Banach J. Math. Anal. 9 (2015), no. 3, 153–163
http://doi.org/10.15352/bjma/09-3-11
ISSN: 1735-8787 (electronic)
http://projecteuclid.org/bjma

BESSEL MULTIPLIERS IN HILBERT C^*–MODULES

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Communicated by D. Bakić

Abstract. In this paper we introduce Bessel multipliers, g-Bessel multipliers and Bessel fusion multipliers in Hilbert C^*–modules and we show that they share many useful properties with their corresponding notions in Hilbert and Banach spaces. We show that various properties of multipliers are closely related to their symbols and Bessel sequences, especially we consider multipliers when their Bessel sequences are modular Riesz bases and we see that in this case multipliers can be composed and inverted. We also study bounded below multipliers and generalize some of the results obtained for fusion frames in Hilbert spaces to Hilbert C^*–modules.

1. Introduction

Frames for Hilbert spaces were first introduced by Duffin and Schaeffer [9] in 1952 to study some problems in nonharmonic Fourier series, reintroduced in 1986 by Daubechies, Grossmann and Meyer [8]. Fusion frames [7] and g-frames [18] are important generalizations of frames.

Hilbert C^*–modules are generalizations of Hilbert spaces by allowing the inner product to take values in a C^*–algebra rather than in the field of complex numbers. In [10] Frank and Larson presented a general approach to the frame theory in Hilbert C^*–modules. Also, the first author and B. Khosravi introduced fusion frames and g-frames in Hilbert C^*–modules (see [13]).

Bessel multipliers in Hilbert spaces were introduced by Balazs in [4]. Bessel multipliers are operators that are defined by a fixed multiplication pattern which

Date: Received: Aug. 23, 2014; Accepted: Oct. 26, 2014.

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2010 *Mathematics Subject Classification.* 46L08; Secondary 42C15, 46H25, 47A05.

Key words and phrases. Hilbert C^*–module, Bessel sequence, Bessel multiplier, modular Riesz basis.

is inserted between the analysis and synthesis operators. Bessel multipliers have useful applications, for example in [5] they are used for solving approximation problems. Bessel fusion multipliers and g-Bessel multipliers in Hilbert spaces were introduced in [3] and [16], respectively. Also multipliers were introduced for p-Bessel sequences in Banach spaces (see [17]). In this paper we generalize these notions to Hilbert C^*–modules. First in the following section, we have a brief review of the definitions and basic properties of frames, fusion frames and g-frames in Hilbert C^*–modules.

2. Frames, fusion frames and g-frames in Hilbert C^*–modules

Suppose that $\mathfrak{A}$ is a unital C^*–algebra and E is a left $\mathfrak{A}$–module such that the linear structures of $\mathfrak{A}$ and E are compatible. E is a pre-Hilbert $\mathfrak{A}$–module if E is equipped with an $\mathfrak{A}$–valued inner product $\langle .,. \rangle : E \times E \longrightarrow \mathfrak{A}$, such that

(i) $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$, for each $\alpha, \beta \in \mathbb{C}$ and $x, y, z \in E$;
(ii) $\langle ax, y \rangle = a \langle x, y \rangle$, for each $a \in \mathfrak{A}$ and $x, y \in E$;
(iii) $\langle x, y \rangle = \langle y, x \rangle^*$, for each $x, y \in E$;
(iv) $\langle x, x \rangle \geq 0$, for each $x \in E$ and if $\langle x, x \rangle = 0$, then $x = 0$.

For each $x \in E$, we define $\|x\| = \|\langle x, x \rangle\|^{\frac{1}{2}}$ and $|x| = \langle x, x \rangle^{\frac{1}{2}}$. If E is complete with $\|.\|$, it is called a *Hilbert $\mathfrak{A}$–module* or a *Hilbert C^*–module* over $\mathfrak{A}$. We call $\mathcal{Z}(\mathfrak{A}) = \{a \in \mathfrak{A} : ab = ba, \forall b \in \mathfrak{A}\}$, the *center* of $\mathfrak{A}$. Note that if $a \in \mathcal{Z}(\mathfrak{A})$, then $a^* \in \mathcal{Z}(\mathfrak{A})$, and if a is an invertible element of $\mathcal{Z}(\mathfrak{A})$, then $a^{-1} \in \mathcal{Z}(\mathfrak{A})$, also if a is a positive element of $\mathcal{Z}(\mathfrak{A})$, since $a^{\frac{1}{2}}$ is in the closure of the set of polynomials in a, we have $a^{\frac{1}{2}} \in \mathcal{Z}(\mathfrak{A})$. Let E and F be Hilbert $\mathfrak{A}$–modules. An operator $T : E \longrightarrow F$ is called *adjointable* if there exists an operator $T^* : F \longrightarrow E$ such that $\langle T(x), y \rangle = \langle x, T^*(y) \rangle$, for each $x \in E$ and $y \in F$. Every adjointable operator T is bounded and $\mathfrak{A}$–linear (that is, $T(ax) = aT(x)$ for each $x \in E$ and $a \in \mathfrak{A}$). We denote the set of all adjointable operators from E into F by $\mathfrak{L}(E, F)$. Note that $\mathfrak{L}(E, E)$ is a C^*–algebra and it is denoted by $\mathfrak{L}(E)$. For a unital C^*–algebra $\mathfrak{A}$, $\ell^2(I, \mathfrak{A})$ which is defined by

$$\ell^2(I, \mathfrak{A}) = \left\{ \{a_i\}_{i \in I} \subseteq \mathfrak{A} : \sum_{i \in I} a_i {a_i}^* converges\ in\ \|.\| \right\},$$

