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INTERPOLATION CLASSES AND MATRIX MEANS

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Abstract. Using a 'local' integral representation of a matrix connection of order n corresponding to an interpolation function of the same order, for each integer n, we can describe an injective map from the class of matrix connections of order n to the class of positive n-monotone functions on $(0,\infty)$ and the range of this corresponding covers the class of interpolation functions of order $2n$. In particular, the space of symmetric connections is isomorphic to the space of symmetric positive n-monotone functions. Moreover, we show that, for each n, the class of n-connections extremely contains that of $(n+2)$-connections.

1. Introduction

Throughout the paper, let us denote $\mathbb{R}_+$ the subset $(0,\infty)$ of the real line $\mathbb{R}$, M_n the algebra of square matrices of order n with coefficients in $\mathbb{C}$ and M_n^+ the cone of positive semi-definite matrices in M_n. The order relation $A \le B$ on the set of all self-adjoint matrices means that $B - A \ge 0$. A *n-monotone* function on $[0,\infty)$ is a function which preserves the order on the set of all $n \times n$ positive semi-definite matrices. Moreover, if f is n-monotone for all $n \in \mathbb{N}$, then f is called *operator monotone*.

With a view to studying electrical network connections, Anderson and Duffin [5] introduced the concept of parallel sum of two positive semi-definite matrices. Subsequently, in [6] Anderson and Trapp have extended the notions of parallel addition and shorted operation to bounded linear positive operators on a Hilbert

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space H and showed their important applications in operator theory. In the paper [12] Kubo and Ando developed an axiomatic theory of operator means. This theory has found a number of applications in operator theory and quantum information theory. In particular, Petz [17] connected the theory of monotone metrics with the theory of operator means by Kubo and Ando. He proved that an operator monotone function $f : [0,\infty) \longrightarrow [0,\infty)$ satisfying the symmetry condition

$$f(t) = tf(t^{-1}), \quad t \geq 0 \tag{1.1}$$

is related to a Morozova-Chentsov function which gives a monotone metric on the quantum state which consists of $n \times n$ density matrices.

Restricting the definition of operator means from [12] on the set of positive matrices of order n, we can consider *matrix means* of positive matrices of order n.

Definition 1.1. A binary operation σ on M_n^+, $(A, B) \mapsto A\sigma B$ is called a *matrix connection of order* n (or n*-connection*) if it satisfies the following properties:

(I) $A \leq C$ and $B \leq D$ imply $A\sigma B \leq C\sigma D$.
(II) $C(A\sigma B)C \leq (CAC)\sigma(CBC)$.
(III) $A_n \downarrow A$ and $B_n \downarrow B$ imply $A_n\sigma B_n \downarrow A\sigma B$

where $A_n \downarrow A$ means that $A_1 \geq A_2 \geq \ldots$ and A_n converges strongly to A.

A *mean* is a normalized connection, i.e. $1\sigma 1 = 1$. An *operator connection* means a connection of every order. A *n-semi-connection* is a binary operation on M_n^+ satisfying the conditions (II) and (III).

In [12], by using the representation of operator monotone functions on $[0,\infty)$, Kubo and Ando showed that there exists an affine order-isomorphism from the class of connections onto the class of positive operator monotone functions. The following natural question is one of the motivations of our study: *Does there exist an injective affine order-homomorphism from the class of n-connections to the class of positive n-monotone functions on* $[0,\infty)$? To study this question, the approach in [12] could not be used, since it is not clear if there is an integral representation of n-monotone functions. We need another candidates replacing n-monotone functions.

A function $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ is called an *interpolation function of order n* ([1]) if for any $T, A \in M_n$ with $A > 0$ and $T^*T \leq 1$

$$T^*AT \leq A \quad \Longrightarrow \quad T^*f(A)T \leq f(A).$$

We denote by $\mathcal{C}_n$ the class of all interpolation functions of order n on $\mathbb{R}_+$.

