{2} c\varepsilon, \qquad x \in X_0,$$

and consequently $d(Jg, Jh) \le c/2$.

Thus we have shown that

$$d(Jg, Jh) \le \frac{1}{2} d(g, h), \qquad g, h \in \mathcal{M}.$$

Hence J is a strictly contractive mapping on $\mathcal{M}$. Putting $y = x$ in (2.1) we obtain

$$\left\| \frac{1}{2} f(2x) - f(x) \right\| \le \frac{1}{2} \varepsilon, \qquad x \in X_0,$$

which means that $d(f, Jf) \leq 1/2$.

From the fixed point alternative we deduce the existence of a mapping $T\colon X \to Y$, which is a fixed point of J (that is $T(2x) = 2T(x)$ for all $x \in X_0$). Since $\lim_{n\to\infty} d(J^n f, T) = 0$, we easily conclude that

$$\lim_{n\to\infty} \frac{f(2^n x)}{2^n} = T(x), \qquad x \in X_0.$$

The mapping T is the unique fixed point of J in the set $\{g \in \mathcal{M} : d(f, g) < \infty\}$. Hence T is the unique fixed point of J such that $\|f(x) - T(x)\| \leq K\varepsilon$ for some $K > 0$ and for all $x \in X$. Again, by applying the fixed point alternative theorem we obtain

$$d(f, T) \leq 2d(f, Jf) \leq 1,$$

and so

$$\|f(x) - T(x)\| \leq \varepsilon, \qquad x \in X_0.$$

Replacing x, y by $2^n x, 2^n y$, respectively, in (2.1), we get

$$\|2^{-n} f(2^n(x+y)) - 2^{-n} f(2^n x) - 2^{-n} f(2^n y)\| \leq 2^{-n}\varepsilon$$

for all $x, y \in X_0$ with $x + y \in X_0$. Taking the limit as $n \to \infty$ we obtain the additivity of T on X_0. □

2.4. **Invariant Mean Method.** The idea of applying invariant means to the stability problems is due to Székelyhidi [113] and [114]. One of the advantages of this approach is that one is able to prove stability results on groups or semigroups which are not necessarily abelian, but satisfies an essentially weaker condition of being amenable (we define this notion below). Before we give the details, let us recall briefly its history. In 1982 Székelyhidi [113] presented an alternative proof of a result of Albert and Baker concerning stability of monomials. He was assuming that the domain of investigated mappings is a commutative semigroup, but he applied the technique of invariant means and therefore the assumptions of commutativity in his result can be replaced by a weaker assumption of amenability. During The Twenty-Second International Symposium on Functional Equations in Oberwolfach in 1984, Forti [45] showed that if $F_{\{a,b\}}$ is a free group generated by two generators a, b, then there exists a mapping $f\colon F_{\{a,b\}} \to \mathbb{R}$ which satisfies

$$f(x+y) - f(x) - f(y) \in \{-1, 0, 1\}, \quad x, y \in F_{\{a,b\}}$$

and for which there exist no constant $\varepsilon > 0$ and no additive mapping $T\colon F_{\{a,b\}} \to \mathbb{R}$ such that

$$|f(x) - T(x)| \leq \varepsilon, \quad x \in F_{\{a,b\}}.$$

Therefore, if the underlying group is not abelian, then the stability theorem needs not to hold. During the same meeting Székelyhidi [114] showed that the Cauchy equation is stable for complex-valued mappings defined on an amenable group. Note that the free group of two generators is a standard example of a group which is not amenable. Therefore, one can ask about a deeper relation between the Hyers-Ulam stability and the amenability of the domain. For further discussion and some more examples of non-stability results the reader is referred to [9, 46].

Assume that $(S,\cdot)$ is an arbitrary semigroup. Let $B(S,)$ denote the Banach space of all bounded complex-valued functions on S equipped with the supremum norm. If $f \in B(S,)$, then by ${}_xf$ and f_x we mean the left and right translations of f given by:

$$ {}_xf(y) = f(xy), \quad f_x(y) = f(yx), \quad x, y \in S. $$

It is clear that for every $f \in B(S,)$ the functions ${}_xf$ and f_x are bounded, i.e. ${}_xf, f_x \in B(S,)$. Frequently, we will abuse the notation and identify constants with constant mappings belonging to $B(S,)$.

A positive linear functional $M\colon B(S,) \to$ such that $M(1) = 1$ is called left [right] invariant mean, if $M({}_xf) = M(f)$ [$M(f_x) = M(f)$, resp.] for every $f \in B(S,)$ and every $x \in S$. If M is simultaneously left and right invariant mean, then we call it simply invariant mean. It is known that if a semigroup admits both left and right invariant means, then it admits an invariant mean. Moreover, if S is a group, then S admits a left invariant mean if and only if it admits a right invariant mean. Finally, semigroup S is called [left/ right] amenable, if it admits a [left/ right, resp.] invariant mean. It is clear that for groups all the three notions coincide. The class of amenable groups or semigroups is quite large. In particular, every abelian semigroup, every finite group and every solvable group are amenable. Moreover, every subgroup of amenable group is amenable. For more details concerning these notions the reader is referred to [58].

Now, we present the basic result from [114], which illustrates the technique of invariant means in stability problems.

Theorem 2.6. *Suppose that $(S,\cdot)$ is a left or right amenable semigroup and $f\colon S \to$ is a mapping for which there exists $\varepsilon > 0$ such that*

$$ |f(xy) - f(x) - f(y)| \leq \varepsilon $$

for all $x, y \in G$. Then there exists a unique additive mapping $T\colon S \to$ such that

$$ |T(x) - f(x)| \leq \varepsilon $$

for all $x \in G$.

Proof. We will consider first the case that S is right amenable. Let us note that for every fixed element $x \in G$ the function $f_x - f$ belongs to $B(S,)$. Indeed, for every $y \in S$ we have

$$ \begin{aligned} |f_x(y) - f(y)| &= |f(yx) - f(y)| \\ &\leq |f(yx) - f(y) - f(x))| + |f(x)| \\ &\leq \varepsilon + |f(x)|. \end{aligned} $$

Therefore, the map $T\colon S \to$ is well defined by $T(x) = M(f_x - f)$. Next, we will check that T is additive:

$$ \begin{aligned} T(x) + T(y) &= M(f_x - f) + M(f_y - f) \\ &= M([f_x - f]_y) + M(f_y - f) \\ &= M(f_{xy} - f_y) + M(f_y - f) \\ &= M(f_{xy} - f_y + f_y - f) \\ &= M(f_{xy} - f) = T(xy). \end{aligned} $$

Note that above we have used the fact that M is right-invariant. Finally,

$$\begin{aligned}
|T(x) - f(x)| &= |M(f_x - f) - f(x)| \\
&= |M(f_x - f - f(x))| \\
&\leq \|M\| \|f_x - f - f(x)\| \\
&= \sup_{y \in G} |f(yx) - f(y) - f(x)| \leq \varepsilon.
\end{aligned}$$

In the case that S is left amenable, one can modify the foregoing reasoning and argue as follows. For every $x \in G$ we have ${}_xf - f \in B(S,)$. To see this, fix $y \in S$ and check that

$$\begin{aligned}
|{}_xf(y) - f(y)| &= |f(xy) - f(y)| \\
&\leq |f(xy) - f(x) - f(y))| + |f(x)| \\
&\leq \varepsilon + |f(x)|.
\end{aligned}$$

Therefore, the map $T\colon S \to$ is well defined by $T(x) = M({}_xf - f)$. Moreover, using the fact that M is right-invariant we get that T is additive:

$$\begin{aligned}
T(x) + T(y) &= M({}_xf - f) + M({}_yf - f) \\
&= M({}_xf - f) + M({}_x[{}_yf - f]) \\
&= M({}_xf - f +_{xy} f -_x f) \\
&= M({}_{xy}f - f) = T(xy).
\end{aligned}$$

Moreover,

$$\begin{aligned}
|T(x) - f(x)| &= |M({}_xf - f) - f(x)| \\
&= |M({}_xf - f - f(x))| \\
&\leq \|M\| \|{}_xf - f - f(x)\| \\
&= \sup_{y \in G} |f(xy) - f(y) - f(x)| \leq \varepsilon.
\end{aligned}$$

The uniqueness of T can be proved in a standard way, as in the proof of Theorem 2.1. □

The technique presented in the foregoing proof has been developed further and applied to several stability problems (see e.g. [7, 8]). We will mention here two more contributions, in which the authors deal with conditional stability problems of approximate additivity almost everywhere or on some large sets.

Gajda [50] dealt with the case when the stability inequality is satisfied almost everywhere (in an abstract sense) and the unknown mapping takes values in a vector lattice. Before we formulate Gajda's result, we need some definitions. If $(G, \cdot)$ is a group, then a non-empty family $\mathcal{I}$ of subsets of G is called a proper linearly invariant set ideal (p.l.i. ideal for short), if

($\mathcal{I}1$) $G \notin \mathcal{I}$,
($\mathcal{I}2$) if $A, B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$,
($\mathcal{I}3$) if $A \in \mathcal{I}$, $B \subset G$ and $B \subset A$, then $B \in \mathcal{I}$,

($\mathcal{I}4$) if $A \in \mathcal{I}$ and $a \in G$, then the following sets belong to $\mathcal{I}$:

$$A \cdot a = \{xa : x \in A\}, \quad a \cdot A = \{ax : x \in A\}, \quad A^{-1} = \{x^{-1} : x \in A\}.$$

We say that some condition $\mathcal{C}(x)$ defined for $x \in G$ holds $\mathcal{I}$-almost everywhere ($\mathcal{I}$-a.e. for short), if there exists a set $A \in \mathcal{I}$ such that $\mathcal{C}(x)$ holds for every $x \in G \setminus A$. Given a p.l.i. ideal $\mathcal{I}$ of subsets of G, we can define a p.l.i. ideal $\Omega(\mathcal{I})$ in the product group $G \times G$ in the following way. A set $N \in G \times G$ belongs to $\Omega(\mathcal{I})$, if we have $\{y \in G : (x,y) \in N\} \in \mathcal{I}$ for $\mathcal{I}$-a.e. $x \in G$. If S is a subsemigroup of G and $\mathcal{I}$ is a p.l.i. ideal of subsets of G, then one can define a natural p.l.i. ideal of subsets of S as follows:

$$\mathcal{I}|_S = \{A \cap S : A \in \mathcal{I}\}.$$

If $(X, \leq)$ is a vector lattice, then we call it boundedly complete, if every non-empty upper bounded subset of X has a supremum in X.

Now, we are ready to quote Theorem 3.1 from [50].

Theorem 2.7. *Assume that $(G, \cdot)$ is a group and S is a subsemigroup of G such that S is left amenable and $G = S \cdot S^{-1} = \{x \cdot y^{-1} : x, y \in S\}$. Let $\mathcal{I}$ be a p.l.i. ideal of subsets of G such that $S \notin \mathcal{I}$. Further, let $(X, \leq)$ be a boundedly complete vector lattice and let $V \subset X$ be a non-empty bounded set. Assume that $f \colon S \to X$ is a mapping such that*

$$f(x+y) - f(x) - f(y) \in V$$

for $\Omega(\mathcal{I})$-a.e. $(x,y) \in S \times S$. Then, there exists a unique additive mapping $T \colon G \to X$ such that

$$\inf V \leq T(x) - f(x) \leq \sup V$$

holds for $\mathcal{I}$-a.e. $x \in S$.

The invariant mean technique has been used by Cabello-Sánchez [30] for a related conditional stability problem. He studied mappings from a commutative or amenable group into the real line or into a Banach space which satisfies a stability inequality on a big subset of the domain. If G is an amenable group, then a subset $B \subset G$ is called big, if there exists an invariant mean M for G such that $M(1_B) = 1$, where 1_B denotes the characteristic function of B. Below we quote main results from [30]. In the first theorem three stability problems are covered.

Theorem 2.8. *Assume that $(G, +)$ is a commutative group, $B \subseteq G$ is a big subset of G, $\rho \colon G \to \mathbb{R}$ is a nonnegative "control" mapping and $K \geq 0$ is a real constant. Assume further that a mapping $f \colon B \to \mathbb{R}$ satisfies <u>one</u> of the following conditions:*

(a) $|f(x+y) - f(x) - f(y)| \leq K(\rho(x) + \rho(y) - \rho(x+y))$ *for every $x, y \in B$ such that $x + y \in B$,*
(b) $|f(x+y) - f(x) - f(y)| \leq K\rho(x)$ *for every $x, y \in B$ such that $x + y \in B$,*
(c) $\left| f\left(\sum_{i=1}^{n} x_i\right) - \sum_{i=1}^{n} f(x_i) \right| \leq K \sum_{i=1}^{n} \rho(x_i)$ *for every $x_i \in B$ such that $\sum_{i=1}^{n} x_i \in B$ and for all $n \in \mathbb{N}$.*

Then, there exists an additive mapping $T\colon G \to \mathbb{R}$ such that

$$|f(x) - T(x)| \leq K\rho(x), \quad x \in B.$$

In the second theorem stability of (b) is extended for mappings defined on an amenable group and having values into a Banach space.

Theorem 2.9. *Assume that $(G, +)$ is an amenable group, $B \subseteq G$ is a big subset of G, $\rho\colon G \to \mathbb{R}$ is a nonnegative "control" mapping, $K \geq 0$ is a real constant and Y is a Banach space which is complemented in its second dual by a projection π. Assume further that a mapping $f\colon B \to Y$ satisfies*

$$\|f(x + y) - f(x) - f(y)\| \leq K\rho(x)$$

for every $x, y \in B$ such that $x + y \in B$. Then, there exists an additive mapping $T\colon G \to Y$ such that

$$\|f(x) - T(x)\| \leq K\|\pi\|\rho(x), \quad x \in B.$$

3. Hyperstability

In this section we focus on the issue of hyperstability, a particular case of superstability. For suitable references, more detailed discussion on similarities and differences between those two notions and numerous examples of results obtained for various functional equations we refer to [28]. Here we confine our considerations only to the additive Cauchy equation.