is a Hilbert $\mathfrak{A}$–module with inner product $\langle \{a_i\}_{i \in I}, \{b_i\}_{i \in I} \rangle = \sum_{i \in I} a_i b_i^*$.
As usual $\ell^\infty(I, \mathfrak{A})$ is the set $\left\{ \{a_i\}_{i \in I} \subseteq \mathfrak{A} : sup\{\|a_i\| : i \in I\} < \infty \right\}$, and $C_0(I, \mathfrak{A})$ is the set of all $\{a_i\}_{i \in I} \subseteq \mathfrak{A}$ with this property that for each $\varepsilon > 0$ there exists a finite set $K \subseteq I$ with $sup\{\|a_i\| : i \in I - K\} \leq \varepsilon$.
Let E and F be Hilbert $\mathfrak{A}$–modules. For each $x \in E, y \in F$, the operator $\theta_{x,y} : F \longrightarrow E$ is defined by $\theta_{x,y}(z) = \langle z, y \rangle x$. It is easy to check that $\theta_{x,y} \in \mathfrak{L}(F, E)$, with $(\theta_{x,y})^* = \theta_{y,x}$. We say that an operator $T \in \mathfrak{L}(F, E)$ is *compact* if it is in the closed linear subspace of $\mathfrak{L}(F, E)$ spanned by $\{\theta_{x,y} : x \in E, y \in F\}$.
A Hilbert $\mathfrak{A}$–module E is *finitely generated* if there exists a finite set $\{x_1, \ldots, x_n\} \subseteq E$ such that every element $x \in E$ can be expressed as an $\mathfrak{A}$–linear combination $x = \sum_{i=1}^n a_i x_i, a_i \in \mathfrak{A}$. A Hilbert $\mathfrak{A}$–module E is *countably generated* if there

exists a countable set $\{x_i\}_{i\in I} \subseteq E$ such that E equals the norm-closure of the $\mathfrak{A}$–linear hull of $\{x_i\}_{i\in I}$. For more details about Hilbert C^*–modules, see [15].

Let E be a Hilbert $\mathfrak{A}$–module. A family $\{f_i\}_{i\in I} \subseteq E$ is a *frame* for E, if there exist real constants $0 < A \leq B < \infty$, such that for each $x \in E$,

$$A\langle x, x\rangle \leq \sum_{i\in I} \langle x, f_i\rangle\langle f_i, x\rangle \leq B\langle x, x\rangle. \tag{2.1}$$

The numbers A and B are called the *lower* and *upper bound* of the frame, respectively. In this case we call it an (A, B) *frame*. If only the second inequality is required, we call it a *Bessel sequence*. If the sum in (2.1) converges in norm, the frame is called *standard*.

For a standard Bessel sequence $\mathcal{F} = \{f_i\}_{i\in I}$ with an upper bound B, the operator $T_{\mathcal{F}} : E \longrightarrow \ell^2(I, \mathfrak{A})$ which is defined by $T_{\mathcal{F}}(x) = \{\langle x, f_i\rangle\}_{i\in I}$ is called the *analysis operator* of $\mathcal{F}$. It is adjointable with $T^*_{\mathcal{F}}(\{a_i\}_{i\in I}) = \sum_{i\in I} a_i f_i$ and $\|T_{\mathcal{F}}\| \leq \sqrt{B}$. $T^*_{\mathcal{F}}$ is the *synthesis operator* of $\mathcal{F}$. Now we define the operator $S_{\mathcal{F}} : E \longrightarrow E$ by $S_{\mathcal{F}}(x) = T^*_{\mathcal{F}} T_{\mathcal{F}}(x) = \sum_{i\in I}\langle x, f_i\rangle f_i$. If $\mathcal{F}$ is a standard (A, B) frame, then $A.Id_E \leq S_{\mathcal{F}} \leq B.Id_E$. The operator $S_{\mathcal{F}}$ is called the *frame operator* of $\mathcal{F}$. Let $\mathcal{F} = \{f_i\}_{i\in I}$ and $\mathcal{G} = \{g_i\}_{i\in I}$ be standard Bessel sequences in E. Then we say that $\mathcal{G}$ is an *alternate dual* or a *dual* of $\mathcal{F}$, if $x = \sum_{i\in I}\langle x, f_i\rangle g_i$ or equivalently $x = \sum_{i\in I}\langle x, g_i\rangle f_i$, for each $x \in E$ (see [12, Proposition 3.8]).

It is easy to see that if $\mathcal{F}$ is an (A, B) standard frame, then $\widetilde{\mathcal{F}} = \{S^{-1}_{\mathcal{F}} f_i\}_{i\in I}$ is an $(\frac{1}{B}, \frac{1}{A})$ standard frame with $x = \sum_{i\in I}\langle x, S^{-1}_{\mathcal{F}} f_i\rangle f_i = \sum_{i\in I}\langle x, f_i\rangle S^{-1}_{\mathcal{F}} f_i$, for each $x \in E$. Hence $\widetilde{\mathcal{F}} = \{S^{-1}_{\mathcal{F}} f_i\}_{i\in I}$ is a dual of $\mathcal{F}$ called the *canonical dual* of $\mathcal{F}$.

Note that a closed submodule M of E is *orthogonally complemented* if $E = M \oplus M^\perp$. In this case $\pi_M \in \mathfrak{L}(E, M)$, where $\pi_M : E \longrightarrow M$ is the orthogonal projection onto M. Now we recall some definitions from [13].

Suppose that $\{\omega_i : i \in I\} \subseteq \mathfrak{A}$ is a family of weights, i.e., each ω_i is a positive, invertible element from the center of $\mathfrak{A}$, and $\{W_i : i \in I\}$ is a family of orthogonally complemented submodules of E. Then $\{(W_i, \omega_i)\}_{i\in I}$ is a *fusion frame* if there exist real constants $0 < A \leq B < \infty$ such that

$$A\langle x, x\rangle \leq \sum_{i\in I} \omega_i^2 \langle \pi_{W_i}(x), \pi_{W_i}(x)\rangle \leq B\langle x, x\rangle,$$

for each $x \in E$. In this case we call it an (A, B) *fusion frame*. If we only require to have the upper bound, then $\{(W_i, \omega_i)\}_{i\in I}$ is called a *Bessel fusion sequence* with upper bound B.