Remark 1.2. Let $P(\mathbb{R}_+)$ be a set of all Pick functions on $\mathbb{R}_+$, P' the set of all positive Pick functions on $\mathbb{R}_+$, i.e., functions of the form

$$h(s) = \int_{[0,\infty]} \frac{(1+t)s}{1+ts} d\rho(t), \quad s > 0,$$

where ρ is some positive Radon measure on $[0,\infty]$. For $n \in \mathbb{N}$ denote by P'_n the set of all strictly positive n-monotone functions. The following properties can be found in [1], [2],[3], [11], [14] or [4], :

(i) $P' = \cap_{n=1}^{\infty} P'_n$, $P' = \cap_{n=1}^{\infty} \mathcal{C}_n$;
(ii) $\mathcal{C}_{n+1} \subseteq \mathcal{C}_n$;
(iii) $P'_{n+1} \subseteq \mathcal{C}_{2n+1} \subseteq \mathcal{C}_{2n} \subseteq P'_n$, $P'_n \subsetneq \mathcal{C}_n$
(iv) $\mathcal{C}_{2n} \subsetneq P'_n$ [16];
(v) A function $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ belongs to $\mathcal{C}_n$ if and only if $\dfrac{t}{f(t)}$ belongs to $\mathcal{C}_n$ [4, Proposition 3.5].

The following useful characterization of a function in $\mathcal{C}_n$ is due to Donoghue (see [9], [8]), and to Ameur (see [1]).

Theorem 1.3. [4, Corollary 2.4] *A function $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ belongs to $\mathcal{C}_n$ if and only if for every n-set $\{\lambda_i\}_{i=1}^n \subset \mathbb{R}_+$ there exists a positive Pick function h on $\mathbb{R}$, such that*

$$f(\lambda_i) = h(\lambda_i) \quad \textit{for} \quad i = 1, \ldots, n.$$

As a consequence, Ameur gave a ‘local’ integral representation of every function in $\mathcal{C}_n$ as follows.

Theorem 1.4. [2, Theorem 7.1] *Let A be a positive definite matrix in M_n and $f \in \mathcal{C}_n$. Then there exists a positive Radon measure $\rho_{\sigma(A)}$ on $[0, \infty]$ such that*

$$f(A) = \int_{[0,\infty]} A(1+s)(A+s)^{-1} d\rho_{\sigma(A)}(s),$$

where $\sigma(A)$ is the set of eigenvalues of A.

Applying this representation, we give a ‘local’ integral formula for a connection of order n corresponding to a n-monotone function on $(0, \infty)$ (hence, an interpolation function of order n) via the formula (2.1) (Lemma 2.1). Furthermore, this ‘local’ formula also establishes, for each interpolation function f of order $2n$, a connection σ of order n corresponding to the given interpolation function f. Therefore, it shows that the map from the n-connections to the interpolation functions of order n is injective with the range containing the interpolation functions of order $2n$. Moreover, we also show that the class of 1-connections is isomorphic to the class of interpolation functions of order 2 and as much as properties we know in the space of n-connections also hold in the space $\mathcal{C}_{2n}$ of interpolation functions of order $2n$ (Proposition 3.1 and Proposition 2.8). This gives a hope that the class of n-connections is isomorphic to the class $\mathcal{C}_{2n}$.

An interesting and well-studied class of n-connections is the symmetric one, since the corresponding representation functions f should satisfy (1.1). Using the definition of symmetric connections, we can also give a corresponding concept for interpolation functions and n-monotone functions. It is shown that the space of n-connections is strictly subset of the space of positive n-monotone functions on $(0, \infty)$ (Corollary 2.9). However, restricting on the symmetric functions, the space of symmetric n-monotone functions is the same as that of symmetric n-connections (Theorem 2.10).