The following theorem has been proved in [23] (it corresponds to the earlier results of Aoki, Rassias and Gajda).

Theorem 3.1. *Let E_1 and E_2 be normed spaces, $X \subset E_1 \setminus \{0\}$ be nonempty, $c \geq 0$ and $p < 0$. Assume that*

$$-X := \{-x : x \in X\} = X \tag{3.1}$$

and there exists a positive integer m_0 with

$$nx \in X, \qquad x \in X, n \in \mathbb{N}, n \geq m_0. \tag{3.2}$$

Then every mapping $g\colon E_1 \to E_2$ with

$$\|g(x + y) - g(x) - g(y)\| \leq c(\|x\|^p + \|y\|^p), \qquad x, y \in X, x + y \in X, \tag{3.3}$$

is additive on X, i.e.,

$$g(x + y) = g(x) + g(y), \qquad x, y \in X, x + y \in X. \tag{3.4}$$

In the case $X = E_1 \setminus \{0\}$, such outcome follows easily from an earlier (and more general) [73, Theorem 5].

Modifying accordingly the terminology used in [74] (see also [28]), we can describe the result contained in Theorem 3.1 as the property of φ-hyperstability of the conditional equation (3.4) for $\varphi(x, y) \equiv c(\|x\|^p + \|y\|^p)$. A more precise definition of hyperstability of equation (3.4) can be patterned on [28, Definition 7] and stated as follows.

Definition 3.2. Let $(A,+)$ be a groupoid (i.e. a nonempty set equipped with a binary operation $+\colon X\times X\to X$), $X\subset A$ be nonempty, (Y,d) be a metric space, and $\varepsilon\in{\mathbb{R}_+}^{X^2}$. We say that equation (3.4) is $\varepsilon-$hyperstable provided every mapping $g\colon X\to Y$ satisfying the inequality

$$d(g(x+y),g(x)+g(y))\le\varepsilon(x,y),\qquad x,y\in X, x+y\in X, \tag{3.5}$$

fulfils equation (3.4).

Below we present an analogous complementary result on a restricted domain for $p\ge 0$ by using a method suggested by G.L. Forti (it is a modification of the direct method). To this end we need to recall the following observation (cf. [16, Theorem 6.1]).

Theorem 3.3. *Assume that (Y,d) is a complete metric space, K is a nonempty set, $\lambda\in\mathbb{R}_+$, and $f\colon K\to Y$, $\Psi\colon Y\to Y$, $a\colon K\to K$, $h\colon K\to\mathbb{R}_+$ are mappings such that*

$$\begin{aligned} d(\Psi\circ f\circ a(x),f(x))&\le h(x),\qquad x\in K,\\ d(\Psi(x),\Psi(y))&\le\lambda d(x,y),\qquad x,y\in Y,\end{aligned} \tag{3.6}$$

and

$$H(x):=\sum_{i=0}^{\infty}\lambda^i h\left(a^i(x)\right)<\infty,\qquad x\in K. \tag{3.7}$$

Then, for every $x\in K$, the limit

$$F(x):=\lim_{n\to\infty}\Psi^n\circ f\circ a^n(x)$$

exists and $F\colon K\to Y$ is the unique mapping such that $\Psi\circ F\circ a=F$ and

$$d(f(x),F(x))\le H(x),\qquad x\in K.$$

Now, we are in a position to prove the following complement to Theorem 3.1.

Theorem 3.4. *Let E_1 and E_2 be normed spaces, $X\subset E_1\setminus\{0\}$ be nonempty, $c\ge 0$ and $p\ge 0$, $p\ne 1$. Assume that the following two conditions hold.*

(i) *If $p<1$, then $2X:=\{2x : x\in X\}\subset X$.*
(ii) *If $p>1$, then $X\subset 2X$.*

Then for every mapping $g\colon X\to E_2$ with

$$\|g(x+y)-g(x)-g(y)\|\le c(\|x\|^p+\|y\|^p),\qquad x,y\in X, x+y\in X, \tag{3.8}$$

there is a unique mapping $T\colon X\to E_2$ that is additive on X and such that

$$\|f(x)-T(x)\|\le\frac{2c}{|2-2^p|}\|x\|^p,\qquad x\in X. \tag{3.9}$$

Proof. If we take $x=y$ in (3.8) and, moreover, when $p>1$, replace x by $\frac{1}{2}x$, then in either case we arrive at the inequality

$$\|\Psi\circ f\circ a(x)-f(x)\|\le h(x),\qquad x\in X, \tag{3.10}$$

where

$$\Psi(x):=\begin{cases}2x, & \text{if } p>1;\\ \frac{1}{2}x, & \text{if } p<1,\end{cases}\qquad a(x):=\begin{cases}\frac{1}{2}x, & \text{if } p>1;\\ 2x, & \text{if } p<1,\end{cases}$$

$$h(x) := \begin{cases} 2^{1-p}c\|x\|^p, & \text{if } p > 1; \\ c\|x\|^p, & \text{if } p < 1. \end{cases}$$

So, the assumptions of Theorem 3.3 are fulfilled with

$$\lambda := \begin{cases} 2, & \text{if } p > 1; \\ \frac{1}{2}, & \text{if } p < 1. \end{cases}$$

Moreover, it is easy to check that

$$H(x) = \frac{2c}{|2 - 2^p|}\|x\|^p, \qquad x \in X,$$

where H is given by (3.7). Hence, by Theorem 3.3, there is a unique mapping $T\colon X \to E_2$ such that $T = \Psi \circ T \circ a$ and (3.9) holds. Moreover,

$$T(x) = \lim_{n\to\infty} \Psi^n \circ f \circ a^n(x), \qquad x \in X.$$

It remains to show that T is additive on X.

So, fix $z, w \in X$ with $z + w \in X$. Then, by (3.9),

$$\begin{aligned} \|\Psi^n \circ f \circ a^n(x+y) - \Psi^n \circ f \circ a^n(x) - \Psi^n \circ f \circ a^n(y)\| \\ \leq \lambda^n \|f \circ a^n(x+y) - f \circ a^n(x) - f \circ a^n(y)\| \\ \leq \lambda^n c(\|a^n(x)\|^p + \|a^n(y)\|^p), \end{aligned}$$

whence, letting $n \to \infty$, we deduce that $T(x+y) = T(x) + T(y)$, because

$$\lim_{n\to\infty} \lambda^n \|a^n(z)\|^p = 0, \qquad z \in X.$$

Since every mapping $T_0\colon X \to E_2$ that is additive on X, fulfils also the condition $T_0 = \Psi \circ T_0 \circ a$, the statement on uniqueness of T in Theorem 3.4 results from the analogous statement in Theorem 3.3. □

Theorems 3.1 and 3.4 and some earlier observations allow us to formulate the subsequent theorem.

Theorem 3.5. *Let E_1 and E_2 be normed spaces and $c \geq 0$ and $p \neq 1$ be fixed real numbers. Assume also that $f\colon E_1 \to E_2$ is a mapping with*

$$\|f(x+y) - f(x) - f(y)\| \leq c(\|x\|^p + \|y\|^p), \qquad x, y \in E_1 \setminus \{0\}. \tag{3.11}$$

If $p \geq 0$ and E_2 is complete, then there exists a unique additive $T\colon E_1 \to E_2$ such that

$$\|f(x) - T(x)\| \leq \frac{c\|x\|^p}{|1 - 2^{p-1}|}, \qquad x \in X_0, \tag{3.12}$$

where

$$X_0 := \begin{cases} E_1 \setminus \{0\}, & \text{if } p = 0; \\ E_1, & \text{if } p \neq 0; \end{cases}$$

moreover, if for each $x \in E_1$, the mapping $f_x\colon \mathbb{R} \ni t \mapsto f(tx)$ is continuous at least at one point or bounded above or below on a set with positive inner Lebesgue measure, then T is linear.

If $p < 0$, then f is additive.

Proof. Write $X := E_1 \setminus \{0\}$. First consider the case $p < 0$. Then Theorem 3.1 implies that f is additive on X. It remains to show that

$$f(x+0) = f(x) + f(0), \qquad x \in E_1,$$

i.e., that $f(0) = 0$. So fix $z \in X$. Then

$$f(z) = f((z+z) - z) = f(2z) + f(-z) = 2f(z) + f(-z),$$

which means that $f(-z) = -f(z)$. Consequently, by (3.11),

$$\|f(0)\| = \|f(z-z) - f(z) - f(-z)\| \leq 2c\|z\|^p, \qquad z \in X,$$

whence $f(0) = 0$.

Now, assume that $p > 0$. Then, on account of Theorem 3.4, there exists a unique mapping $T_0 \colon X \to E_2$ that is additive on X and such that (3.9) holds. Define $T \colon E_1 \to E_2$ by $T(x) = T_0(x)$ for $x \in X$ and $T(0) = 0$.

As before (for f) we show that $T(-z) = -T(z)$ for each $z \in X$, which implies the additivity of T. Next, by (3.11) and (3.9), we get

$$\begin{aligned} \|f(0)\| \leq \|f(z-z) - f(z) - f(-z)\| + \|f(z) - T(z)\| + \|f(-z) - T(-z)\| \\ \leq \mu\|z\|^p, \qquad z \in X, \end{aligned}$$

with some $\mu > 0$. Hence $f(0) = 0$, which completes the proof of (3.12).

It remains to consider the case, where for each $x \in E_1$ the mapping $f_x \colon \mathbb{R} \ni t \mapsto f(tx)$ is continuous at least at one point or bounded above or below on a set with positive inner Lebesgue measure. Then (3.12) implies that the mapping $\mathbb{R} \ni t \mapsto T(tx)$ is bounded, above or below, on a set with positive inner Lebesgue measure and, consequently, it is linear. □

Remark 3.6. We cannot have $X_0 = E_1$ in (3.12) for $p = 0$ in the general situation. For instance, let $c > 0$, $f(x) = 2c$ for $x \in E_1 \setminus \{0\}$ and $f(0) = 6c$. Then it is easily seen that (3.11) holds, $T(x) \equiv 0$ is the only additive mapping satisfying (3.12), but

$$\|f(0) - T(0)\| = \|f(0)\| = 6c > 2c = \frac{c\|0\|^0}{|1 - 2^{-1}|},$$

with $0^0 := 1$. However, this can be amended if in the case $p = 0$ we replace condition (3.11) by

$$\|f(x+y) - f(x) - f(y)\| \leq 2c, \qquad x, y \in D, \tag{3.13}$$

with some $D \subset X^2$ such that $(E_1 \setminus \{0\})^2 \subset D$ and $\{x \in E_1 : (x, 0) \in D\} \neq \emptyset$. Then, taking $y = 0$ in (3.13), we have $\|f(0)\| \leq 2c$, which yields $X_0 = E_1$ in (3.12) also for $p = 0$.

Remark 3.7. Let E_2 be a normed space, $X \subset \mathbb{R}$ be nonempty, and $p > 0$. Take $w_0 \in E_2 \setminus \{0\}$. Define $f, T \colon \mathbb{R} \to E_2$ by

$$f(x) = \frac{|x|^p}{|1 - 2^{p-1}|} w_0, \qquad T(x) = 0, \qquad x \in X.$$

It has been proved in [22] that

$$\big|\, |x+y|^p - |x|^p - |y|^p \big| \leq |1 - 2^{p-1}|(|x|^p + |y|^p), \qquad x, y \in \mathbb{R}.$$

Consequently

$$\|f(x+y)-f(x)-f(y)\| \le \|w_0\|(|x|^p+|y|^p), \qquad x,y\in\mathbb{R}.$$

Moreover,

$$\|f(x)-T(x)\| = \|f(x)\| = \|w_0\|\frac{|x|^p}{|1-2^{p-1}|}, \qquad x\in\mathbb{R}.$$

This shows that estimations (3.9) and (3.12) are optimal in Theorems 3.4 and 3.5, respectively, when $E_1=\mathbb{R}$.

Moreover, Theorem 3.5 disproves the conjecture of Th.M. Rassias and J. Tabor [93] stating that this is also the case for $p<0$.

Remark 3.8. According to Definition 3.2, the second statement of Theorem 3.5, for $p<0$, can be described as the φ-hyperstability of equation (1.1) for $\varphi(x,y)\equiv c(\|x\|^p+\|y\|^p)$. It seems to be interesting that it is not true without condition (3.1) (or any suitable hypothesis replacing it). In fact, let $p<0$, $a\ge 0$, $I=(a,\infty)$ and $f,T\colon I\to\mathbb{R}$ be given by $T(x)=0$ and $f(x)=x^p$ for $x\in I$. Then

$$|f(x)-T(x)| = x^p, \qquad x\in I.$$

Next, fix $x,y\in I$. Suppose that $x\le y$. Then

$$(x+y)^p \le (2x)^p = 2^p x^p \le x^p \le x^p+y^p$$

and, consequently,

$$|f(x+y)-f(x)-f(y)| = x^p+y^p-(x+y)^p \le x^p+y^p.$$

Without condition (3.1) we "only" have the following (see [25, Theorem 1.3]).

Theorem 3.9. *Let E_1 and E_2 be normed spaces, $X\subset E_1\setminus\{0\}$ be nonempty, E_2 be complete, $c\ge 0$ and $p<0$. Assume that there is $m_0\in\mathbb{N}$ such that (3.2) holds and $f\colon X\to E_2$ is a mapping satisfying (3.3). Then there exists a unique mapping $T\colon X\to E_2$ with*

$$T(x+y)=T(x)+T(y), \qquad x,y\in X, x+y\in X, \tag{3.14}$$

$$\|f(x)-T(x)\|\le c\|x\|^p, \qquad x\in X. \tag{3.15}$$

The example in Remark 3.8 shows that for $p<0$ estimation (3.15) is optimal under the assumptions of Theorem 3.9. However, with a somewhat different (though still natural) form of the function φ, φ-hyperstability still holds even without (3.1). Namely, in [24, Theorem 1.3] the subsequent result has been proved.