Note that if $\{E_i : i \in I\}$ is a sequence of Hilbert $\mathfrak{A}$–modules, then $\oplus_{i\in I} E_i$ which is the set $\Big\{\{x_i\}_{i\in I} : x_i \in E_i, \{\langle x_i, x_i\rangle^{\frac{1}{2}}\}_{i\in I} \in \ell^2(I, \mathfrak{A})\Big\}$, is a Hilbert $\mathfrak{A}$–module with pointwise operations and $\mathfrak{A}$-valued inner product $\langle\{x_i\}_{i\in I}, \{y_i\}_{i\in I}\rangle = \sum_{i\in I}\langle x_i, y_i\rangle$.

A sequence $\Lambda = \{\Lambda_i \in \mathfrak{L}(E, E_i) : i \in I\}$ is called a *g-frame* for E with respect to $\{E_i : i \in I\}$ if there exist real constants $A, B > 0$ such that

$$A\langle x, x\rangle \leq \sum_{i\in I} \langle \Lambda_i x, \Lambda_i x\rangle \leq B\langle x, x\rangle,$$

for each $x \in E$. In this case we call it an (A, B) *g-frame*. If only the second-hand inequality is required, then Λ is called a *g-Bessel sequence*. Note that *standard* fusion frames and g-frames are defined similar to the standard frames. For a standard g-Bessel sequence Λ, the operator $T_\Lambda : E \longrightarrow \oplus_{i\in I} E_i$ which is defined by $T_\Lambda(x) = \{\Lambda_i x\}_{i\in I}$ is called the *analysis operator* of Λ. T_Λ is adjointable with $T^*_\Lambda(\{x_i\}_{i\in I}) = \sum_{i\in I} \Lambda^*_i(x_i)$, for each $\{x_i\}_{i\in I} \in \oplus_{i\in I} E_i$. Now we define the operator $S_\Lambda : E \longrightarrow E$ by $S_\Lambda x = T^*_\Lambda T_\Lambda(x) = \sum_{i\in I} \Lambda^*_i \Lambda_i(x)$. If Λ is a standard (A, B) g-frame, then $A.Id_E \le S_\Lambda \le B.Id_E$. The operator S_Λ is called the *g-frame operator* of Λ. For more results about frames, fusion frames and g-frames in Hilbert C^*–modules, see [10, 1, 2, 13, 19].

In this paper all C^*–algebras are unital and all Hilbert C^*–modules are finitely or countably generated. All frames, fusion frames, g-frames and Bessel sequences are standard.

3. Bessel multipliers, G-Bessel multipliers and Bessel fusion multipliers

In this section, we introduce Bessel multipliers, g-Bessel multipliers and Bessel fusion multipliers in Hilbert C^*–modules and we get some of their basic properties. We also generalize some of the results obtained for Bessel multipliers in Hilbert and Banach spaces to Hilbert C^*–modules. We begin with the following proposition:

Proposition 3.1. *Let $m = \{m_i\}_{i\in I} \in \ell^\infty(I, \mathfrak{A})$ with $m_i \in \mathcal{Z}(\mathfrak{A})$, for each $i \in I$. Then the operator $\mathcal{M}_m$ defined on $\oplus_{i\in I} E_i$ by $\mathcal{M}_m(\{x_i\}_{i\in I}) = \{m_i x_i\}_{i\in I}$ is well-defined, adjointable with $\mathcal{M}^*_m = \mathcal{M}_{m^*}$ and $\|\mathcal{M}_m\| \le \|m\|_\infty$, where $m^* = \{m^*_i\}_{i\in I}$.*

Proof. Let $\{x_i\}_{i\in I} \in \oplus_{i\in I} E_i$. Then

$$0 \le \langle m_i x_i, m_i x_i\rangle = \langle x_i, x_i\rangle^{\frac{1}{2}} m_i m^*_i \langle x_i, x_i\rangle^{\frac{1}{2}} \le \|m\|^2_\infty \langle x_i, x_i\rangle,$$

for each $i \in I$. Thus for each finite subset $K \subseteq I$, we have

$$\sum_{i\in K} \langle m_i x_i, m_i x_i\rangle = \sum_{i\in K} \langle x_i, x_i\rangle^{\frac{1}{2}} m_i m^*_i \langle x_i, x_i\rangle^{\frac{1}{2}} \le \|m\|^2_\infty \sum_{i\in K} \langle x_i, x_i\rangle.$$

Hence $\sum_{i\in I} \langle m_i x_i, m_i x_i\rangle$ converges in norm and

$$\Big\| \sum_{i\in I} \langle m_i x_i, m_i x_i\rangle \Big\|^{\frac{1}{2}} \le \|m\|_\infty \Big\| \sum_{i\in I} \langle x_i, x_i\rangle \Big\|^{\frac{1}{2}}.$$