2. Interpolation functions and Means of positive matrices

In [12], there is an affine order-isomorphism from the set of connections onto the set of operator monotone functions. In this section, we describe the similar relation between the connections of order n and $\mathcal{C}_n \supsetneq \mathcal{C}_{2n}$. Note that every positive semi-definite matrix can be obtained as a limit of a decreasing sequence of positive definite matrices, from now on, we can always assume that connections are defined on positive definite matrices.

2.1. **From n-connections to P_n'.** For any n-connection σ, the matrix $I_n\sigma(tI_n)$ is a scalar by [12, Theorem 3.2], and so we can define a function f on $(0,\infty)$ by

$$f(t)I_n = I_n\sigma(tI_n),$$

where I_n is the identity in M_n.

Claim: $f \in P_n' \subsetneq \mathcal{C}_n$. Indeed, as in the proof of [12, Theorem 3.2], using the property (I) of the definition of connection, f is a n-monotone function on $(0,\infty)$.

Injectivity: Let σ_1 and σ_2 be two n-connections. Then there correspond two functions f_1 and f_2 belonging to $\mathcal{C}_n$, where $f_i(t)I_n = I_n\sigma_i(tI_n)$ $(i = 1, 2)$. Suppose that $f_1 = f_2$ then we have, for any $A > 0$ and $B > 0$ of order n,

$$\begin{aligned} A\sigma_1 B &= A^{\frac{1}{2}}(I_n\sigma_1 A^{\frac{-1}{2}}BA^{\frac{-1}{2}})A^{\frac{1}{2}} \qquad ([12, (3.8)]) \\ &= A^{\frac{1}{2}}f_1(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})A^{\frac{1}{2}} \\ &= A^{\frac{1}{2}}f_2(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})A^{\frac{1}{2}} \\ &= A\sigma_2 B. \end{aligned}$$

Hence, $\sigma_1 = \sigma_2$ by the continuity of means.

2.2. **From $\mathcal{C}_{2n}$ to n-connections.** Let f be a function belonging to $\mathcal{C}_n$. We can define a binary operation σ on positive definite matrices in M_n by:

$$A\sigma B = A^{\frac{1}{2}}f[A^{\frac{-1}{2}}BA^{\frac{-1}{2}}]A^{\frac{1}{2}}, \quad \forall A, B > 0. \tag{2.1}$$

This operation satisfies the property (III) of the definition of connection. Indeed, let A_n and B_n be two decreasing sequences which converge strongly to A and B, respectively. Then A_n^{-1} and B_n^{-1} converge strongly to A^{-1} and B^{-1}, respectively. Therefore, $A_n^{\frac{-1}{2}}B_nA_n^{\frac{-1}{2}}$ converges strongly to $A^{\frac{-1}{2}}BA^{\frac{-1}{2}}$ and by the continuity of f we get the property (III). In [12], if f is an operator monotone, then the operation σ defined above can be represented as:

$$A\sigma B = \int_{[0,\infty]} \frac{1+s}{s}\{(sA) : B\}d\rho(s), \tag{2.2}$$

where ρ is the Radon measure on $[0,\infty]$ corresponding to f (see [12, Theorem 3.4]). Unfortunately, in the case f belongs to $\mathcal{C}_n$ considered here, we do not know the existence of the measure ρ satisfying the representation (2.2). However, we can have such the representation of σ at "locally" as follows.

Lemma 2.1. *Let f be a function in $\mathcal{C}_n$ and A, B positive matrices of order n. Then there exists a Radon measure on the spectrum of $A^{\frac{-1}{2}}BA^{\frac{-1}{2}}$ such that the binary operation σ determined by (2.1) can be represented as the integral (2.2).*