Theorem 3.10. *Let E_1 and E_2 be normed spaces, $X\subset E_1\setminus\{0\}$ be nonempty, $c\ge 0$ and p,q be real numbers with $p+q<0$. Assume that there is an $m_0\in\mathbb{N}$ such that (3.2) holds. Then every $g\colon X\to E_2$, satisfying the inequality*

$$\|g(x+y)-g(x)-g(y)\|\le c\|x\|^p\|y\|^q, \qquad x,y\in X, x+y\in X, \tag{3.16}$$

is additive on X.

Theorem 3.10 corresponds to the investigations in [89], where results analogous to those in Theorem 3.5 have been proved, with the factor $\|x\|^p + \|y\|^p$ replaced by $\|x\|^p \cdot \|y\|^q$ for $p, q \in \mathbb{R}$ with $p + q \neq 1$. For information on investigations of condition (1.4) with functions φ of some other forms we refer to [16, 28].

Clearly, in connection with those outcomes, presented above, a natural question arises: when for a mapping $T_0\colon E_1 \to E_2$ that is additive on $X \subset E_1$, there is an additive $T\colon E_1 \to E_2$ with $T(x) = T_0(x)$ for $x \in X$? For some information on this issue we refer the reader to, e.g., [71, Theorem 1.1, Chapter XVIII], [96, Chapter 4], and [109, pp. 143-144].

4. Stability of the inhomogeneous Cauchy equation

One may ask whether analogous results to those presented in the preceding sections can be proved for the inhomogeneous Cauchy equation

$$g(x + y) = g(x) + g(y) + d(x, y), \tag{4.1}$$

with a suitably defined function d. The equation has drawn attention of several authors and been studied already for various spaces and forms of d (see [38, 26] for references). Some general results (with suitable examples) concerning that issue have been presented in [26]. In particular, the following theorem has been proved.

Theorem 4.1. *Let E_1 and E_2 be normed spaces, $d\colon E_1^2 \to E_2$ and $c, p \in \mathbb{R}$. Assume that (4.1) admits a solution $f_0\colon E_1 \to E_2$. Then the following three statements are valid.*

(a) *If $p \geq 0$, $p \neq 1$, and E_2 is complete, then for every mapping $f\colon E_1 \to E_2$, satisfying*

$$\|f(x + y) - f(x) - f(y) - d(x, y)\| \leq c(\|x\|^p + \|y\|^p), \qquad x, y \in E_1 \setminus \{0\}, \tag{4.2}$$

there exists a unique solution $g\colon E_1 \to E_2$ of (4.1) such that

$$\|f(x) - g(x)\| \leq \frac{c\|x\|^p}{|2^{p-1} - 1|}, \qquad x \in E_1 \setminus \{0\}. \tag{4.3}$$

Moreover, that estimate is optimal when $E_1 = \mathbb{R}$; namely there exists a mapping $f\colon \mathbb{R} \to E_2$ such that

$$\|f(x + y) - f(x) - f(y) - d(x, y)\| \leq c(|x|^p + |y|^p), \qquad x, y \in \mathbb{R},$$

$$\|f(x) - f_0(x)\| = \frac{c|x|^p}{|2^{p-1} - 1|}, \qquad x \in \mathbb{R}. \tag{4.4}$$

(b) *If $p < 0$, then every $f\colon E_1 \to E_2$ satisfying (4.2) is a solution of (4.1).*

(c) *If $E_1 = E_2 = \mathbb{R}$, then for each real $c_0 > 0$ there is $f\colon \mathbb{R} \to \mathbb{R}$ such that*

$$|f(x + y) - f(x) - f(y) - d(x, y)| \leq c_0(|x| + |y|), \qquad x, y \in \mathbb{R}, \tag{4.5}$$

and, for each solution $h\colon \mathbb{R} \to \mathbb{R}$ of (4.1), we have

$$\sup_{x \in \mathbb{R} \setminus \{0\}} \frac{|f(x) - h(x)|}{|x|} = \infty \tag{4.6}$$

Below we show how to obtain a similar outcome for the situation described in Theorem 1.3.

Theorem 4.2. *Suppose that $(G,+)$ is an abelian semigroup, $D \subset G$ is nonempty, $2D \subset D$, E is a Banach space, $d\colon G^2 \to E$, $\varphi\colon G\times G \to [0,\infty)$ satisfies (1.7) and equation (4.1) admits a solution $g\colon G \to E$. Let $f\colon D \to E$ fulfil the inequality*

$$\|f(x+y) - f(x) - f(y) - d(x,y)\| \le \varphi(x,y)$$

for all $x, y \in D$ with $x+y \in D$. Then there is a unique mapping $T\colon D \to E$ with

$$T(x+y) = T(x) + T(y) + d(x,y), \qquad x, y \in D, x+y \in D, \tag{4.7}$$

$$\|f(x) - T(x)\| \le \widetilde{\varphi}(x,x), \qquad x \in D.$$

Proof. Write $f_0 := f - g$. Then

$$\|f_0(x+y) - f_0(x) - f_0(y)\| \le \varphi(x,y), \qquad x, y \in D, x+y \in D,$$

whence by Theorem 1.3 there exists a unique mapping $T_0\colon D \to E$ that is additive on D (i.e., $T_0(x+y) = T_0(x) + T_0(y)$ for $x, y \in D$ with $x+y \in D$) and

$$\|f_0(x) - T_0(x)\| \le \widetilde{\varphi}(x,x), \qquad x \in D.$$

Let $T\colon = T_0 + g$. Then it is easily seen that (4.7) holds and

$$\|f(x) - T(x)\| \le \widetilde{\varphi}(x,x), \qquad x \in D.$$

Assume that $T'\colon D \to E$ is also a solution of (4.7) with

$$\|f(x) - T'(x)\| \le \widetilde{\varphi}(x,x), \qquad x \in D.$$

Write $T_0' := T' - g$. Then T_0' is additive on D and

$$\|f_0(x) - T_0'(x)\| \le \widetilde{\varphi}(x,x), \qquad x \in D,$$

which means that $T_0 = T_0'$ and, consequently, $T = T'$. □

The assumption that equation (4.1) admits a solution $g\colon G \to E$ cannot be omitted (see [26, Remark 2.3 and Example]). Clearly, the assumption means that

$$d(x,y) = h(x+y) - h(x) - h(y), \qquad x, y \in G, \tag{4.8}$$

with some mapping $h\colon G \to E$ (i.e., d is a coboundary), which is equivalent (see, e.g., [111]) to the fact that d is a symmetric cocycle, i.e., for every $x, y, z \in G$

$$d(x,y) = d(y,x), \quad d(x+y,z) + d(x,y) = d(x,y+z) = d(y,z). \tag{4.9}$$

Condition (4.8) shows that we can find numerous natural examples of functions d with which equation (4.1) has a solution. For more information and further references on cocycles we refer to [111].

5. Stability of functional equations on generalized spheres

In this part of the paper we give results concerning stability of functional equations for functions from X into Y with condition $\gamma(x) = \gamma(y)$ for defined below X, Y and γ. A classical example of such function γ in a normed space X is, simply, a norm in X, whence the name "generalized spheres".

Below we give two sets of assumptions for function γ: first – which has more algebraic nature, and second – with more topological nature. Even though there are functions which satisfy both sets of conditions (e.g., a norm in at least three-dimensional inner product space with orthogonality defined by a standard way by means of an inner product), none of the families satisfying one of the sets of assumptions is contained in the second one.

Consider two sets of conditions:

- **(a)** Let X be a real linear space, $\dim X \geq 2$, Z arbitrary nonempty set and let $\gamma\colon X \to Z$ be a function such that:
  - $(\mathrm{a})_1$ for all linearly independent $x, y \in X$ there exist linearly independent $u, v \in \operatorname{lin}\{x, y\}$ such that $\gamma(u+v) = \gamma(u-v)$;
  - $(\mathrm{a})_2$ for all $x, y \in X$, if $\gamma(x+y) = \gamma(x-y)$ then $\gamma(\alpha x + y) = \gamma(\alpha x - y)$ for $\alpha \in \mathbb{R}$;
  - $(\mathrm{a})_3$ for all $x \in X$ and $\lambda \in \mathbb{R}_+ := (0, \infty)$, there exists $y \in X$ such that $\gamma(x+y) = \gamma(x-y)$ and $\gamma((\lambda+1)x) = \gamma((1-\lambda)x + 2y)$.
- **(b)** Let $(X, +), (Z, +)$ be topological groups and let $\prec \subset Z \times Z$ be a connected binary relation on Z (i.e., for all $x, y \in Z$ we have $x \prec y$ or $y \prec x$ or $x = y$) with conditions:
  - $(\mathrm{b})_1$ for all $x \in Z$ relation $0 \prec x$ implies $-x \prec 0$;
  - $(\mathrm{b})_2$ sets $\{x \in Z : \ x \prec 0\}$ and $\{x \in Z : \ 0 \prec x\}$ are disjoint and open in Z.

  Moreover, let $\gamma\colon X \to Z$ be a continuous function satisfying
  - $(\mathrm{b})_3$ for all $x, y \in X$, if $\gamma(x) \prec \gamma(y)$, then the set $\{t \in X : \ \gamma(x+t) = \gamma(x-t) = \gamma(y)\}$ is nonempty and connected.

For further investigations let us make some notations for our assumptions:

- **(I)** X is a real linear space with $\dim X \geq 2$, $(Y, +)$ is a real Banach space, Z is a nonempty set, and $\gamma\colon X \to Z$ is an even function satisfying (a).
- **(II)** X is a real linear space with $\dim X \geq 2$, $(Y, +)$ is a real Banach space, $(Z, +)$ is a topological group equipped with a binary connected relation $\prec \subset Z \times Z$ with properties $(\mathrm{b})_1$, $(\mathrm{b})_2$ and $\gamma\colon X \to Z$ is an even continuous function, satisfying $(\mathrm{b})_3$.

In order to make the above assumptions more clear, we give some examples (see [57]). Observe that any constant function γ trivially satisfies each of sets of conditions (a), (b). A conditional equation becomes then simply an unconditional one. It is also easy to see that if γ satisfies (a), so do function $\phi \circ \gamma$ for any injection ϕ.

Example 5.1. Let X be a real inner product space, $\dim X \geq 2$, $Z = \mathbb{R}$ and $\gamma(x) := \|x\|$ for $x \in X$. Then function γ satisfies (a).

Example 5.2. Let X, Z be real linear spaces and let $A\colon X^2 \to Z$ be a bilinear and symmetric mapping such that

- ⋄ for any $\lambda \in \mathbb{R}$ and $x \in X$ there exists $y \in X$ with $A(x,y) = 0$ and $A(y,y) = \lambda A(x,x)$;
- ⋄ for any linearly independent vectors $x, y \in X$ there exist linearly independent vectors $u, v \in \operatorname{lin}\{x,y\}$ such that $A(u,v) = 0$.

Then $\gamma\colon X \to Z$ defined as $\gamma(x) := A(x,x)$ for all $x \in X$ satisfies (a).

Example 5.3. Let X be a real linear normed space with $\dim X \geq 3$. If $Z = \mathbb{R}$, $\prec \subset \mathbb{R} \times \mathbb{R}$ stands for $<$, and $\gamma\colon X \to \mathbb{R}$ be defined as $\gamma(x) := \|x\|$ for $x \in X$, then such γ and relation $\prec$ satisfy (b) (see Szab [112]).

Example 5.4. Let X be a real linear space. Assume that H is a real inner product space with $\dim H \geq 3$, and $L\colon X \to H$ is a linear surjection. Let $Z := \mathbb{R}$, $\prec := <$ and $\gamma\colon X \to \mathbb{R}$ be defined by $\gamma(x) := \|L(x)\|$ for all $x \in X$. Then (b) holds.

It is worth mentioning that, in fact, in (b) the connectedness of the set $\{t \in X : \gamma(x+t) = \gamma(x-t) = \gamma(y)\}$ and the continuity of function γ are used only (see Ger and Sikorska [57]) for the existence of a solution t of the following system of equations

$$\begin{aligned} &\gamma(t) = \gamma(y),\\ &\gamma(2x - t) = \gamma(y),\\ &\gamma(y + t) = \gamma(2x + y - t), \end{aligned} \tag{5.1}$$

for all x, y such that $\gamma(x) \prec \gamma(y)$. In many instances we may obtain a solution of such system directly.

5.1. **Cauchy equation.** In [57], it was shown that under assumptions (I) or (II), $f\colon X \to Y$ is a solution of a conditional equation (1.11) if and only if f is additive, and if γ satisfies

$$\gamma(x) = \gamma(y) \quad \Longrightarrow \quad \gamma(2x) = \gamma(2x) \tag{$(\gamma)_2$}$$

and for $f\colon X \to Y$ and $\varepsilon \geq 0$ we have

$$\gamma(x) = \gamma(y) \quad \Longrightarrow \quad \|f(x+y) - f(x) - f(y)\| \leq \varepsilon,$$

then there exists a uniquely determined additive function $g\colon X \to Y$ such that $\|f(x) - g(x)\| \leq \varepsilon$ for all $x \in X$.

In order to give more clear statements of the next results me make still some notations.

Let D be a nonempty subset of $X \times X$. Consider functions $\varphi\colon X \times X \to [0,\infty)$ and $\gamma\colon X \to Z$ such that either

1° (i) series $\displaystyle\sum_{n=1}^{\infty} 2^{-n}\varphi(2^{n-1}x, 2^{n-1}x)$ is convergent for all $x \in X$; denote this sum by $\Phi(x)$;
(ii) $\lim\limits_{n\to\infty} 2^{-n}\varphi(2^n x, 2^n y) = 0$ for all $(x,y) \in D$;
(iii) function γ satisfies the condition $(\gamma)_2$;

or

2° (i) series $\sum_{n=1}^{\infty} 2^{n-1}\varphi(2^{-n}x, 2^{-n}x)$ is convergent for all $x \in X$; denote this sum by $\Phi(x)$;
(ii) $\lim_{n\to\infty} 2^n \varphi(2^{-n}x, 2^{-n}y) = 0$ for all $(x, y) \in D$;
(iii) function γ satisfies the condition

$$\gamma(x) = \gamma(y) \quad \Longrightarrow \quad \gamma\left(\tfrac{1}{2}x\right) = \gamma\left(\tfrac{1}{2}y\right). \tag{$(\gamma)_{1/2}$}$$

First stability result reads as follows (see Sikorska [103]).