Therefore $\mathcal{M}_m$ is a well-defined and bounded operator with $\|\mathcal{M}_m\| \le \|m\|_\infty$. Similarly we can see that $\{m^*_i x_i\}_{i\in I} \in \oplus_{i\in I} E_i$ and the operator $\mathcal{M}_{m^*} : \oplus_{i\in I} E_i \longrightarrow \oplus_{i\in I} E_i$ which is defined by $\mathcal{M}_{m^*}(\{x_i\}_{i\in I}) = \{m^*_i x_i\}_{i\in I}$ is well-defined and bounded. Now for each $x = \{x_i\}_{i\in I}, y = \{y_i\}_{i\in I} \in \oplus_{i\in I} E_i$, we have

$$\langle \mathcal{M}_m x, y\rangle = \sum_{i\in I} m_i \langle x_i, y_i\rangle = \sum_{i\in I} \langle x_i, m^*_i y_i\rangle = \langle x, \mathcal{M}_{m^*} y\rangle,$$

so $\mathcal{M}^*_m = \mathcal{M}_{m^*}$. $\square$

In this note m is always a sequence $\{m_i\}_{i\in I} \in \ell^\infty(I,\mathfrak{A})$ with $m_i \in \mathcal{Z}(\mathfrak{A})$, for each $i \in I$. Each sequence with these properties is called a *symbol*. We denote the set of all symbols by $N(I,\mathfrak{A})$ which is a C^*–subalgebra of $\ell^\infty(I,\mathfrak{A})$.
Note that a unital C^*–algebra $\mathfrak{A}$ is a Hilbert $\mathfrak{A}$–module with the inner product $\langle a,b\rangle = ab^*$, so if $E_i = \mathfrak{A}$, for each $i \in I$, then $\oplus_{i\in I} E_i = \ell^2(I,\mathfrak{A})$. Hence in this case $\mathcal{M}_m$ is an adjointable operator on $\ell^2(I,\mathfrak{A})$.

Definition 3.2. Let E_1 and E_2 be Hilbert $\mathfrak{A}$-modules, and let $\mathcal{F} = \{f_i\}_{i\in I} \subseteq E_1$ and $\mathcal{G} = \{g_i\}_{i\in I} \subseteq E_2$ be standard Bessel sequences. We call the operator $M_{m,\mathcal{G},\mathcal{F}} : E_1 \longrightarrow E_2$ which is defined by $M_{m,\mathcal{G},\mathcal{F}} = T^*_{\mathcal{G}}\mathcal{M}_m T_{\mathcal{F}}$, the *Bessel multiplier* for the Bessel sequences $\mathcal{F}$ and $\mathcal{G}$. It is easy to see that $M_{m,\mathcal{G},\mathcal{F}}(x) = \sum_{i\in I} m_i\langle x, f_i\rangle g_i$. We denote $M_{m,\mathcal{F},\mathcal{F}}$ by $M_{m,\mathcal{F}}$.

The following theorem is a generalization of parts (1), (2), (3) of Theorem 6.1 in [4], part (1)(a) of Theorem 8.1 in [4] and Lemmas 3.1, 3.6 and Theorem 3.7 in [17] to Hilbert C^*–modules.

Theorem 3.3.
(i) *$M_{m,\mathcal{G},\mathcal{F}} \in \mathfrak{L}(E_1,E_2)$ and $M^*_{m,\mathcal{G},\mathcal{F}} = M_{m^*,\mathcal{F},\mathcal{G}}$. Also $\|M_{m,\mathcal{G},\mathcal{F}}\| \le \sqrt{BD}\|m\|_\infty$, where B and D are the upper bounds of $\mathcal{F}$ and $\mathcal{G}$, respectively.*
(ii) *If $m \in C_0(I,\mathfrak{A})$, then $M_{m,\mathcal{G},\mathcal{F}}$ is a compact operator.*
(iii) *If $\{m^{(\ell)}\}_\ell \subseteq N(I,\mathfrak{A})$ and $m^{(\ell)} \longrightarrow m$, then $\|M_{m^{(\ell)},\mathcal{G},\mathcal{F}} - M_{m,\mathcal{G},\mathcal{F}}\| \longrightarrow 0$.*

Proof. (i) From Proposition 3.1, it is clear that $M_{m,\mathcal{G},\mathcal{F}} \in \mathfrak{L}(E_1,E_2)$, and $M^*_{m,\mathcal{G},\mathcal{F}} = T^*_{\mathcal{F}}\mathcal{M}_{m^*}T_{\mathcal{G}} = M_{m^*,\mathcal{F},\mathcal{G}}$, also we have $\|M_{m,\mathcal{G},\mathcal{F}}\| \le \|T^*_{\mathcal{G}}\|\|\mathcal{M}_m\|\|T_{\mathcal{F}}\| \le \sqrt{BD}\|m\|_\infty$.
(ii) For each $x \in E_1$ and $i \in I$, we have

$$\theta_{g_i,m_i^* f_i}(x) = \langle x, m_i^* f_i\rangle g_i = m_i\langle x, f_i\rangle g_i,$$

so $M_{m,\mathcal{G},\mathcal{F}}(x) = \sum_{i\in I}\theta_{g_i,m_i^* f_i}(x)$. Let $\varepsilon > 0$. Then there exists a finite set $K \subseteq I$ such that $\|\{m_i\}_{i\in I-K}\|_\infty \le \frac{\varepsilon}{\sqrt{BD}}$. Therefore

$$\left\| M_{m,\mathcal{G},\mathcal{F}}(x) - \sum_{i\in K}\theta_{g_i,m_i^* f_i}(x)\right\| \le \|\{m_i\}_{i\in I-K}\|_\infty\sqrt{BD}\|x\| \le \varepsilon\|x\|.$$

This means that $M_{m,\mathcal{G},\mathcal{F}}$ is a compact operator.
(iii) We have

$$\|M_{m,\mathcal{G},\mathcal{F}} - M_{m^{(\ell)},\mathcal{G},\mathcal{F}}\| = \|T^*_{\mathcal{G}}(\mathcal{M}_m - \mathcal{M}_{m^{(\ell)}})T_{\mathcal{F}}\| \le \sqrt{BD}\|m^{(\ell)} - m\|_\infty \longrightarrow 0,$$

and the result follows. □

Corollary 3.4. *Let $\mathcal{F} = \{f_i\}_{i\in I}$ be a Bessel sequence for E. If $m_i = m_i^*$ (resp. $m_i \ge 0$), for each $i \in I$, then $M_{m,\mathcal{F}}$ is a self-adjoint (resp. positive) operator.*