Proof. By Theorem 1.4, there exists a Radon measure $\rho = \rho_{\sigma(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})}$ on $[0, \infty]$ such that

$$f[A^{\frac{-1}{2}}BA^{\frac{-1}{2}}] = \int_0^\infty A^{\frac{-1}{2}}BA^{\frac{-1}{2}}(1+s)(A^{\frac{-1}{2}}BA^{\frac{-1}{2}} + s)^{-1}d\rho(s),$$

where $\sigma(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})$ is the set of eigenvalues of $A^{\frac{-1}{2}}BA^{\frac{-1}{2}}$. Substituting this equality into (2.1), we have

$$\begin{aligned}
A\sigma B &= A^{\frac{1}{2}}\int_0^\infty [A^{\frac{-1}{2}}BA^{\frac{-1}{2}}](1+s)(A^{\frac{-1}{2}}BA^{\frac{-1}{2}} + s)^{-1}d\rho(s)A^{\frac{1}{2}} \\
&= \int_0^\infty BA^{\frac{-1}{2}}(1+s)(A^{\frac{-1}{2}}BA^{\frac{-1}{2}} + s)^{-1}A^{\frac{1}{2}}d\rho(s) \\
&= \int_0^\infty (1+s)\left(A^{\frac{-1}{2}}(A^{\frac{-1}{2}}BA^{\frac{-1}{2}} + s)A^{\frac{1}{2}}B^{-1}\right)^{-1}d\rho(s) \\
&= \int_0^\infty (1+s)(A^{-1} + sB^{-1})^{-1}d\rho(s) \\
&= \int_0^\infty \frac{1+s}{s}\{(sA) : B\}d\rho(s).
\end{aligned}$$

□

Corollary 2.2. *Let f be a positive function on $(0, \infty)$ belonging to $\mathcal{C}_n$. Then there is a semi-connection of order n, σ, such that $f(t)I_n = I_n\sigma(tI_n)$ for $t > 0$.*

Proof. We can define a binary σ by the formula (2.1). Because of the continuity of f (see Remark 2.3 below), we imply that σ has the property (III) in the definition. By Lemma 2.1, there exists a Radon measure ρ such that

$$A\sigma B = \int_{[0,\infty]} \frac{1+s}{s}\{(sA) : B\}d\rho(s)$$

For any positive definite matrix C of order n,

$$\begin{aligned}
C(A\sigma B)C &= \int_{[0,\infty]} \frac{1+s}{s}C\{(sA) : B\}Cd\rho(s) \\
&= \int_{[0,\infty]} \frac{1+s}{s}\{(sCAC) : CBC\}d\rho(s) \\
&= (CAC)\sigma(CBC).
\end{aligned}$$

□

In the proof above, we need the continuity of $f \in \mathcal{C}_n$. Actually, we follow the definition of interpolation function in [4] and the continuity is the prior assumption for any function. However, even if we did not assume the continuity of the functions under consideration, we have

Remark 2.3. If $f \in \mathcal{C}_n(I)$ for $n > 2$ then f is continuous on I.

Proof. In order to prove the remark, we use the following facts.

(i) Any convex function on an open interval is continuous. (c.f. [15, Theorem 1.3.3]) We may assume that $I = (-1, 1)$.

(ii) If $f \in \mathcal{C}_3$, then $g(t) = (t+1)f(t)$ is convex (see the below), and f is continous.

To prove the remark, we do the same step in the proof of [7, Theorem V. 3.6]. Indeed, since $f \in \mathcal{C}_3$, for a finite set S of any three points $t_1, t_2, \lambda t_1 + (1-\lambda)t_2 \in I$ $(0 < \lambda < 1)$ there exists an operator monotone function h such that $f = h$ on S. Since $g_1(t) = (t+1)h(t)$ is operator convex on $(-1,1)$ by [7, Lemma V. 3. 5], we have

$$\begin{aligned} g(\lambda t_1 + (1-\lambda)t_2) &= g_1(\lambda t_1 + (1-\lambda)t_2) \\ &\leq \lambda g_1(t_1) + (1-\lambda)g_1(t_2) \\ &= \lambda g(t_1) + (1-\lambda)g(t_2) \end{aligned}$$

and $g(t) = (t+1)f(t)$ is convex. So, $g(t) = (t+1)f(t)$ is continuous. Since $(t+1)$ is positive on $(-1,1)$, f is continuous on $(-1,1)$.