Theorem 5.5. *Let $(X, +)$ be a uniquely 2-divisible abelian group, Y let be a real Banach space, and Z a given nonempty set. Let, moreover, $D := \{(x, y) \in X \times X : \gamma(x) = \gamma(y)\}$, $\varphi\colon X \times X \to [0, \infty)$ and $\gamma\colon X \to Z$ satisfies either 1°, or 2°. If $f\colon X \to Y$ fulfils*

$$\gamma(x) = \gamma(y) \quad \Longrightarrow \quad \|f(x + y) - f(x) - f(y)\| \le \varphi(x, y), \tag{5.2}$$

then there exists a uniquely determined solution $g\colon X \to Y$ of (1.11) such that

$$\|f(x) - g(x)\| \le \Phi(x), \quad x \in X.$$

A general statement of the theorem allows us to apply it for various functions φ. The most often used in the literature forms are $\varphi(x, y) := \varepsilon(\|x\|^p + \|y\|^p)$ and $\varphi(x, y) := \varepsilon\|x + y\|^p$. Below we give such immediate application of Theorem 5.5.

Corollary 5.6. *Let X be a real normed space, Y be a real Banach space and $f\colon X \to Y$ satisfies the condition*

$$\|x\| = \|y\| \quad \Longrightarrow \quad \|f(x + y) - f(x) - f(y)\| \le \varepsilon\|x\|^p$$

for some $\varepsilon > 0$ and $p > 1$. Then there exists a uniquely determined function $g\colon X \to Y$ satisfying (1.8) and such that

$$\|f(x) - g(x)\| \le \frac{\varepsilon\|x\|^p}{2^p - 2}, \quad x \in X.$$

Moreover, if we use results in [26], the we can obtain the following even more general result.

Corollary 5.7. *Let X be a real normed space, Y be a real Banach space, $d\colon X^2 \to Y$ be a symmetric cocycle and $f\colon X \to Y$ satisfies the condition*

$$\|x\| = \|y\| \quad \Longrightarrow \quad \|f(x + y) - f(x) - f(y) - d(x, y)\| \le \varepsilon\|x\|^p$$

for some $\varepsilon > 0$ and $p > 1$. Then there exists a uniquely determined function $g\colon X \to Y$ satisfying the equation

$$\|x\| = \|y\| \quad \Longrightarrow \quad g(x + y) = g(x) + g(y) + d(x, y) \tag{5.3}$$

and such that

$$\|f(x) - g(x)\| \le \frac{\varepsilon\|x\|^p}{2^p - 2}, \quad x \in X.$$

Proof. The reasoning is actually analogous as for Theorem 4.2, but for the convenience of readers we present it here. Let $h : X \to Y$ be a solution of the equation

$$h(x+y) = h(x) + h(y) + d(x,y)$$

and $f_0 := f - h$. Then

$$\|x\| = \|y\| \quad \Longrightarrow \quad \|f_0(x+y) - f_0(x) - f_0(y)\| \leq \varepsilon \|x\|^p,$$

whence by Corollary 5.6 there exists a unique solution $T_0\colon X \to E$ of (1.8) with

$$\|f_0(x) - T_0(x)\| \leq \frac{\varepsilon \|x\|^p}{2^p - 2}, \qquad x \in X.$$

Let $T\colon = T_0 + g$. Then it is easily seen that T is a solution of (5.3) and

$$\|f(x) - T(x)\| \leq \frac{\varepsilon \|x\|^p}{2^p - 2}, \qquad x \in X.$$

Assume that $T'\colon D \to E$ is also a solution of (5.3) with

$$\|f(x) - T'(x)\| \leq \frac{\varepsilon \|x\|^p}{2^p - 2}, \qquad x \in X.$$

Write $T_0' := T' - g$. Then T_0' is a solution to (1.8) and

$$\|f_0(x) - T_0'(x)\| \leq \frac{\varepsilon \|x\|^p}{2^p - 2}, \qquad x \in X,$$

which means that $T_0 = T_0'$ and, consequently, $T = T'$. □

Making additional assumptions in Theorem 5.5, we obtain not only the existence but also the form of function g. Namely, we have

Theorem 5.8. *Assume* (I) *or* (II). *If* $D := \{(x,y) \in X \times X : \gamma(x) = \gamma(y)\}$, $\varphi\colon X \times X \to [0,\infty)$, γ *satisfy* 1° *or* 2°, *and* $f\colon X \to Y$ *satisfies for all* $x, y \in X$ *condition* (5.2), *then there exists a uniquely determined additive function* $a\colon X \to Y$ *such that*

$$\|f(x) - a(x)\| \leq \Phi(x), \quad x \in X.$$

5.2. **Pexider equation.** It is interesting to study an alternative of (1.11), where we consider three functions f, g, h instead of the only f in the successor of the implication. We start this part with giving solutions of the conditional Pexider equation of the form

$$\gamma(x) = \gamma(y) \quad \Longrightarrow \quad f(x+y) = g(x) + h(y) \tag{5.4}$$

under assumptions (I) or (II), although in case of solutions of the equation it is enough to assume about Y that it is an abelian group uniquely 2-divisible (see Sikorska [104, Theorem 3.1]).

Theorem 5.9. *Assume* (I) *or* (II)*, and in case* (II) *also* $(\gamma)_{1/2}$ *or* $(\gamma)_2$*. If* $f, g, h\colon X \to Y$ *satisfy for all* $x, y \in X$ *condition* (5.4)*, then for all* $x \in X$*,*

$$\begin{aligned} f(x) &= A(x) + \alpha + \beta, \\ g(x) &= A(x) + \delta(x) + \alpha, \\ h(x) &= A(x) - \delta(x) + \beta, \end{aligned} \tag{5.5}$$

where $\alpha := g(0)$*,* $\beta := h(0)$*,* $A\colon X \to Y$ *is additive,* $\delta\colon X \to Y$ *is an even function with the properties* $\delta(0) = 0$ *and*

$$\gamma(x) = \gamma(y) \quad \Longrightarrow \quad \delta(x) = \delta(y), \tag{5.6}$$

i.e., $\delta = \Lambda \circ \gamma$ *for some* $\Lambda\colon Z \to Y$ *such that* $\Lambda(\gamma(0)) = 0$*.*

Conversely, each triple of functions f, g, h *given by formulas* (5.5) *with arbitrary constants* $\alpha, \beta \in Y$*, arbitrary additive function* $A\colon X \to Y$ *and arbitrary* $\delta\colon X \to Y$ *satisfying* (5.6) *is a solution of* (5.4)*.*

The form (5.5) of solutions of equation (5.4) is significantly different from the usually expected one for a solution of the Pexider equation (i.e., $A + \alpha + \beta$, $A + \alpha$, $A + \beta$, respectively; cf., section 6.2 of this paper).

In what follows we deal with the Hyers-Ulam stability of (5.4) (cf., [104, Theorem 4.1]).

Theorem 5.10. *Assume* (I) *or* (II) *with* γ *which satisfies also* $(\gamma)_2$ *and let* $\varepsilon > 0$*. If* $f, g, h\colon X \to Y$ *fulfil for all* $x, y \in X$ *the condition*

$$\gamma(x) = \gamma(y) \quad \Longrightarrow \quad \|f(x+y) - g(x) - h(y)\| \le \varepsilon,$$

then there exist a uniquely determined additive function $A\colon X \to Y$*, an even function* $\delta\colon X \to Y$ *with* $\delta(0) = 0$*,*

$$\gamma(x) = \gamma(y) \quad \Longrightarrow \quad \delta(x) = \delta(y),$$

and positive constants k, l *such that for all* $x \in X$ *we have*

$$\begin{aligned} \|f(x) - f(0) - A(x)\| &\le k\varepsilon, \\ \|g(x) - g(0) - A(x) - \delta(x)\| &\le l\varepsilon, \\ \|h(x) - h(0) - A(x) + \delta(x)\| &\le l\varepsilon. \end{aligned} \tag{5.7}$$

Since $\|f(0) - g(0) - h(0)\| \le \varepsilon$, on account of (5.7) we have approximations of functions f, g, h by $A + \alpha + \beta$, $A + \delta + \alpha$, $A - \delta + \beta$, respectively, for $\alpha := g(0)$, $\beta := h(0)$, and whence the Pexider equation (5.4) is stable in the Hyers-Ulam sense.

Function δ in the above theorem is not uniquely determined. We define it with help of the axiom of choice. However, in some particular instances it can be defined by a precise formula (see Sikorska [104, Remark 4.2]).

Studying general stability, we present a result concerning the conditional equation of the form

$$\gamma(x) = \gamma(y) \quad \Longrightarrow \quad f(x+y) = g(x) + g(y)$$

in a special case (for the results in more general form see Sikorska [104, Theorem 5.1 – Theorem 5.6]).

Corollary 5.11. *Let X be a real inner product space with $\dim X \geq 2$ and let Y be a real Banach space. If functions $f, g\colon X \to Y$ satisfy for all $x, y \in X$ the condition*

$$\|x\| = \|y\| \quad \Longrightarrow \quad \|f(x+y) - g(x) - g(y)\| \leq \varepsilon \|x+y\|^p ,$$

where $p > 1$ and ε is some non-negative constant, then there exists a uniquely determined additive function $A\colon X \to Y$ such that for all $x \in X$,

$$\|f(x) - f(0) - A(x)\| \leq \left(\frac{12}{1-2^{1-p}} + 1\right) \varepsilon \|x\|^p,$$

$$\|g(x) - g(0) - A(x)\| \leq \frac{6 \cdot 2^p}{1-2^{1-p}} \varepsilon \|x\|^p.$$

5.3. **Some applications.** Results concerning stability in the sense of Hyers-Ulam, or more generally in the sense of Bourgin, may be applied to separation problems, that is, to so called "sandwich theorems" (see, e.g., Kranz [70], Gajda and Kominek [51]).

We start with quite general result (see Fechner [41]).

Theorem 5.12. *Assume that $(S, +)$ is a semigroup, $D \subset S \times S$, $p\colon S \to \mathbb{R}$ and $q\colon S \to \mathbb{R}$ satisfy*

$$(x,y) \in D \quad \Longrightarrow \quad p(x+y) \leq p(x) + p(y), \tag{5.8}$$

$$(x,y) \in D \quad \Longrightarrow \quad q(x+y) \geq q(x) + q(y), \tag{5.9}$$

$q \leq p$ and $\|p-q\|_{\sup} < +\infty$. If for every $x \in S$ implies that

$$(x,x) \in D \tag{5.10}$$

and the conditional Cauchy equation (1.13) is stable on D, then there exists a solution $f\colon S \to \mathbb{R}$ of (1.13) such that $q \leq f \leq p$.

Proof. Fix $(x,y) \in D$ arbitrarily and check that

$$p(x+y) - p(x) - p(y) \leq 0$$

and

$$\begin{aligned} p(x+y) - p(x) - p(y) &\geq q(x+y) - p(x) - p(y) \\ &\geq q(x+y) - q(x) - q(y) - 2\|p-q\|_{\sup}. \end{aligned}$$

Thus, after letting $\varepsilon := 2\|p-q\|_{\sup}$ we arrive at

$$(x,y) \in D \quad \Longrightarrow \quad |p(x+y) - p(x) - p(y)| \leq \varepsilon.$$

From our assumptions it follows that there exist a $\delta > 0$ and a solution $f\colon S \to \mathbb{R}$ of (1.13) such that $\|p - f\|_{\sup} \leq \delta$.

Now, by the use of (5.10) jointly with (1.13), (5.8) and (5.9) we obtain

$$f(2x) = 2f(x), \quad p(2x) \leq 2p(x), \quad q(2x) \geq 2q(x), \quad x \in S.$$

On the other hand, we have

$$q(x) - \delta \leq p(x) - \delta \leq f(x) \leq p(x) + \delta, \quad x \in S,$$

and thus

$$\begin{aligned} 2^n q(x) - \delta \leq q(2^n x) - \delta \leq f(2^n x) = 2^n f(x) \leq p(2^n x) + \delta \\ \leq 2^n p(x) + \delta, \quad x \in S. \end{aligned}$$

Divide this estimations side by side by 2^n to get

$$q(x) + \frac{1}{2^n}\delta \leq f(x) \leq p(x) + \frac{1}{2^n}\delta, \quad x \in S.$$

Now, tend with n to $+\infty$ to deduce that

$$q(x) \leq f(x) \leq p(x), \quad x \in S.$$

□

By Theorem 5.5 we get (see Fechner [41]).

Corollary 5.13. *Assume that $(S, +)$ is abelian semigroup, Z is a nonempty set, and $\gamma\colon S \to Z$ satisfies the condition $(\gamma)_2$. Further, let $p\colon S \to \mathbb{R}$ and $q\colon S \to \mathbb{R}$ satisfy*

$$\gamma(x) = \gamma(y) \quad \Longrightarrow \quad p(x+y) \leq p(x) + p(y),$$

$$\gamma(x) = \gamma(y) \quad \Longrightarrow \quad q(x+y) \geq q(x) + q(y).$$

$q \leq p$ and $\|p - q\|_{\sup} < +\infty$. Then there exists a solution $f\colon S \to \mathbb{R}$ of (1.12) such that $q \leq f \leq p$.

Now, basing on Theorem 5.8 for some special case of φ we give two other "sandwich" results (see Fechner and Sikorska [44] for this and more general results).