Proof. If $m_i = m_i^*$, for each $i \in I$, then by the above theorem $M^*_{m,\mathcal{F}} = M_{m^*,\mathcal{F}} = M_{m,\mathcal{F}}$. Hence $M_{m,\mathcal{F}}$ is self-adjoint. Now let $m_i \ge 0$, for each $i \in I$. Since $m_i \ge 0$, we have $\langle x, f_i\rangle m_i\langle f_i, x\rangle = \langle x, f_i\rangle m_i\langle x, f_i\rangle^* \ge 0$, for each $i \in I$ and $x \in E$. Thus

$$\langle x, M_{m,\mathcal{F}}x\rangle = \left\langle x, \sum_{i\in I} m_i\langle x, f_i\rangle f_i\right\rangle = \sum_{i\in I}\langle x, f_i\rangle m_i\langle f_i, x\rangle \ge 0,$$

and the result follows from Lemma 4.1 in [15]. □

Recall from Example 3.2 in [13] that if $W = \{(W_i, \omega_i)\}_{i\in I}$ is a standard Bessel fusion sequence (resp. standard fusion frame) for E, then $\Lambda_W = \{\omega_i \pi_{W_i}\}_{i\in I}$ is a standard g-Bessel sequence (resp. standard g-frame) for E with respect to $\{W_i\}_{i\in I}$.

Definition 3.5. Let $\Lambda = \{\Lambda_i\}_{i\in I}$ and $\Gamma = \{\Gamma_i\}_{i\in I}$ be standard g-Bessel sequences for E with respect to $\{E_i\}_{i\in I}$. Then the operator $M_{m,\Gamma,\Lambda} : E \longrightarrow E$ which is defined by $M_{m,\Gamma,\Lambda} = T^*_\Gamma \mathcal{M}_m T_\Lambda$ is called the *g-Bessel multiplier* for the g-Bessel sequences Λ and Γ. We have $M_{m,\Gamma,\Lambda}(x) = \sum_{i\in I} m_i \Gamma^*_i \Lambda_i(x)$. Also if $W = \{(W_i, \omega_i)\}_{i\in I}$ and $V = \{(V_i, \upsilon_i)\}_{i\in I}$ are standard Bessel fusion sequences for E, we call the operator $M_{m,V,W}(x) = M_{m,\Lambda_V,\Lambda_W}(x) = \sum_{i\in I} m_i \upsilon_i \omega_i \pi_{V_i} \pi_{W_i}(x)$, the *Bessel fusion multiplier* for W and V.

Similar to the proof of Theorem 3.3, we can see that $M_{m,\Gamma,\Lambda} \in \mathfrak{L}(E)$, $M^*_{m,\Gamma,\Lambda} = M_{m^*,\Lambda,\Gamma}$ and $\|M_{m,\Gamma,\Lambda}\| \le \|m\|_\infty \sqrt{BD}$, where B and D are upper bounds of Λ and Γ, respectively. Also, we have $M_{m,V,W} \in \mathfrak{L}(E)$, $M^*_{m,V,W} = M_{m^*,W,V}$ and $\|M_{m,V,W}\| \le \|m\|_\infty \sqrt{BD}$, where B and D are upper bounds of W and V, respectively. We denote $M_{m,\Lambda,\Lambda}$ by $M_{m,\Lambda}$ and if $m_i = 1_{\mathfrak{A}}$, for each $i \in I$, then $M_{m,V,W}$ is denoted by M_{VW}.

Remark 3.6. Let $m = \{m_i\}_{i\in I}$ be a symbol and let $\mathcal{F} = \{f_i\}_{i\in I}, \mathcal{G} = \{g_i\}_{i\in I} \subseteq E$ be standard Bessel sequences. It was shown in Example 3.2 in [13] that if $\Lambda_i, \Gamma_i : E \longrightarrow \mathfrak{A}$ are defined by $\Lambda_i(x) = \langle x, f_i\rangle$, $\Gamma_i(x) = \langle x, g_i\rangle$, then $\Lambda = \{\Lambda_i\}_{i\in I}$ and $\Gamma = \{\Gamma_i\}_{i\in I}$ are standard g-Bessel sequences. Now we have

$$M_{m,\Gamma,\Lambda}(x) = \sum_{i\in I} m_i \Gamma^*_i \Lambda_i(x) = \sum_{i\in I} m_i \langle x, f_i\rangle g_i = M_{m,\mathcal{G},\mathcal{F}}(x).$$

Note that if $a \in \mathcal{Z}(\mathfrak{A})$ and $T \in \mathfrak{L}(E, F)$, then the operator $aT : E \longrightarrow F$ which is defined by $(aT)(x) = aT(x)$ is adjointable with $(aT)^* = a^* T^*$.
The following proposition and corollary are generalizations of Theorem 4.5 in [6] to Hilbert C^*–modules:

Proposition 3.7. *Suppose that $m = \{m_i\}_{i\in I}$ is a symbol such that each m_i is a weight and there exists a positive number α with $\alpha 1_{\mathfrak{A}} \le m_i$, for each $i \in I$. Then the following are equivalent:*

(i) *$\Lambda = \{\Lambda_i\}_{i\in I}$ is a standard g-frame.*
(ii) *$\{m_i^{\frac{1}{2}} \Lambda_i\}_{i\in I}$ is a standard g-frame.*
(iii) *$M_{m,\Lambda}$ is a bounded, positive and invertible operator.*

Proof. Since $\alpha 1_{\mathfrak{A}} \le m_i \le \|m\|_\infty 1_{\mathfrak{A}}$, for each $i \in I$, we have

$$\alpha \langle \Lambda_i(x), \Lambda_i(x)\rangle \le \langle m_i^{\frac{1}{2}} \Lambda_i(x), m_i^{\frac{1}{2}} \Lambda_i(x)\rangle \le \|m\|_\infty \langle \Lambda_i(x), \Lambda_i(x)\rangle.$$