□

Now we can state the main theorem of this section.

Theorem 2.4. *For any natural number n there is an injective map Σ from the set of matrix connections of order n to $P_n' \supset \mathcal{C}_{2n}$ associating each connection σ to the function f_σ such that $f_\sigma(t)I_n = I_n\sigma(tI_n)$ for $t > 0$. Furthermore, the range of this map contains $\mathcal{C}_{2n}$.*

Proof. We have only to prove that the range of the map Σ contains C_{2n}. For any $f \in \mathcal{C}_{2n}$, since $\mathcal{C}_{2n} \subset \mathcal{C}_n$, by Corollary 2.2, there is a semi-connection σ_f defined by the formula (2.1) and $f(t)I_n = I_n\sigma_f(tI_n)$ on $(0,\infty)$. Since $f \in \mathcal{C}_{2n}$, by Theorem 1.4 we have that for any $0 < A \leq C$ and $0 < B \leq D$ there exists a Radon measure ρ on $\sigma(A^{\frac{-1}{2}}BA^{\frac{-1}{2}}) \cup \sigma(C^{\frac{-1}{2}}DC^{\frac{-1}{2}})$ such that

$$A\sigma_f B = \int_{[0,\infty]} \frac{1+s}{s}\{(sA) : B\}d\rho(s),$$

$$C\sigma_f D = \int_{[0,\infty]} \frac{1+s}{s}\{(sC) : D\}d\rho(s).$$

Since $\{(sA) : B\} \leq \{(sC) : D\}$, the condition (I) satisfies. Hence σ_f is a connection of order n. Since $\Sigma(\sigma_f)(t)I_n = I_n\sigma_f(tI_n) = f(t)I_n$ for any $t \in \mathbb{R}^+$, we are done.

□

Remark 2.5. Since $P_n' \subsetneq \mathcal{C}_n$, the map associating each connection of order n to a function in $\mathcal{C}_n$ as above is not surjective.

2.3. **Decreasing inclusion of the connections of order n.** Via the usual embedding of M_n into M_{n+1}, it is straightforward to check that the classes of connections of order n is decreasing. It is natural to ask the following question: *Is there a matrix mean σ_n of the order n on M_n such that σ_n is not of order $n+1$?*

The following observation gives partially affirmative data to the above question.

Proposition 2.6.

(1) *For any* $n \geq 2$ *there is a matrix mean* σ_n *of order* n *which is not of order* $n+2$.
(2) *There is a matrix mean* σ_1 *of order* 1 *which is not of order* 2.

Proof. (1): Take $f \in \mathcal{C}_{2n} \setminus P'_{n+2}$ (actually, we take $f \in P'_{n+1} \setminus P'_{n+2}$). Note that we can take such a function as $f(0) = 0$. Then we have a matrix mean σ_f of order n such that $f(t)I_n = I_n\sigma_f(tI_n)$ for $t \in \mathbb{R}^+$ by Theorem 2.4. Suppose on the contrary that σ_f is a matrix mean of order $(n+2)$.

From Theorem 2.4 there is a $(n+2)$-monotone function g such that $g(s)I_{n+2} = I_{n+2}\sigma_f(sI_{n+2})$ for $s \in \mathbb{R}^+$. For any $A \in M_n^+$ we set $\tilde{A} = \mathrm{diag}(A, O_2) \in M_{n+2}^+$. Then $g(\tilde{A}) = \mathrm{diag}(g(A), g(O_2))$. Therefore

$$\begin{aligned}\mathrm{diag}(g(A), O_2) &= \mathrm{diag}(I_n, O_2)g(\tilde{A})\,\mathrm{diag}(I_n, O_2)\\ &= \mathrm{diag}(I_n, O_2)(I_{n+2}\sigma_f\tilde{A})\,\mathrm{diag}(I_n, O_2)\\ &= \mathrm{diag}(I_n, O_2)I_{n+2}\sigma_f\tilde{A}\,\mathrm{diag}(I_n, O_2) \qquad ([12, (3.6)])\\ &= \mathrm{diag}(I_nf(A), O_2)\ (f(0)=0)\\ &= \mathrm{diag}(f(A), O_2)\end{aligned}$$

This means that $f(x) = g(x)$ for $x \in \mathbb{R}^+$, hence $f \in P'_{n+2}$. This is a contradiction to the assumption that $f \notin P'_{n+2}$.