Corollary 5.14. *Assume that X is a real inner product space with $\dim X \geq 2$, $r > 1$, $\lambda \geq 0$, $p\colon X \to \mathbb{R}$ and $q\colon X \to \mathbb{R}$ satisfy conditions $q \leq p$,*

$$\begin{aligned} \|x\| = \|y\| \quad &\Longrightarrow \quad p(x+y) \leq p(x) + p(y), \\ \|x\| = \|y\| \quad &\Longrightarrow \quad q(x+y) \geq q(x) + q(y), \end{aligned} \tag{5.11}$$

and at least one from the approximations

$$\begin{aligned} |q(x) - p(x)| &\leq \lambda \|x\|^r, \quad x \in X, \\ |q(x) - p(-x)| &\leq \lambda \|x\|^r, \quad x \in X. \end{aligned} \tag{5.12}$$

Then there exists a unique additive function $A\colon X \to \mathbb{R}$ such that $q \leq A \leq p$.

Corollary 5.15. *Assume that X is a real inner product space with $\dim X \geq 2$, $r \in (0, 1)$, $\lambda \geq 0$, $p\colon X \to \mathbb{R}$ and $q\colon X \to \mathbb{R}$ satisfy relations $p \leq q$, (5.11) and at least one from the approximations (5.12). Then there exists a unique additive function $A\colon X \to \mathbb{R}$ such that $p \leq A \leq q$.*

5.4. **Exponential equation.** Let X be a real linear normed space, and Y a semigroup with a neutral element. We will consider $f\colon X \to Y$ satisfying a conditional equation of the form

$$\|x\| = \|y\| \quad \Longrightarrow \quad f(x+y) = f(x)f(y). \tag{5.13}$$

In the case, where the target space is a group (not necessarily abelian), the form of solutions of (5.13) is known (see Alsina and Garcia-Roig [3], Szabó [112], Ger and Sikorska [57], Brzdęk [18]). Namely, such solutions (under some assumptions on the dimension of X) satisfy equation

$$f(x+y) = f(x)f(y), \quad x, y \in X \tag{5.14}$$

unconditionally. An additional assumption allows us to solve the equation in the class of functions with values in semigroups. The key result is the following lemma (see Sikorska [106]).

Lemma 5.16. *Let X be a real normed space, $\dim X \geq 2$, $(Y, \cdot)$ be an abelian semigroup with a neutral element, and let $I \subset Y$ stand for a subgroup of all invertible elements in Y. If $f\colon X \to Y$ satisfies (5.13), then either $f(X) \subset I$ or $f(X) \cap I = \emptyset$.*

Usually, a result as above is one of the main tool while proving facts concerning exponential functions. In our case it allows us to extend the result for functions with values in a group to semigroups.

Theorem 5.17. *Let X be a real normed space with $\dim X \geq 2$, $(Y, \cdot)$ be an abelian semigroup with a neutral element. Assume that $f\colon X \to Y$ is a solution of (5.13) and there exists $x_0 \in X$ such that $f(x_0)$ is invertible in Y. If one of the following conditions is satisfied:*

(i) *X is an inner product space,*
(ii) $\dim X \geq 3$,

then f satisfies (5.14).

The example below (cf., Brzdęk [18]) shows that Theorem 5.17 is not true in case where $\dim X = 1$. Indeed, consider a function $f\colon \mathbb{R} \to \mathbb{R}$ given by the formula

$$f(x) = \begin{cases} 1, & \text{if} \quad x = 0, \\ a^{2^n}, & \text{if} \quad x \in (2^n, 2^{n+1}]\,,\ n \in \mathbb{N}, \\ f(-x)^{-1}, & \text{if} \quad x < 0, \end{cases}$$

for arbitrary $a > 0$ and $a \neq 1$, which satisfy (5.13), but does not satisfy (5.14). The case where $\dim X = 2$ and the norm in X does not come from any inner product, remains open.

Next result concerns Hyers-Ulam stability of (5.13). The forms of inequalities appearing in the assumptions of the theorem have their motivation in Ger's [53], and consideration of conditional forms was suggested by Chudziak (in case of the Gołąb-Schinzel equation) in [34]. Below we join these two approaches (see Sikorska [106]).

Theorem 5.18. *Let X be a real normed space with* $\dim X \geq 2$. *If* $f\colon X \to \mathbb{K}$, *where* $\mathbb{K} \in \{\mathbb{R}, \}$, *satisfies*

$$\Big(\|x\| = \|y\|,\ f(x)f(y) \neq 0\Big) \quad\Longrightarrow\quad \left|\frac{f(x+y)}{f(x)f(y)} - 1\right| \leq \varepsilon \tag{5.15}$$

and

$$\Big(\|x\| = \|y\|,\ f(x+y) \neq 0\Big) \quad\Longrightarrow\quad \left|\frac{f(x)f(y)}{f(x+y)} - 1\right| \leq \varepsilon \tag{5.16}$$

with some nonnegative constant $\varepsilon < 1$ and f does not vanish on X, then there exists a uniquely determined function $g\colon X \to (0, \infty)$ satisfying (5.13) *and*

$$\frac{1}{1+\varepsilon} \leq \left|\frac{f(x)}{g(x)}\right| \leq 1 + \varepsilon, \quad x \in X. \tag{5.17}$$

Moreover, if $\mathbb{K} = \mathbb{R}$, then for all $x \in X$,

$$\left|\frac{f(x)}{g(x)} - 1\right| \leq \varepsilon \quad \textit{and} \quad \left|\frac{g(x)}{f(x)} - 1\right| \leq \varepsilon.$$

If we assume that X is an inner product space or $\dim X \geq 3$, then g in Theorem 5.18 is (unconditionally) exponential, that is, it satisfies (5.14).

In the case when $(X, +)$ is a cancellative abelian semigroup and $f\colon X \to \setminus\{0\}$ satisfies (5.15) and (5.16) unconditionally, Ger and Šemrl [55] proved that the expression $\left|\frac{f(x)}{g(x)} - 1\right|$ may be approximated by a constant (depending on ε) with property that it converges to zero while ε tends to zero. It remains open, whether similar approximation can be achieved in case of Theorem 5.18 (and later on, in Theorem 5.23).

The results concerning orthogonally exponential function on general spheres come from Sikorska [105]. We start with solutions of a suitable equation.

$$\gamma(x) = \gamma(y) \quad\Longrightarrow\quad f(x+y) = f(x)f(y). \tag{5.18}$$

Assume that X is a real linear space, $\dim X \geq 2$, and an even function γ has properties $(\mathrm{a})_1$–$(\mathrm{a})_3$ (see beginning of section 5).

First we give solutions of (5.18).

Theorem 5.19. *Let $(Y, \cdot)$ be an abelian semigroup with a neutral element. If $f\colon X \to Y$ is a solution of* (5.18) *and there exist $x_0 \in X \setminus \{0\}$ such that $f(x_0)$ is invertible in Y, then f is a group homomorphism.*

Next theorem brings results concerning stability of (5.18).

Theorem 5.20. *If $f\colon X \to \mathbb{K}$ satisfies*

$$\Big[\gamma(x) = \gamma(y),\ f(x)f(y) \neq 0\Big] \quad\Longrightarrow\quad \left|\frac{f(x+y)}{f(x)f(y)} - 1\right| \leq \varepsilon$$

and

$$\Big[\gamma(x) = \gamma(y),\ f(x+y) \neq 0\Big] \quad\Longrightarrow\quad \left|\frac{f(x)f(y)}{f(x+y)} - 1\right| \leq \varepsilon$$

for some nonnegative $\varepsilon < 1$ *and* f *does not vanish on* X*, then there exists a uniquely determined function* $g\colon X \to (0,\infty)$ *such that for all* $x, y \in X$*,*

$$\gamma(x) = \gamma(y) \quad \Longrightarrow \quad g(x+y) = g(x)g(y),$$

and for all $x \in X$*,*

$$\left|\frac{f(x)}{g(x)} - 1\right| \le \varepsilon + 2 \quad \text{and} \quad \left|\frac{g(x)}{f(x)} - 1\right| \le \varepsilon + 2.$$

Moreover, if $\mathbb{K} = \mathbb{R}$*, the for all* $x \in X$*,*

$$\left|\frac{f(x)}{g(x)} - 1\right| \le \varepsilon \quad \text{and} \quad \left|\frac{g(x)}{f(x)} - 1\right| \le \varepsilon.$$

Similarly as in the classical version of the Cauchy equation, we may consider a pexiderized version of the exponential equation, namely a conditional equation of the form

$$\|x\| = \|y\| \quad \Longrightarrow \quad f_1(x+y) = f_2(x)f_3(y) \tag{5.19}$$

(see Sikorska [106]).

Assume that X is a real normed space with $\dim X \ge 2$. We start with giving solutions of (5.19).

Theorem 5.21. *Let* $(Y, \cdot)$ *be a uniquely* 2*-divisible abelian semigroup with a neutral element. Assume that* $f_1, f_2, f_3\colon X \to Y$ *are solutions of* (5.19) *and there exists* $x_0 \in X$ *such that* $f_1(x_0)$ *is invertible in* Y*. If one of the following conditions is fulfilled:*

(i) X *is an inner product space,*
(ii) $\dim X \ge 3$,

then there exists a uniquely determined exponential function $g\colon X \to I$ *(that is,* g *satisfies* (5.14)*), an even function* $\delta\colon X \to I$ *constant on spheres (i.e.,* $\delta(x) = \delta(y)$ *whenever* $\|x\| = \|y\|$*) and* $\alpha, \beta \in I$ *such that*

$$f_1 = \alpha\beta g, \quad f_2 = \alpha g\delta, \quad f_3 = \beta g \delta^{-1}.$$

Studying stability of (5.19) for $f_1, f_2, f_3\colon X \to \mathbb{K}$, we consider conditions

$$\Big(\|x\| = \|y\|,\ f_2(x)f_3(y) \ne 0\Big) \quad \Longrightarrow \quad \left|\frac{f_1(x+y)}{f_2(x)f_3(y)} - 1\right| \le \varepsilon \tag{5.20}$$

and

$$\Big(\|x\| = \|y\|,\ f_1(x+y) \ne 0\Big) \quad \Longrightarrow \quad \left|\frac{f_2(x)f_3(y)}{f_1(x+y)} - 1\right| \le \varepsilon \tag{5.21}$$

for some $\varepsilon \ge 0$.

Similarly as in the previous case, the key-result is the following lemma.

Lemma 5.22. *If* $f_1, f_2, f_3\colon X \to \mathbb{K}$ *satisfy conditions* (5.20) *and* (5.21) *with some nonnegative* $\varepsilon < 1$*, then either* $f_1(X) = \{0\}$*, or* $0 \notin f_1(X)$*.*

Theorem 5.23. *Let* $f_1, f_2, f_3\colon X \to \mathbb{K}$ *satisfy* (5.20) *and* (5.21) *with some nonnegative* $\varepsilon < 1$ *and let* f_1 *does not vanish on* X*. If one of the following conditions is fulfilled:*

(i) *X is an inner product space,*
(ii) $\dim X \geq 3$,

then there exist functions $g_1, g_2, g_3 \colon X \to (0, \infty)$ and positive constants c_1, c_2, c_3 such that for all $x, y \in X$,

$$\|x\| = \|y\| \quad \Longrightarrow \quad g_1(x+y) = g_2(x)g_3(y)$$

and for all $x \in X$, $i \in \{1, 2, 3\}$,

$$\frac{1}{(1+\varepsilon)^{c_i}} \leq \left|\frac{f_i(x)}{g_i(x)}\right| \leq (1+\varepsilon)^{c_i}. \tag{5.22}$$

In particular, there exist an additive function $A \colon X \to \mathbb{R}$, an even function $\delta \colon X \to \mathbb{R}$ constant on spheres and vanishing at zero, and positive constants α, β such that for all $x \in X$

$$\begin{aligned} g_1(x) &= \exp\big(A(x) + \alpha + \beta\big), \\ g_2(x) &= \exp\big(A(x) + \delta(x) + \alpha\big), \\ g_3(x) &= \exp\big(A(x) - \delta(x) + \beta\big). \end{aligned}$$

Moreover, if $\mathbb{K} = \mathbb{R}$, then f_1 is of a constant sign, and for all $x \in X$,

$$\left|\frac{\mu f_1(x)}{g_1(x)} - 1\right| \leq (1+\varepsilon)^{c_1} - 1 \quad \text{and} \quad \left|\frac{g_1(x)}{\mu f_1(x)} - 1\right| \leq (1+\varepsilon)^{c_1} - 1$$

for some $\mu \in \{-1, 1\}$.

It follows from (5.22) that

$$\left|\frac{f_i(x)}{g_i(x)} - 1\right| \leq \left|\frac{f_i(x)}{g_i(x)}\right| + 1 \leq (1+\varepsilon)^{c_i} + 1, \quad x \in X,\ i \in \{1, 2, 3\},$$

which means that the approximating constant tends to 2, while ε tending to zero (see also (5.17)). Constants c_1, c_2, c_3 in the assertion of Theorem 5.23 can be derived from Sikorska [104] (see also Sikorska [106]).

6. General orthogonal stability

In this part of the paper we present results concerning conditional functional equations with condition $(x, y) \in D = \{(x, y) : x \perp y\}$. The orthogonality relation will be understood either as the Birkhoff orthogonality (or, more generally, orthogonality relation in an orthogonality space), or as a generalization of the James orthogonality, i.e., with $D = \{(x, y) : \gamma(x+y) = \gamma(x-y)\}$.

6.1. **Cauchy equation.**

We start this part with presenting a stability result concerning (1.12). In [57] it was shown that under some assumptions on γ, every odd solution of (1.12) is additive.

In what follows, we give results on a general stability of equation (1.12) (see Sikorska [103]).

For the next theorems, instead of conditions (I) and (II), consider:

(I)' X is a real linear space with $\dim X \geq 2$, $(Y, +)$ is a real Banach space, Z is a nonempty set, and $\gamma\colon X \to Z$ is an even function satisfying $(a)_3$ with $\lambda = 1$,

(II)' X is a real linear space with $\dim X \geq 2$, $(Y, +)$ is a real Banach space, $(Z, +)$ is a topological group equipped with a binary connected relation $\prec \subset Z \times Z$, $\gamma\colon X \to Z$ is even and the following conditions hold: for every $x \in X$ we have $\gamma(0) \prec \gamma(x)$ or $\gamma(0) = \gamma(x)$ and for every $x \in X$ there exists $y \in X$ such that $\gamma(x) = \gamma(y)$ and $\gamma(x+y) = \gamma(x-y)$.