Therefore Λ is a standard g-frame if and only if $\{m_i^{\frac{1}{2}} \Lambda_i\}_{i\in I}$ is a standard g-frame. Let $\{m_i^{\frac{1}{2}} \Lambda_i\}_{i\in I}$ be a standard g-frame. For each $x \in E$, we have

$$\sum_{i\in I} (m_i^{\frac{1}{2}} \Lambda_i)^* (m_i^{\frac{1}{2}} \Lambda_i)(x) = \sum_{i\in I} m_i \Lambda^*_i \Lambda_i(x) = M_{m,\Lambda}(x).$$

This means that $M_{m,\Lambda}$ is the g-frame operator of $\{m_i^{\frac{1}{2}}\Lambda_i\}_{i\in I}$. Hence it is positive and invertible, so (ii) $\Longrightarrow$ (iii). Note that (iii) $\Longrightarrow$ (ii) can be obtained from the following inequality and Lemma 4.1 in [15],

$$\|M_{m,\Lambda}^{-1}\|^{-1} Id_E \leq M_{m,\Lambda} \leq \|M_{m,\Lambda}\| Id_E.$$

□

Since frames are special cases of g-frames, we have the following result:

Corollary 3.8. *Suppose that $m = \{m_i\}_{i\in I}$ is a symbol such that each m_i is a weight and there exists a positive number α with $\alpha 1_{\mathfrak{A}} \leq m_i$, for each $i \in I$. Then the following are equivalent:*

(i) *$\mathcal{F} = \{f_i\}_{i\in I}$ is a standard frame.*
(ii) *$\{m_i^{\frac{1}{2}} f_i\}_{i\in I}$ is a standard frame.*
(iii) *$M_{m,\mathcal{F}}$ is a bounded, positive and invertible operator.*

Proposition 3.9. *Let Λ and Γ be standard g-Bessel sequences.*

(i) *If $M_{m,\Gamma,\Lambda}$ is bounded below, then Λ is a standard g-frame.*
(ii) *If there exists $A > 0$, such that $A\|x\|^2 \leq \|\langle M_{m,\Gamma,\Lambda}x, x\rangle\|$, for each $x \in E$, then Γ and Λ are standard g-frames.*

Proof. (i) Suppose that $R > 0$ such that $R\|x\| < \|M_{m,\Gamma,\Lambda}x\|$, for each nonzero element $x \in E$ and D is a Bessel bound for Γ. Now for each $x \in E$, we can choose some $y \in E$ with $\|y\| = 1$ and $R\|x\| \leq \|\langle M_{m,\Gamma,\Lambda}x, y\rangle\|$. Hence by using the Cauchy-Schwarz inequality in Hilbert C^*–modules, we have

$$\begin{aligned}
R\|x\| &\leq \|\langle M_{m,\Gamma,\Lambda}x, y\rangle\| = \left\|\sum_{i\in I}\langle \Lambda_i x, m_i^*\Gamma_i y\rangle\right\| \\
&\leq \left\|\sum_{i\in I}\langle \Lambda_i x, \Lambda_i x\rangle\right\|^{\frac{1}{2}} \left\|\sum_{i\in I} m_i m_i^*\langle \Gamma_i y, \Gamma_i y\rangle\right\|^{\frac{1}{2}} \\
&\leq \sqrt{D}\|m\|_\infty \left\|\sum_{i\in I}\langle \Lambda_i x, \Lambda_i x\rangle\right\|^{\frac{1}{2}}.
\end{aligned}$$

Therefore $\|\sum_{i\in I}\langle \Lambda_i x, \Lambda_i x\rangle\| \geq \frac{R^2}{D\|m\|_\infty^2}\|x\|^2$. Now by Theorem 3.1 in [19], Λ is a standard g-frame.
(ii) For each $x \in E$, we have $A\|x\|^2 \leq \|\langle M_{m,\Gamma,\Lambda}x, x\rangle\| \leq \|M_{m,\Gamma,\Lambda}x\|\|x\|$. Since $M_{m,\Gamma,\Lambda}^* = M_{m^*,\Lambda,\Gamma}$, we can also obtain that $A\|x\|^2 \leq \|M_{m^*,\Lambda,\Gamma}x\|\|x\|$. Hence $M_{m,\Gamma,\Lambda}$ and $M_{m^*,\Lambda,\Gamma}$ are bounded below. Therefore by part (i), Λ and Γ are standard g-frames. □

Proposition 3.10. *Let Λ and Γ be standard g-Bessel sequences.*

(i) *If there exist $\lambda_1 < 1$, $\lambda_2 > -1$ such that $\|x - M_{m,\Gamma,\Lambda}x\| \leq \lambda_1\|x\| + \lambda_2\|M_{m,\Gamma,\Lambda}x\|$, for each $x \in E$, then Λ is a standard g-frame.*
(ii) *If there exists $\lambda \in [0, 1)$ such that $\|x - M_{m,\Gamma,\Lambda}x\| \leq \lambda\|x\|$, for each $x \in E$, then Λ and Γ are standard g-frames.*

Proof. (i) We have $\|x - M_{m,\Gamma,\Lambda}x\| \geq \|x\| - \|M_{m,\Gamma,\Lambda}x\|$. Hence

$$\lambda_1\|x\| + \lambda_2\|M_{m,\Gamma,\Lambda}x\| \geq \|x\| - \|M_{m,\Gamma,\Lambda}x\|,$$

so $\|M_{m,\Gamma,\Lambda}x\| \geq \frac{1-\lambda_1}{1+\lambda_2}\|x\|$. Now the result follows from Proposition 3.9.
(ii) We have

$$\|x - M_{m^*,\Lambda,\Gamma}x\| = \|(Id_E - M_{m,\Gamma,\Lambda})^*x\| \leq \|Id_E - M_{m,\Gamma,\Lambda}\|\|x\| \leq \lambda\|x\|.$$

Therefore by using part (i), we obtain that Λ and Γ are standard g-frames. □

Now as a consequence of our results, we can obtain the generalizations of Theorems 3.1, 3.4 and Corollary 3.5 in [11] to Hilbert C^*–modules.