(2): Take $f \in \mathcal{C}_2 \backslash P'_2$ (see [4, Proposition 3.4]).

From Corollary 3.3 there is a mean σ_f of order 1. We know, then, σ_f is not of order 2. Indeed, if σ_f is of order 2, there is a 2-monotone h such that $h(t)I_2 = I_2\sigma_f(tI_2)$ from the argument in Section 3.1. Then since $f(t) = h(t)$ for $t \in R^+$, f is 2-monotone, and a contradiction. Therefore, σ_f is not of order 2. □

We can give here another proof of Proposition 2.6.

Proof. Denote by Σ_n the image of the class of connections of order n via the map in Theorem 2.4 for each n. Therefore, Σ_n is isomorphic to the class of n-connections (so the sequence $\{\Sigma_n\}$ is decreasing) and $\Sigma_n \subseteq P_n^{'}$. From now on, we will identify the space of n-connections with Σ_n.

(1): On account of Remark 1.2 and Theorem 2.4, we obtain the following inclusion:

$$\begin{aligned}\Sigma_{n+2} \subseteq P_{n+2}^{'} \subseteq \mathcal{C}_{2(n+1)+1} \subseteq \mathcal{C}_{2(n+1)} \subseteq \Sigma_{n+1}\\ \subseteq P_{n+1}^{'} \subseteq \mathcal{C}_{2n+1} \subseteq \mathcal{C}_{2n} \subseteq \Sigma_n.\end{aligned}$$

And since $P_{n+2}^{'} \subsetneq P_{n+1}^{'}$, we imply that $\Sigma_{n+2} \subsetneq \Sigma_n$.

(2): Using Remark 1.2 again and Corollary 3.3, we get

$$\Sigma_2 \subseteq P_2^{'} \subseteq \mathcal{C}_3 \subseteq \mathcal{C}_2 = \Sigma_1.$$

By $P_2^{'} \neq \mathcal{C}_3$ [4, Proposition 3.14], we then have the statement. □

Remark 2.7. From the second proof of Proposition 2.6, we highlight the inclusion: For each natural number n,

$$\mathcal{C}_{2(n+1)} \subseteq \Sigma_{n+1} \subseteq P_{n+1}^{'} \subseteq \mathcal{C}_{2n+1} \subseteq \mathcal{C}_{2n} \subseteq \Sigma_n \subseteq P_n^{'}.$$

2.4. **Symmetric connections.** As the same in [12], we can recall some notations and properties of connections as follows. Let σ be a n-connection. The *transpose* σ', the *adjoint* σ^* and the *dual* $\sigma^\perp$ of σ are defined by

$$A\sigma' B = B\sigma A, \quad A\sigma^* B = (A^{-1}\sigma B^{-1})^{-1}, \quad \sigma^\perp = \sigma'^*.$$

A connection is called *symmetric* if it equals to its transpose. Denoted by Σ_n^{sym} the set of n-monotone representing functions of symmetric n-connections, i.e., Σ_n^{sym} is the image of the set of all symmetric n-connections via the canonical map in Theorem 2.4. Then, using the same argument as in [12], we can state the following properties for any n-connection:

(1) $\sigma + \sigma'$ and $\sigma(:)\sigma'$ are symmetric.
(2) $\omega_l(\sigma)\omega_r = \sigma$; $\omega_r(\sigma)\omega_l = \sigma'$, where $A\omega_l B = A$ and $A\omega_r B = B$.
(3) The n-monotone representing function of the n-connection $\sigma(\tau)\rho$ is indeed $f(x)g[h(x)/f(x)]$, where f, g, h are the representing functions of σ, τ, ρ in Theorem 2.4, respectively.
(4) σ is symmetric if and only if its n-monotone representing function f is *symmetric*, that is, $f(x) = xf(x^{-1})$.