Moreover, for a given nonempty set $D \subset X \times X$, consider still the following properties of functions $\varphi\colon X \times X \to [0, \infty)$ and $\gamma\colon X \to Z$:

3° (i) series $\sum_{n=0}^{\infty} 4^{1-n}\varphi(2^{n-1}x, 2^{n-1}x)$ is convergent for every $x \in X$; denote its sum by $\Psi(x)$;
(ii) $\lim_{n\to\infty} 4^{-n}\varphi(2^n x, 2^n y) = 0$ for all $(x, y) \in D$;
(iii) function γ satisfies $(\gamma)_2$;

and

4° (i) series $\sum_{n=1}^{\infty} 4^{n}\varphi(2^{-n}x, 2^{-n}x)$ is convergent for every $x \in X$; denote its sum by $\Psi(x)$;
(ii) $\lim_{n\to\infty} 4^{n}\varphi(2^{-n} x, 2^{-n} y) = 0$ for all $(x, y) \in D$;
(iii) function γ satisfies condition $(\gamma)_{1/2}$.

Theorem 6.1. *Assume* (I)' *or* (II)'*. Let* $D := \{(x, y) \in X \times X : \gamma(x+y) = \gamma(x-y)\}$*,* $\varphi\colon X \times X \to [0, \infty)$ *and* γ *satisfy one of the conditions* 1° *or* 2° *and one of the conditions* 3° *or* 4°*, and in each case assume that*

(iv) *there exists* $M \geq 1$ *such that for all* $x, y \in X$*, if* $\gamma(2x) = \gamma(2y)$ *and* $\gamma(x+y) = \gamma(x-y)$*, then*

$$\max\{\varphi(x, y), \varphi(x, -y), \varphi(x+y, x-y), \varphi(x+y, y-x)\} \leq M\varphi(x, x).$$

If $f\colon X \to Y$ *fulfils for all* $x, y \in X$ *condition*

$$\gamma(x+y) = \gamma(x-y) \quad \Longrightarrow \quad \|f(x+y) - f(x) - f(y)\| \leq \varphi(x, y), \tag{6.1}$$

then there exists a uniquely determined function $g\colon X \to Y$ *satisfying* (1.12) *and such that*

$$\|f(x) - g(x)\| \leq \frac{1}{2}M\big[3\Phi(x) + 3\Phi(-x) + \Psi(x) + \Psi(-x)\big], \quad x \in X.$$

The above theorem is a pure stability result, that is, it gives a positive answer for a question about the existence of an approximation function satisfying condition $\gamma(x+y) = \gamma(x-y)$. It does not tell, however, anything about the form of solutions of such conditional equation. In a general situation, we do not know the form of solutions of (1.12). However, if we assume (I) or (II) and oddness of f, (cf., [103, Theorem 2.3]) we get the approximation by an additive function. Also in a special case $\gamma := \|\cdot\|$, on account of Szabó's results [112], the form of an approximating function is known.

Remark 6.2. If X is a real linear normed space, $\dim X \geq 3$, $\gamma := \|\cdot\|$ and $\varphi(x,y) := \varepsilon(\|x\|^p + \|y\|^p)$, then we obtain a result concerning the generalized stability of orthogonal additivity with orthogonality relation defined in the sense of James (see Sikorska [102, Theorem 2.9]). Similar approach is used in case of orthogonal additivity with the Birkhoff orthogonality relation (see [102, Theorem 2.4]).

Although, the method of splitting a function into its odd and even parts works almost in all situations, and also condition (iv) is given in quite general form, in some particular situations one can proceed in a different way what leads usually to much better approximations. We show it on an example of the Birkhoff orthogonality and $\varphi(x,y) = \varepsilon\left(\|x\|^p + \|y\|^p\right)$ with $p < 1$.

Theorem 6.3. *Assume X is a real normed space with $\dim X \geq 2$ and with Birkhoff orthogonality, and Y is a real Banach space. If $f\colon X \to Y$ satisfies*

$$x \perp y \quad \Longrightarrow \quad \|f(x+y) - f(x) - f(y)\| \leq \varepsilon\left(\|x\|^p + \|y\|^p\right) \tag{6.2}$$

with some $\varepsilon \geq 0$ and $p < 1$, then there exists an orthogonally additive function $g\colon X \to Y$ such that

$$\|f(x) - g(x)\| \leq \frac{1}{1 - 2^{p-1}}\,\alpha\,\varepsilon\|x\|^p, \quad x \in X_p,$$

where X_p stands for X if $p \geq 0$ (with $0^0 := 1$) and $X \setminus \{0\}$ if $p < 0$ and

$$\alpha = \begin{cases} \frac{1}{2}\left(4 + 3\cdot 2^p + 2\cdot 3^p + 4^p\right) & \text{if } p \geq 0 \\ 2 + 3\cdot 2^{-p} & \text{if } p < 0. \end{cases} \tag{6.3}$$

Proof. Let us observe first some properties of vectors which are orthogonal in the Birkhoff sense. Assume that for two vectors x and y we have $x \perp y$ and $x+y \perp x-y$. From the definition of the orthogonality, if $x \perp y$, then $\|x\| \leq \|x+y\|$ and $\|x\| \leq \|x-y\|$ (for $\lambda = 1$ and $\lambda = -1$, respectively), and, analogously, if $x+y \perp x-y$ then $\|x+y\| \leq \|2x\|$ and $\|x+y\| \leq \|2y\|$. From these relations and the triangle inequality we have additionally: $\|y\| \leq 3\|x\|$, $\|x-y\| \leq 4\|x\|$, $\|x\| \leq 2\|y\|$.

In the case when p is a nonnegative real number, we have the approximations

$$\|y\|^p \leq 3^p\|x\|^p, \quad \|x+y\|^p \leq 2^p\|x\|^p, \quad \|x-y\|^p \leq 4^p\|x\|^p, \tag{6.4}$$

otherwise,

$$\|y\|^p \leq 2^{-p}\|x\|^p, \quad \|x+y\|^p \leq \|x\|^p, \quad \|x-y\|^p \leq \|x\|^p. \tag{6.5}$$

Take now $x \in X$. There exists $y \in X$ such that $x \perp y$ and $x+y \perp x-y$. As in [43] it was done in case ε now we obtain approximations

$$\begin{aligned}\|3f(2x) - 8f(x) - f(-2x)\| &\leq 4\varepsilon\left(\|x+y\|^p + \|x-y\|^p\right) \\ &\quad + 8\varepsilon\left(\|x\|^p + \|y\|^p\right) + 8\varepsilon\left(\left\|\frac{x+y}{2}\right\|^p + \left\|\frac{x-y}{2}\right\|^p\right).\end{aligned}$$

By means of (6.4) or (6.5), in respective cases we obtain

$$\left\| f(x) - \frac{3}{8} f(2x) + \frac{1}{8} f(-2x) \right\| \le \alpha\, \varepsilon \|x\|^p, \quad x \in X_p, \tag{6.6}$$

where α is given by (6.3).

Using induction we get

$$\left\| f(x) - \frac{2^n+1}{2 \cdot 4^n} f(2^n x) + \frac{2^n-1}{2 \cdot 4^n} f(-2^n x) \right\| \le \frac{2^{n(p-1)}-1}{2^{p-1}-1} \alpha\, \varepsilon \|x\|^p, \ \ x \in X_p,\ n \in \mathbb{N}.$$

A standard procedure allows us to define

$$g(x) := \lim_{n\to\infty} \left[\frac{2^n+1}{2 \cdot 4^n} f(2^n x) - \frac{2^n-1}{2 \cdot 4^n} f(-2^n x) \right], \quad x \in X_p,$$

and to obtain the desired approximation. □

One can easily observe that setting $p := 0$ in the above theorem gives a result from [43]. In fact, the orthogonality relation which appears in [43] is defined quite generally, so that the results can be used also in case of condition defined by means of function γ. Namely, we have the following.

Theorem 6.4. *Assume* (I)' *or* (II)' *and let* γ *satisfy* $(\gamma)_2$. *If* $f\colon X \to Y$ *satisfies for some* $\varepsilon > 0$ *and all* $x, y \in X$ *the condition*

$$\gamma(x+y) = \gamma(x-y) \quad \Longrightarrow \quad \|f(x+y) - f(x) - f(y)\| \le \varepsilon,$$

then there exists a function $g\colon X \to Y$ *fulfilling* (1.12) *and such that*

$$\|f(x) - g(x)\| \le 5\varepsilon, \quad x \in X.$$

Using above results we may prove various kinds of "sandwich" theorems, where we separate orthogonally subadditive and orthogonally superadditive functions. We give here one example (cf., Fechner and Sikorska [42, Proposition 3]). More examples and more general forms of the theorem can be found in [42].

Theorem 6.5. *Let* X *be a real normed space,* $\dim X \ge 2$, *with Birkhoff orthogonality. Assume that* $p, q\colon X \to \mathbb{R}$ *satisfy*

$$x \perp y \quad \Longrightarrow \quad p(x+y) \le p(x) + p(y),$$
$$x \perp y \quad \Longrightarrow \quad q(x+y) \ge q(x) + q(y)$$

and

$$q(x) \le p(x) \quad \text{for all } x \in X.$$

If $p(x) - q(x) \le c\|x\|^r$ *for all* $x \in X$, *where* c, r *are positive constants and* $r > 2$, *then there exists a unique orthogonally additive mapping* $f\colon X \to \mathbb{R}$ *such that with some positive constant* d,

$$q(x) - d\|x\|^r \le f(x) \le p(x) + d\|x\|^r \quad \text{for all } x \in X.$$

With some additional assumptions imposed on functions p and q we get the approximation $q \le f \le p$ in the above theorem (see [42, Theorem 3]).

Some other results on orthogonally superadditive functions can be found in Fechner [40].

6.2. **Pexider equation.** In what follows we present results concerning a conditional Pexider equation with a condition $\gamma(x+y)=\gamma(x-y)$ (see Sikorska [104]). We start with solutions of the equation.

Theorem 6.6. *Let $(X,+)$, $(Y,+)$ be abelian groups, Z a given nonempty set, and $\gamma\colon X\to Z$ an even function. If $f,g,h\colon X\to Y$ are solutions of*

$$\gamma(x+y)=\gamma(x-y) \quad\Longrightarrow\quad f(x+y)=g(x)+h(y), \tag{6.7}$$

then there exist a uniquely determined solution $F\colon X\to Y$ of (1.12) and constants $\alpha,\beta\in Y$ such that

$$\begin{aligned} f(x)&=F(x)+\alpha+\beta, \quad x\in X,\\ g(x)&=F(x)+\alpha, \quad x\in X,\\ h(x)&=F(x)+\beta, \quad x\in X. \end{aligned}$$

Basing on Theorem 6.4, we go on with Hyers-Ulam stability of (6.7).

Theorem 6.7. *Assume* (I)' *or* (II)' *and let γ satisfy $(\gamma)_2$. If $f,g,h\colon X\to Y$ satisfy for some $\varepsilon>0$ and all $x,y\in X$ a condition*

$$\gamma(x+y)=\gamma(x-y) \quad\Longrightarrow\quad \|f(x+y)-g(x)-h(y)\|\le\varepsilon,$$

then there exists a uniquely determined function $F\colon X\to Y$ satisfying (1.12) and such that

$$\begin{aligned} \|f(x)-g(0)-h(0)-F(x)\|&\le 15\varepsilon, \quad x\in X,\\ \|g(x)-g(0)-F(x)\|&\le 16\varepsilon, \quad x\in X,\\ \|h(x)-h(0)-F(x)\|&\le 16\varepsilon, \quad x\in X. \end{aligned}$$

The part devoted to the conditional Pexider equation we finish with generalized stability result.

Theorem 6.8. *Assume* (I)' *or* (II)'. *Let $\psi\colon X\times X\to[0,\infty)$ be a given function and $\varphi(x,y):=\psi(x,y)+\psi(x,0)+\psi(0,y)$ for all $x,y\in X$. Assume moreover, that φ and γ satisfy one of the conditions 1° or 2° and one of the conditions 3° or 4°, and condition* (iv) *(see Theorem 6.1). If $f,g,h\colon X\to Y$ satisfy for all $x,y\in X$*

$$\gamma(x+y)=\gamma(x-y) \quad\Longrightarrow\quad \|f(x+y)-g(x)-h(y)\|\le\psi(x,y),$$

then there exist a uniquely determined function $F\colon X\to Y$ satisfying (1.12) and such that

$$\begin{aligned} \|f(x)-g(0)-h(0)-F(x)\|&\le \frac{1}{2}M\big[3\Phi(x)+3\Phi(-x)+\Psi(x)+\Psi(-x)\big],\\ \|g(x)-g(0)-F(x)\|&\le \frac{1}{2}M\big[3\Phi(x)+3\Phi(-x)+\Psi(x)+\Psi(-x)\big]+\psi(x,0),\\ \|h(x)-h(0)-F(x)\|&\le \frac{1}{2}M\big[3\Phi(x)+3\Phi(-x)+\Psi(x)+\Psi(-x)\big]+\psi(0,x), \end{aligned}$$

for all $x\in X$, where Φ and Ψ are defined in 1°(i) *or* 2°(i) *and in* 3°(i) *or* 4°(i), *respectively.*

6.3. **Exponential equation.** In what follows we put our attention to stability problem for a conditional exponential equation. We will investigate the equation both in classical and pexiderized forms.

Let $(X,+)$ be a groupoid with a neutral element 0 and let $(Y,\cdot)$ be a semigroup with a neutral element. Consider a function $f\colon X\to Y$ satisfying a quite general conditional equation of the form

$$(x,y)\in D \quad\Longrightarrow\quad f(x+y)=f(x)f(y), \tag{6.8}$$

where D is a nonempty subset of $X\times X$. Later on we will specify the properties of D.