Corollary 3.11. *Let $W = \{(W_i, \omega_i)\}_{i\in I}$ and $V = \{(V_i, \upsilon_i)\}_{i\in I}$ be standard Bessel fusion sequences.*

(i) *If M_{VW} is bounded below, then W is a standard fusion frame.*
(ii) *If there exist $\lambda_1 < 1$ and $\lambda_2 > -1$ such that*

$$\|x - M_{VW}x\| \leq \lambda_1\|x\| + \lambda_2\|M_{VW}x\|,$$

for each $x \in E$, then W is a standard fusion frame.
(iii) *If there exists $\lambda \in [0, 1)$ such that $\|x - M_{VW}x\| \leq \lambda\|x\|$, for each $x \in E$, then V and W are standard fusion frames.*

4. Modular Riesz bases and Bessel multipliers

Riesz bases and modular Riesz bases in Hilbert C^*–modules were introduced in [10] and [14], respectively. In this section, we get some properties of multipliers when the Bessel sequences are modular Riesz bases.

A standard frame $\{f_i\}_{i\in I}$ for E is a *modular Riesz basis* if it has the following property:
if an $\mathfrak{A}$-linear combination $\sum_{i\in K} a_i f_i$ with coefficients $\{a_i : i \in K\} \subseteq \mathfrak{A}$ and $K \subseteq I$ is equal to zero, then $a_i = 0$, for each $i \in K$.

Lemma 4.1. *Let $\mathcal{F} = \{f_i\}_{i\in I} \subseteq E$ be a modular Riesz basis. Then $\mathcal{F}$ and $\widetilde{\mathcal{F}} = \{\widetilde{f}_i\}_{i\in I}$ are biorthogonal, where $\widetilde{f}_i = S_{\mathcal{F}}^{-1} f_i$.*

Proof. Let $i_0 \in I$. Then we have $f_{i_0} = \sum_{i\in I}\langle f_{i_0}, S_{\mathcal{F}}^{-1} f_i\rangle f_i$, so $\sum_{i\in I} a_i f_i = 0$ where $a_i = \langle f_{i_0}, S_{\mathcal{F}}^{-1} f_i\rangle$, for each $i \neq i_0$ and $a_{i_0} = \langle f_{i_0}, S_{\mathcal{F}}^{-1} f_{i_0}\rangle - 1_{\mathfrak{A}}$. Since $\mathcal{F}$ is a modular Riesz basis, the equality $\sum_{i\in I} a_i f_i = 0$ implies that $\langle f_{i_0}, S_{\mathcal{F}}^{-1} f_i\rangle = 0$, for every $i \neq i_0$ and $\langle f_{i_0}, S_{\mathcal{F}}^{-1} f_{i_0}\rangle = 1_{\mathfrak{A}}$. This means that $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ are biorthogonal. □

The following result is analogous to Corollaries 7.3 and 7.6 in [4].

Proposition 4.2. (i) *Suppose that $\mathcal{F} = \{f_i\}_{i\in I}$, $\Psi = \{h_i\}_{i\in I}$ are Bessel sequences in E_1 and $\mathcal{G} = \{g_i\}_{i\in I}$, $\Phi = \{\xi_i\}_{i\in I}$ are Bessel sequences in E_2. If Φ and $\mathcal{G}$ are biorthogonal, then $M_{m,\mathcal{F},\mathcal{G}} \circ M_{m',\Phi,\Psi} = M_{m.m',\mathcal{F},\Psi} = M_{m'.m,\mathcal{F},\Psi}$, where $m = \{m_i\}_{i\in I}$ and $m' = \{m'_i\}_{i\in I}$ are symbols with $m.m' = \{m_i m'_i\}_{i\in I}$.*
(ii) *If $\mathcal{F} = \{f_i\}_{i\in I}$ is a modular Riesz basis, then $M_{m,\widetilde{\mathcal{F}},\mathcal{F}} \circ M_{m',\widetilde{\mathcal{F}},\mathcal{F}} = M_{m',\widetilde{\mathcal{F}},\mathcal{F}} \circ M_{m,\widetilde{\mathcal{F}},\mathcal{F}} = M_{mm',\widetilde{\mathcal{F}},\mathcal{F}}$.*

Proof. (i) For each $x \in E_1$, we have

$$\begin{aligned} M_{m,\mathcal{F},\mathcal{G}} \circ M_{m',\Phi,\Psi}(x) &= M_{m,\mathcal{F},\mathcal{G}}(\sum_{k\in I} m'_k \langle x, h_k\rangle \xi_k) \\ &= \sum_{i\in I}\sum_{k\in I} m_i m'_k \langle x, h_k\rangle \langle \xi_k, g_i\rangle f_i = \sum_{i\in I} m_i m'_i \langle x, h_i\rangle f_i \\ &= M_{m.m',\mathcal{F},\Psi}(x) = M_{m'.m,\mathcal{F},\Psi}(x). \end{aligned}$$

(ii) We can obtain the result by using part (i) and the above lemma. □

Our next result is analogous to [4, Lemma 7.1, Propositions 7.2 and 7.7] and [17, Propositions 3.3, 3.4 and 3.5].