Each n-connection corresponds to a positive n-monotone function belonging to Σ_n by Theorem 2.4. Therefore, combining with the observation above, we get the following.

Proposition 2.8. *Let $f(x), g(x), h(x)$ belong to Σ_n. Then the following statements hold true:*

(i) $k(x) = xf(x^{-1})$, $f^*(x) = f(x^{-1})^{-1}$, $\frac{x}{f(x)}$, $f(x)g[h(x)/f(x)]$, $af(x) + bg(x)$ *all belong to* Σ_n*;*
(ii) $f(x) + k(x)$, $\frac{f(x)k(x)}{f(x)+k(x)}$ *all belong to* Σ_n^{sym}.

Proof. By the hypothesis, there are n-connections σ, τ, ρ such that their representing functions are $f(x), g(x), h(x)$, respectively. Then the statements follow from the the fact that the functions $k(x) = xf(x^{-1})$, $f^*(x) = f(x^{-1})^{-1}$, $\frac{x}{f(x)}$, $af(x) + bg(x)$, $f(x)g[h(x)/f(x)]$, $f(x) + k(x)$, $\frac{f(x)k(x)}{f(x)+k(x)}$ are the representing functions of n-connections σ', σ^*, $\sigma^\perp$, $a\sigma + b\tau$, $\sigma(\tau)\rho$, $\sigma + \sigma'$, $\sigma(:)\sigma'$, respectively. ☐

Corollary 2.9.

$$\mathcal{C}_{2n} \subseteq \Sigma_n \subsetneq P'_n.$$

Proof. We only need to show that $\Sigma_n \neq P'_n$ for $n > 1$. Suppose on the contrary that $\Sigma_n = P'_n$. Let

$$p(x) = \sum_{k=1}^{2n-1} \frac{1}{k!} x^k.$$

Then $p(x)$ belongs to $P'_n(0, \alpha_n)$ for some $\alpha_n > 0$ (see [13]). Let ϕ be the operator monotone isomorphism from $(0, \alpha_n)$ to $(0, \infty)$ defined by

$$\phi(x) = \frac{x}{\alpha_n - x}.$$

Then $p \circ \phi^{-1}$ belongs to P'_n. By the assumption, $p \circ \phi^{-1} \in \Sigma_n$. Then

$$x(p\circ\phi^{-1})(x^{-1}) = xp(\frac{\alpha_n}{1+x}) = \sum_{k=1}^{2n-1}\frac{\alpha_n^k}{k!}\frac{x}{(1+x)^k}$$

is in Σ_n by Proposition 2.8. In particular, $xp(\frac{\alpha_n}{1+x})$ is monotone; this is impossible if $n>1$. Indeed, the first derivative of the function $\frac{x}{(1+x)^k}$ is $\frac{1+(1-k)x}{(1+x)^{k+1}}$ and is negative for sufficiently large x when $k\geq 2$. □

But if we restrict our attention to the class of the symmetric, we get the following equality.

Theorem 2.10.

$$\Sigma_n^{sym} = P_n'^{sym},$$

where $P_n'^{sym}$ is the set of all symmetric functions in P_n'.

Proof. The inclusion $\Sigma_n^{sym} \subset P_n'^{sym}$ is trivial by Theorem 2.4.