Denote hypotheses:

($\mathcal{H}$) If $\varphi\colon X\to\mathbb{R}$ satisfies (6.8), then either $\varphi(X\setminus\{0\})=\{0\}$, or $0\notin\varphi(X)$.

($\mathcal{C}$) If $\varphi\colon X\to\mathbb{R}$, then the conditional equation $\varphi(x+y)=\varphi(x)+\varphi(y)$ for all $(x,y)\in D$ is stable in a sense, that if

$$(x,y)\in D \quad\Longrightarrow\quad |\varphi(x+y)-\varphi(x)-\varphi(y)|\le\varepsilon$$

for some $\varepsilon\ge 0$, then there exist a function $\psi\colon X\to\mathbb{R}$ and a positive constant c such that $\psi(x+y)=\psi(x)+\psi(y)$ for all $(x,y)\in D$ and $|\varphi(x)-\psi(x)|\le c\varepsilon$ for all $x\in X$.

($\mathcal{P}$) If $\varphi_1,\varphi_2,\varphi_3\colon X\to\mathbb{R}$, then the conditional equation $\varphi_1(x+y)=\varphi_2(x)+\varphi_3(y)$ for all $(x,y)\in D$ is stable in a sense, that if

$$(x,y)\in D \quad\Longrightarrow\quad |\varphi_1(x+y)-\varphi_2(x)-\varphi_3(y)|\le\varepsilon$$

for some $\varepsilon\ge 0$, then there exist functions $\psi_1,\psi_2,\psi_3\colon X\to\mathbb{R}$ and positive constants c_1,c_2,c_3 such that $\psi_i(x+y)=\psi_i(x)+\psi_i(y)$ dla $(x,y)\in D$ and $|\varphi_i(x)-\psi_i(x)|\le c_i\varepsilon$ for all $x\in X$, $i\in\{1,2,3\}$.

With various sets X and D we get various solutions of (6.8). We give here some known examples.

- ◇ For $X=\mathbb{R}^n$ and $D=X\times X$ results can be found in [1] or [2].
- ◇ If $(X,(\cdot|\cdot))$ is a real inner product space, and $D=\{(x,y)\in X\times X : (x|y)=0\}$, then solutions can be found in [13].
- ◇ If $(X,\perp)$ is an orthogonality space, and $D=\{(x,y)\in X\times X : x\perp y\}$, the results are collected in [20].
- ◇ If $(X,\|\cdot\|)$ is a real normed space, $\dim X\ge 2$ and $D=\{(x,y)\in X\times X : \|x+y\|=\|x-y\|\}$, then see [21].

Similar results may be also found in [12, 17, 19].

We will deal now with stability of (6.8). Namely, consider a function $f\colon X\to\mathbb{K}$ with $\mathbb{K}\in\{\mathbb{R},\}$ fulfilling inequalities

$$\Big[(x,y)\in D,\ f(x)f(y)\ne 0\Big] \quad\Longrightarrow\quad \left|\frac{f(x+y)}{f(x)f(y)}-1\right|\le\varepsilon \tag{6.9}$$

and

$$\Big[(x,y)\in D,\ f(x+y)\ne 0\Big] \quad\Longrightarrow\quad \left|\frac{f(x)f(y)}{f(x+y)}-1\right|\le\varepsilon \tag{6.10}$$

for some $\varepsilon\ge 0$ (see Sikorska [105]).

Let $\mathbb{K} \in \{\mathbb{R}, \}$. Similarly, as before, the main results will be preceded by a key-result for further considerations.

Lemma 6.9. *Under the assumption ($\mathcal{H}$), if $f\colon X \to \mathbb{K}$ satisfies (6.9) and (6.10) for some nonnegative constant $\varepsilon < 1$, then either $f(X \setminus \{0\}) = \{0\}$, or $0 \notin f(X)$.*

Theorem 6.10. *Assume ($\mathcal{H}$) i ($\mathcal{C}$). If $f\colon X \to \mathbb{K}$ satisfies (6.9) and (6.10) for some nonnegative constant $\varepsilon < 1$ and f does not vanish on $X \setminus \{0\}$, then there exist $g\colon X \to (0,\infty)$ satisfying (6.8) and a positive constant c such that for all $x \in X$,*

$$\left|\frac{f(x)}{g(x)} - 1\right| \le (1+\varepsilon)^c + 1 \quad \text{and} \quad \left|\frac{g(x)}{f(x)} - 1\right| \le (1+\varepsilon)^c + 1. \tag{6.11}$$

Moreover, if $(X,+)$ is a uniquely 2-divisible abelian group, $\mathbb{K} = \mathbb{R}$ and D is such that

- ⋄ *for every $x \in X$ there exist $y \in X$ such that $(x,y) \in D$ and $(x+y, x-y) \in D$,*
- ⋄ *if $(x,y) \in D$, then $(\pm x, \pm y) \in D$,*

then for all $x \in X$,

$$\left|\frac{f(x)}{g(x)} - 1\right| \le (1+\varepsilon)^c - 1 \quad \text{and} \quad \left|\frac{g(x)}{f(x)} - 1\right| \le (1+\varepsilon)^c - 1.$$

In case where $(X,+)$ is a cancellative abelian semigroup, $Y = \setminus\{0\}$ and $D = X \times X$, Ger and Šemrl [55] showed that expressions in (6.11) may be approximated by constants with a property that if considered as functions of variable ε, they tend to zero while ε tends to zero. In case of Theorem 6.10 it is an open problem whether in a complex case the approximation can be strengthened (cf., comments after Theorem 5.18).

As an immediate consequence of the above result we have a corollary.

Corollary 6.11. *Assume that $(X,\perp)$ is an orthogonality space or a normed linear space with $\dim X \ge 3$ and with James orthogonality. If $f\colon X \to \mathbb{K}$ satisfies*

$$\Big[x \perp y,\ f(x)f(y) \ne 0\Big] \quad \Longrightarrow \quad \left|\frac{f(x+y)}{f(x)f(y)} - 1\right| \le \varepsilon$$

and

$$\Big[x \perp y,\ f(x+y) \ne 0\Big] \quad \Longrightarrow \quad \left|\frac{f(x)f(y)}{f(x+y)} - 1\right| \le \varepsilon$$

for some nonnegative $\varepsilon < 1$, and f does not vanish outside zero, then there exists a unique function $g\colon X \to (0,\infty)$ satisfying for all $x, y \in X$ conditions

$$x \perp y \quad \Longrightarrow \quad g(x+y) = g(x)g(y),$$

$$\left|\frac{f(x)}{g(x)} - 1\right| \le (1+\varepsilon)^5 + 1 \quad \text{and} \quad \left|\frac{g(x)}{f(x)} - 1\right| \le (1+\varepsilon)^5 + 1.$$

Moreover, if $\mathbb{K} = \mathbb{R}$, then for all $x \in X$,

$$\left|\frac{f(x)}{g(x)} - 1\right| \le (1+\varepsilon)^5 - 1 \quad \text{and} \quad \left|\frac{g(x)}{f(x)} - 1\right| \le (1+\varepsilon)^5 - 1.$$

If we define $D := \{(x,y) : \gamma(x+y) = \gamma(x-y)\}$, on account of Brzdęk and Sikorska [29] and Theorem 6.4 we may formulate

Corollary 6.12. *Let X be a linear space,* $\dim X \geq 2$*, Z be a nonempty set, and $\gamma\colon X \to Z$ be an even function fulfilling* (a). *If $f\colon X \to \mathbb{K}$ satisfies*

$$\Big[\gamma(x+y) = \gamma(x-y),\ f(x)f(y) \neq 0\Big] \quad \Longrightarrow \quad \left|\frac{f(x+y)}{f(x)f(y)} - 1\right| \leq \varepsilon$$

and

$$\Big[\gamma(x+y) = \gamma(x-y),\ f(x+y) \neq 0\Big] \quad \Longrightarrow \quad \left|\frac{f(x)f(y)}{f(x+y)} - 1\right| \leq \varepsilon$$

for some nonnegative $\varepsilon < 1$ and $f(x_0) \neq 0$ for some $x_0 \neq 0$, then there exists exactly one $g\colon X \to (0,\infty)$ such that, for all $x, y \in X$,

$$\gamma(x+y) = \gamma(x-y) \quad \Longrightarrow \quad g(x+y) = g(x)g(y),$$

and

$$\left|\frac{f(x)}{g(x)} - 1\right| \leq \delta + 1 \quad \text{and} \quad \left|\frac{g(x)}{f(x)} - 1\right| \leq \delta + 1,$$

where $\delta = (1+\varepsilon)^5$.

Moreover, if $\mathbb{K} = \mathbb{R}$, then for all $x \in X$,

$$\left|\frac{f(x)}{g(x)} - 1\right| \leq \delta - 1 \leq 31\varepsilon \quad \text{and} \quad \left|\frac{g(x)}{f(x)} - 1\right| \leq \delta - 1 \leq 31\varepsilon.$$

The next part of the paper we devote to the pexiderized form of (6.8), that is, to equation of the form

$$(x,y) \in D \quad \Longrightarrow \quad f_1(x+y) = f_2(x)f_3(y). \tag{6.12}$$

Let $(X,+)$ be a groupoid with a neutral element 0 and let $D \subset X \times X$ has the property that $(x,0), (0,x) \in D$ for all $x \in X$. The last condition excludes applying the results in the case where $\gamma(x) = \gamma(y)$ (cf., section 5.4).

We start again with giving solutions of the equation.

Theorem 6.13. *Assume $(\mathcal{H})$. Let $(Y,\cdot)$ be an abelian semigroup with a neutral element. If $f_1, f_2, f_3\colon X \to Y$ are solutions of* (6.12) *and there exists $x_0 \in X\setminus\{0\}$ such that $f_1(x_0)$ is invertible in Y, then there exist a uniquely determined solution $g\colon X \to I$ of* (6.8) *and unique constants $\alpha, \beta \in I$ such that*

$$f_1 = \alpha\beta g, \quad f_2 = \alpha g, \quad f_3 = \beta g.$$

Studying stability of (6.12), for $f_1, f_2, f_3\colon X \to \mathbb{K}$ we will consider

$$\Big[(x,y) \in D,\ f_2(x)f_3(y) \neq 0\Big] \quad \Longrightarrow \quad \left|\frac{f_1(x+y)}{f_2(x)f_3(y)} - 1\right| \leq \varepsilon \tag{6.13}$$

and

$$\Big[(x,y) \in D,\ f_1(x+y) \neq 0\Big] \quad \Longrightarrow \quad \left|\frac{f_2(x)f_3(y)}{f_1(x+y)} - 1\right| \leq \varepsilon \tag{6.14}$$

for some $\varepsilon \geq 0$.

Theorem 6.14. *Assume ($\mathcal{H}$) and ($\mathcal{P}$). If $f_1, f_2, f_3 \colon X \to \mathbb{K}$ satisfy (6.13) and (6.14) for some nonnegative $\varepsilon < 1$ and f_1 does not vanish on $X \setminus \{0\}$, then there exist functions $g_1, g_2, g_3 \colon X \to (0, \infty)$ and positive constants c_1, c_2, c_3 such that*

$$(x, y) \in D \quad \Longrightarrow \quad g_1(x + y) = g_2(x)g_3(y),$$

and for all $x \in X$ and $i \in \{1, 2, 3\}$,

$$\left|\frac{f_i(x)}{g_i(x)} - 1\right| \leq (1 + \varepsilon)^{c_i} + 1 \quad \text{and} \quad \left|\frac{g_i(x)}{f_i(x)} - 1\right| \leq (1 + \varepsilon)^{c_i} + 1. \tag{6.15}$$

Moreover, if $\mathbb{K} = \mathbb{R}$, then there exist $\mu_1, \mu_2, \mu_3 \in \{-1, 1\}$ such that for all $x \in X$, $i \in \{1, 2, 3\}$,

$$\left|\frac{\mu_i f_i(x)}{g_i(x)} - 1\right| \leq (1 + \varepsilon)^{c_i} - 1 \quad \text{and} \quad \left|\frac{g_i(x)}{\mu_i f_i(x)} - 1\right| \leq (1 + \varepsilon)^{c_i} - 1. \tag{6.16}$$

Similarly as before, from the above theorem it follows a result concerning stability in the case when X is an orthogonality space or a normed linear space of dimension greater than or equal 3 with James orthogonality (see Sikorska [105]).

The next result concerns the case where $D := \{(x, y) : \gamma(x + y) = \gamma(x - y)\}$.

Corollary 6.15. *Let X be a linear space, $\dim X \geq 2$, Z be a nonempty set, and $\gamma \colon X \to Z$ be an even function fulfilling* (a)*. Let $f_1, f_2, f_3 \colon X \to \mathbb{K}$ satisfy*

$$\Big[\gamma(x + y) = \gamma(x - y),\ f_2(x)f_3(y) \neq 0\Big] \quad \Longrightarrow \quad \left|\frac{f_1(x + y)}{f_2(x)f_3(y)} - 1\right| \leq \varepsilon,$$

$$\Big[\gamma(x + y) = \gamma(x - y),\ f_1(x + y) \neq 0\Big] \quad \Longrightarrow \quad \left|\frac{f_2(x)f_3(y)}{f_1(x + y)} - 1\right| \leq \varepsilon$$

for some nonnegative $\varepsilon < 1$ and $f(x_0) \neq 0$ for some $x_0 \neq 0$. Then there exist functions $g_1, g_2, g_3 \colon X \to (0, \infty)$ such that

$$\gamma(x + y) = \gamma(x - y) \quad \Longrightarrow \quad g_1(x + y) = g_2(x)g_3(y)$$

and for all $x \in X$, $i \in \{1, 2, 3\}$, (6.15) is fulfilled with $c_1 = 15$ and $c_2 = c_3 = 16$.

Moreover, if $\mathbb{K} = \mathbb{R}$, then there are $\mu_1, \mu_2, \mu_3 \in \{-1, 1\}$ such that for all $x \in X$, $i \in \{1, 2, 3\}$, inequalities (6.16) hold with $c_1 = 15$ and $c_2 = c_3 = 16$.