Theorem 4.3. *Let $\mathcal{F} = \{f_i\}_{i\in I} \subseteq E_1$ and $\mathcal{G} = \{g_i\}_{i\in I} \subseteq E_2$ be (A, B) and (C, D) modular Riesz bases, respectively. Then*

(i) $\sqrt{AC}\|m\|_\infty \le \|M_{m,\mathcal{G},\mathcal{F}}\| \le \sqrt{BD}\|m\|_\infty$.
(ii) *The mapping $m \longrightarrow M_{m,\mathcal{G},\mathcal{F}}$ from $N(I,\mathfrak{A})$ into $\mathfrak{L}(E_1, E_2)$ is injective.*
(iii) *Let a be a positive, invertible element such that $a \le |m_i|$, for each $i \in I$. Then $M_{m,\mathcal{G},\mathcal{F}}$ is invertible and $M^{-1}_{m,\mathcal{G},\mathcal{F}} = M_{m^{-1},\widetilde{\mathcal{F}},\widetilde{\mathcal{G}}}$, where $m^{-1} = \{m_i^{-1}\}_{i\in I}$.*

Proof. (i) By Lemma 4.1, $\mathcal{G}$ and $\widetilde{\mathcal{G}}$ are biorthogonal and since $\widetilde{\mathcal{G}}$ is an $(\frac{1}{D}, \frac{1}{C})$ frame, for each $i_0 \in I$, we have

$$\|m_{i_0}\|^2 = \|m_{i_0} m^*_{i_0}\| = \left\| \sum_{i\in I} \langle m_{i_0} g_{i_0}, \widetilde{g}_i\rangle \langle \widetilde{g}_i, m_{i_0} g_{i_0}\rangle \right\| \le \frac{1}{C}\|m_{i_0} g_{i_0}\|^2.$$

This means that $\sup_{i\in I}\|m_i g_i\| \ge \sqrt{C}\|m\|_\infty$. Also since $\widetilde{\mathcal{F}}$ is an $(\frac{1}{B}, \frac{1}{A})$ frame, we obtain that $\|\widetilde{f}_i\| \le \frac{1}{\sqrt{A}}$, for each $i \in I$. Now we have

$$\begin{aligned} \|M_{m,\mathcal{G},\mathcal{F}}\| \ge \sup_{i\in I}\left\| M_{m,\mathcal{G},\mathcal{F}}(\frac{\widetilde{f}_i}{\|\widetilde{f}_i\|})\right\| &= \sup_{i\in I} \frac{\|\sum_{k\in I} m_k \langle \widetilde{f}_i, f_k\rangle g_k\|}{\|\widetilde{f}_i\|} \\ &= \sup_{i\in I} \frac{\|m_i g_i\|}{\|\widetilde{f}_i\|} \ge \sqrt{AC}\|m\|_\infty. \end{aligned}$$

The inequality $\|M_{m,\mathcal{G},\mathcal{F}}\| \le \|m\|_\infty\sqrt{BD}$ follows from Theorem 3.3.
(ii) Let $m' = \{m'_i\}_{i\in I}$ be a symbol and $M_{m,\mathcal{G},\mathcal{F}} = M_{m',\mathcal{G},\mathcal{F}}$. Then for each $i_0 \in I$, we have $\sum_{i\in I} m_i \langle \widetilde{f_{i_0}}, f_i\rangle g_i = \sum_{i\in I} m'_i \langle \widetilde{f_{i_0}}, f_i\rangle g_i$, so $m_{i_0} g_{i_0} = m'_{i_0} g_{i_0}$. Since $\mathcal{G}$ is a modular Riesz basis, we obtain that $m_{i_0} = m'_{i_0}$.
(iii) We have $a \le |m_i| \le \|m\|_\infty.1_{\mathfrak{A}}$, so $\|m\|^{-1}_\infty \le \||m_i|^{-1}\| \le \|a^{-1}\|$, for each $i \in I$. Because $m_i \in \mathcal{Z}(\mathfrak{A})$ and $|m_i|$ is invertible, it is easy to obtain that m_i is invertible with $|m_i^{-1}| = |m_i|^{-1}$. Therefore $\|m_i^{-1}\| = \||m_i^{-1}|\| = \||m_i|^{-1}\| \le \|a^{-1}\|$, for each $i \in I$. Thus $m^{-1} = \{m_i^{-1}\}_{i\in I} \in \ell^\infty(I,\mathfrak{A})$ and since $m_i^{-1} \in \mathcal{Z}(\mathfrak{A})$, for each $i \in I$,

m^{-1} is a symbol. Now by using Lemma 4.1, for each $x \in E_1$, we have

$$\begin{aligned} M_{m^{-1},\widetilde{\mathcal{F}},\widetilde{\mathcal{G}}} \circ M_{m,\mathcal{G},\mathcal{F}}(x) &= M_{m^{-1},\widetilde{\mathcal{F}},\widetilde{\mathcal{G}}}(\sum_{k\in I} m_k \langle x, f_k\rangle g_k) \\ &= \sum_{i\in I} m_i^{-1} \langle \sum_{k\in I} m_k \langle x, f_k\rangle g_k, S_{\mathcal{G}}^{-1} g_i\rangle S_{\mathcal{F}}^{-1} f_i \\ &= \sum_{i\in I}\sum_{k\in I} m_i^{-1} m_k \langle x, f_k\rangle \langle g_k, S_{\mathcal{G}}^{-1} g_i\rangle S_{\mathcal{F}}^{-1} f_i \\ &= \sum_{i\in I} m_i^{-1} m_i \langle x, f_i\rangle S_{\mathcal{F}}^{-1} f_i = \sum_{i\in I} \langle x, f_i\rangle S_{\mathcal{F}}^{-1} f_i = x. \end{aligned}$$

Similarly we can get $M_{m,\mathcal{G},\mathcal{F}} \circ M_{m^{-1},\widetilde{\mathcal{F}},\widetilde{\mathcal{G}}}(y) = y$, for each $y \in E_2$, and the result follows. □

Acknowledgement. The authors would like to thank the referee for valuable comments and suggestions which improved the manuscript.

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