7. Fuzzy Stability

The results of the following three sections are actually not of conditional type, but we present them here very briefly to stimulate investigations of their various conditional versions.

Mirmostafaei and Moslehian [75] exhibit three reasonable notions of approximately additive functions in fuzzy normed spaces and prove that under some suitable conditions, an approximately additive function f from a space X into a fuzzy Banach space Y can be approximated in a fuzzy sense by an additive mapping $T : X \to Y$. Let us give our notion of a fuzzy norm.

Definition 7.1. Let X be a real linear space. A function $N\colon X\times\mathbb{R}\to[0,1]$ (the so-called fuzzy subset) is said to be a fuzzy norm on X if for all $x,y\in X$ and all $s,t\in\mathbb{R}$,

(N1) $N(x,c)=0$ for $c\le 0$;
(N2) $x=0$ if and only if $N(x,c)=1$ for all $c>0$;
(N3) $N(cx,t)=N(x,\frac{t}{|c|})$ if $c\neq 0$;
(N4) $N(x+y,s+t)\ge\min\{N(x,s),N(y,t)\}$;
(N5) $N(x,.)$ is a non-decreasing function on $\mathbb{R}$ and $\lim_{c\to\infty}N(x,c)=1$.
(N6) if $x\neq 0$, then $N(x,.)$ is a (upper semi)continuous function on $\mathbb{R}$.

The pair (X,N) is called a fuzzy normed linear space. One may regard $N(x,t)$ to be the measure of the probability that the statement 'the norm of x is less than or equal to the real number t' is true. Note that the fuzzy normed linear space (X,N) is exactly a Menger probabilistic normed linear space (X,N,Δ) when $\Delta=\min$; see [33].

Example 7.2. Let $(X,\|\cdot\|)$ be a normed linear space. Then

$$N(x,t)=\begin{cases}\frac{t}{t+\|x\|} & t>0, x\in X\\ 0 & t\le 0, x\in X\end{cases}$$

is a fuzzy norm on X.

Example 7.3. Let $(X,\|\cdot\|)$ be a normed linear space. Then

$$N(x,t)=\begin{cases}0 & t\le 0\\ \frac{t}{\|x\|} & 0<t\le\|x\|\\ 1 & t>\|x\|\end{cases}$$

is a fuzzy norm on X.

Definition 7.4. Let (X,N) be a fuzzy normed linear space. Let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is said to be convergent if there exists $x\in X$ such that $\lim_{n\to\infty}N(x_n-x,t)=1$ for all $t>0$. In that case, x is called the limit of the sequence $\{x_n\}$ and we denote it by $N-\lim x_n=x$. A sequence $\{x_n\}$ in X is called Cauchy if for each $\varepsilon>0$ and each $t>0$ there exists n_0 such that for all $n\ge n_0$ and all $p>0$, we have $N(x_{n+p}-x_n,t)>1-\varepsilon$. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and each complete fuzzy normed space is called a fuzzy Banach space.

The first fuzzy stability theorem reads as follows.

Theorem 7.5. [75] *Let X be a linear space and (Y,N) be a fuzzy Banach space. Let $\varphi : X\times X\to[0,\infty)$ be a control function such that*

$$\widetilde{\varphi}(x,y)=\sum_{n=0}^{\infty}2^{-n}\varphi(2^nx,2^ny)<\infty,\qquad x,y\in X. \tag{7.1}$$

Let $f : X\to Y$ be a uniformly approximately additive function with respect to φ in the sense that

$$\lim_{t\to\infty}N(f(x+y)-f(x)-f(y),t\varphi(x,y))=1 \tag{7.2}$$

uniformly on $X \times X$*. Then the limit* $T(x) := N - \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$ *exists for each* $x \in X$ *and defines an additive mapping* $T : X \to Y$ *such that, for every* $\delta > 0$ *and* $\alpha > 0$ *with*

$$N(f(x+y) - f(x) - f(y), \delta\varphi(x,y)) > \alpha, \qquad x, y \in X, \tag{7.3}$$

we have

$$N\left(T(x) - f(x), \frac{\delta}{2}\widetilde{\varphi}(x,x)\right) > \alpha, \qquad x \in X.$$

Later many mathematicians investigated the stability of various functional equations in some fuzzy senses and we refer to [33] for further information.

8. Stability in non-Archimedean normed spaces

By a *non-Archimedean field* we mean a field K equipped with a function (the so-called valuation) $|\cdot|$ from K into $[0,\infty)$ such that $|r| = 0$ if and only if $r = 0$, $|rs| = |r|\,|s|$, and $|r+s| \le \max\{|r|, |s|\}$ for all $r, s \in K$. Evidently $|n| \le 1$ for all $n \in \mathbb{N}$. Given a vector space X over a field K equipped with a non-Archimedean non-trivial valuation $|\cdot|$, a function $\|\cdot\| : X \to [0,\infty)$ is called a *non-Archimedean norm* if it satisfies the following conditions:

(i) $\|x\| = 0$ if and only if $x = 0$;
(ii) $\|rx\| = |r|\|x\|$ for $r \in K, x \in X$;
(iii) the strong triangle inequality $\|x+y\| \le \max\{\|x\|, \|y\|\}$ for $x, y \in X$.

Then $(X, \|\cdot\|)$ is called a non-Archimedean normed space. By a complete non-Archimedean normed space we mean one in which every Cauchy sequence is convergent and every such a space we call a non-Archimedean Banach space.

To construct an example, fix a prime number p. For any nonzero rational number x, there exists a unique integer $n_x \in \mathbb{Z}$ such that $x = \frac{a}{b}p^{n_x}$, where a and b are integers not divisible by p. Then $|x|_p := p^{-n_x}$ defines a non-Archimedean norm on $\mathbb{Q}$. The completion of $\mathbb{Q}$ with respect to the metric $d(x,y) = |x-y|_p$ is denoted by $\mathbb{Q}_p$, which is called the p-adic number field.

In [6], the authors investigated stability of approximately additive mappings $f : \mathbb{Q}_p \to \mathbb{R}$. They showed that if $f : \mathbb{Q}_p \to \mathbb{R}$ is a continuous mapping for which there exists a fixed ϵ such that $|f(x+y) - f(x) - f(y)| \le \epsilon$ for all $x, y \in \mathbb{Q}_p$, then there exists a unique additive mapping $T : \mathbb{Q}_p \to \mathbb{R}$ such that $|f(x) - T(x)| \le \epsilon$ for all $x \in \mathbb{Q}_p$. In [79], the stability of Cauchy functional equation was investigated in the context of non-Archimedean normed spaces as follows.

Theorem 8.1. *Let* X *be a non-Archimedean Banach space,* $\mathbb{N}_0$ *denote the set of nonnegative integers,* $(H,+)$ *be a commutative semigroup and* $\varphi : H \times H \to [0,\infty)$ *be a function such that*

$$\lim_{n\to\infty} \frac{\varphi(2^n x, 2^n y)}{|2|^n} = 0, \qquad x, y \in H. \tag{8.1}$$

and

$$\widetilde{\varphi}(x) := \sup_{j\in\mathbb{N}_0} \frac{\varphi(2^j x, 2^j x)}{|2|^j}, \qquad x \in X. \tag{8.2}$$

Suppose that $f : H \to X$ *is a mapping satisfying*

$$\|f(x+y) - f(x) - f(y)\| \le \varphi(x,y), \qquad x,y \in H. \tag{8.3}$$

Then there exists a unique additive mapping $T : H \to X$ *such that*

$$\|f(x) - T(x)\| \le \frac{1}{|2|}\widetilde{\varphi}(x), \qquad x \in H. \tag{8.4}$$

That theorem has been formulated in [79] with some additional assumptions, which can be easily derived from (8.1).

Corollary 8.2. *Let* $\rho : [0,\infty) \to [0,\infty)$ *be a function satisfying*

$$\rho(|2|) < |2|, \qquad \rho(|2|t) \le \rho(|2|)\rho(t), \qquad t \ge 0.$$

Let $\delta > 0$*, let* H *be a non-Archimedean normed space and let* $f : H \to X$ *fulfill the inequality*

$$\|f(x+y) - f(x) - f(y)\| \le \delta\left(\rho(\|x\|) + \rho(\|y\|)\right), \qquad x,y \in H.$$

Then there exists a unique additive mapping $T : H \to X$ *such that*

$$\|f(x) - T(x)\| \le \frac{2}{|2|}\delta\rho(\|x\|), \qquad x \in H. \tag{8.5}$$

Remark 8.3. The classical example of such function ρ is the mapping $\rho(t) \equiv t^p$, when $p > 1$ and $|2| < 1$.

For some further examples of results and references on that subject we refer to [16, pp. 22–25].

9. Perturbation

An investigation of approximate homomorphisms of Banach algebras in the framework of the perturbation theory was given in [65]. A pair $(\mathscr{A},\mathscr{B})$ of Banach algebras is said to be AMNM (abbreviation for Approximately Multiplicative implies Near Multiplicative) if for each $\varepsilon > 0$ and $K > 0$, there is $\delta > 0$ such that for every linear bounded operator $T : \mathscr{A} \to \mathscr{B}$ satisfying $\|T(ab) - T(a)T(b)\| \le \delta\|a\|\|b\|$ $(a,b \in \mathscr{A})$ (the so-called δ-multiplicativity) and $\|T\| \le K$, there is a bounded homomorphism $S : \mathscr{A} \to \mathscr{B}$ such that $\|T - S\| \le \varepsilon$; see also [99]. For example,

(i) every pair of finite-dimensional Banach algebras is AMNM;
(ii) If $\mathscr{A} = L^1(G)$ with G locally compact abelian group, then $(\mathscr{A},\mathbb{C})$ is AMNM;
(iii) If $\mathscr{A}$ is an amenable Banach algebra and $\mathscr{B}$ is a dual space, then $(\mathscr{A},\mathscr{B})$ is AMNM.

The pair $(L^1(0,1),\mathbb{C})$ is not however AMNM.

In addition, Šemrl [98] showed that if $= C_{\mathbb{R}}(U), \mathscr{B} = C_{\mathbb{R}}(V)$ for some compact Hausdorff spaces U,V and a δ-multiplicative map $T : \mathscr{A} \to \mathscr{B}$ satisfies $\|T(a+b) - T(a) - T(b)\| \le \delta(\|a\| + \|b\|)$ $(a,b \in \mathscr{A})$, then there exists a linear homomorphism $S : \mathscr{A} \to \mathscr{B}$ such that $\|T - S\| \le \varepsilon$, where $\varepsilon \to 0$ as $\delta \to 0$.

Furthermore, the perturbation of mappings can be related to the Hyers–Ulam stability. Let X, Y be normed linear spaces. A mapping $T : X \to Y$ is said to have the Hyers–Ualm stability if there exists a constant $K \geq 0$ such that

(i) For any y in the range $\mathscr{R}(T)$ of T, $\varepsilon > 0$ and $x \in X$ with $\|T(x) - y\| \leq \varepsilon$, there exists a $x_0 \in X$ such that $T(x_0) = y$ and $\|x - x_0\| \leq K\varepsilon$.

Such a constant $K \geq 0$ is called a Hyers–Ulam stability constant for T. We denote by K_T the infimum of all the Hyers–Ulam stability constants for T. If T is linear then the condition (i) is equivalent to the following condition.

(ii) For any $\varepsilon \geq 0$ and $x \in X$ with $\|Tx\| \leq \varepsilon$ there exists an $x_0 \in X$ such that $Tx_0 = 0$ and $\|x - x_0\| \leq K\varepsilon$.

If $\mathscr{N}(T) := \{x \in X : Tx = 0\}$, then condition (ii) is equivalent to

(iii) For any $x \in X$ there exists a $x_0 \in \mathscr{N}(T)$ such that $\|x - x_0\| \leq K\|Tx\|$.

In [61] the authors proved the following result.

Theorem 9.1. *Let T be a closed operator from the subspace $\mathscr{D}(T)$ of a Hilbert space $\mathscr{H}$ into a Hilbert space $\mathscr{K}$. The following assertions are equivalent:*

(i) *T has the Hyers–Ulam stability;*
(ii) *T has closed range.*

Moreover, if one of the conditions above is true, then $K_T = \gamma(T)^{-1}$, where $\gamma(T) = \sup\{\gamma > 0 : \|Tx\| \geq \gamma\|x\|, \qquad x \in \mathscr{D}(T) \cap (\mathscr{N}(T))^{\perp}\}$.

Let X be a Banach space, M, N be closed linear subspaces of X and set $\delta(M, N) := \inf\{\frac{\operatorname{dist}(x,N)}{\operatorname{dist}(x,M\cap N)} : x \in M, x \notin N\}(\leq 1)$, when $N \setminus M \neq \emptyset$. If $M \subseteq N$, then we set $\delta(M, N) = 1$. Let A and T be operators with their domains in X such that $\mathscr{D}(T) \subseteq \mathscr{D}(A)$, and

$$\|Ax\| \leq a\|x\| + b\|Tx\|, \qquad x \in \mathscr{D}(T), \tag{9.1}$$

where a, b are nonnegative constants. Then A is called T-bounded with T-bounds a, b. A bounded operator A is clearly T-bounded for any T with $\mathscr{D}(T) \subseteq \mathscr{D}(A)$. The following perturbation results is presented in [80].

Theorem 9.2. *In the setting of Hilbert space operators, suppose that A is a T-bounded operator with T-bounds smaller than 1. If T is a closed operator and $S := T + A$, then the following assertions are equivalent:*

(i) *S has the Hyers–Ulam stability;*
(ii) *S has closed range.*

Moreover, if A is closed, A and T have the Hyers–Ulam stability and $\mathscr{R}(S) = \mathscr{R}(A) + \mathscr{R}(T)$, then conditions (i) *and* (ii) *are equivalent with each of the following two assertions:*

(iii) $\delta(\mathscr{R}(A), \mathscr{R}(T)) > 0$*;*
(iv) $\delta(\mathscr{R}(A)^{\perp}, \mathscr{R}(T)^{\perp}) > 0$.

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