Let f be a symmetric function in P_n'. We can define a binary operation on positive definite matrices of order n by

$$A\sigma B = A^{\frac{1}{2}}f[A^{\frac{-1}{2}}BA^{\frac{-1}{2}}]A^{\frac{1}{2}}.$$

For any $B\leq D$, then $A^{\frac{-1}{2}}BA^{\frac{-1}{2}} \leq A^{\frac{-1}{2}}DA^{\frac{-1}{2}}$. Since f is n-monotone and the conjugate action preserves the order on self-adjoint matrices, we obtain

$$A^{\frac{1}{2}}f[A^{\frac{-1}{2}}BA^{\frac{-1}{2}}]A^{\frac{1}{2}} \leq A^{\frac{1}{2}}f[A^{\frac{-1}{2}}DA^{\frac{-1}{2}}]A^{\frac{1}{2}}.$$

This means $A\sigma B\leq A\sigma D$. Since f is symmetric, we also have

$$A\sigma D = D^{\frac{1}{2}}f[D^{\frac{-1}{2}}AD^{\frac{-1}{2}}]D^{\frac{1}{2}}.$$

Using this identity, we can also show that $A\sigma D\leq C\sigma D$ whenever $A\leq C$. Thus, $A\sigma B\leq A\sigma D\leq C\sigma D$ for any positive matrices A,B,C,D with $A\leq C$ and $B\leq D$. □

Remark 2.11. We would like to mention that even $P_{n+1}'\subsetneq P_n'$, but we still do not know whether $P_{n+1}'^{sym}\subsetneq P_n'^{sym}$ holds or not. As the first thought, we can obtain a symmetric function from the polynomial in P_{n+1}' but not in P_n' and such a function is a candidate to show $P_{n+1}'^{sym}\subsetneq P_n'^{sym}$. Unfortunately, this is not true as the following example.

Example 2.12. Let $p(x) = x+\frac{1}{2}x^2+\frac{1}{6}x^3$ be a polynomial which belongs to $P_2'(0,\alpha)$ but does not belong to $P_3'(0,\alpha)$ for some $\alpha>0$ (see [13]). Let $q(x)$ be the symmetrization of p by

$$q(x) = p(x)+xp(x^{-1}).$$

Then q is symmetric. However, we can show that q does not belong to $P_2'(0,\alpha)$. Indeed, the matrix

$$\begin{pmatrix} q'(x) & \frac{1}{2}q''(x) \\ \frac{1}{2}q''(x) & \frac{1}{6}q'''(x) \end{pmatrix}$$

is not positive semi-definite for every $x>0$.

Remark 2.13. Note that a function f on an interval I is n-monotone if and only if the $n \times n$ matrix

$$[f^{(i+j-1)}(t)/(i+j-1)!]$$

is positive for any $t \in I$ (for example see [10, VII Theorem VI and VIII Theorem V]).

3. Toward the conjecture $\mathcal{C}_{2n} = \Sigma_n$

We know that $\mathcal{C}_{2n} \subseteq \Sigma_n \subseteq P_n^{'}$ and $\mathcal{C}_2 = \Sigma_1$ (see Corollary 3.3). Therefore, we may give a conjecture that, for any positive integer n,

$$\mathcal{C}_{2n} = \Sigma_n \text{ and } \Sigma_n^{sym} = \mathcal{C}_{2n}^{sym}.$$

Even we still do not know whether $\mathcal{C}_{2n} = \Sigma_n$ or not, but they have some similar properties. In particular, the properties of the space Σ_n represented in Proposition 2.8 also hold true when we replace Σ_n(resp. Σ_n^{sym}) by $\mathcal{C}_{2n}$ (resp. C_{2n}^{sym}). That is,

Proposition 3.1. *The statements in Proposition 2.8 hold if we replace Σ_n (resp. Σ_n^{sym}) by $\mathcal{C}_{2n}$ (resp. C_{2n}^{sym}).*

Proof. (i): Let S be a subset of $(0, \infty)$ consisting $2n$ points. There exists an operator monotone function $p(x)$ such that p are identified with f on S. Set $p_1(x) = p(x^{-1})^{-1}$, then p_1 is an operator monotone function and p_1 equals to f^* on S. Hence, the function $x/k(x) = f^*(x) \in \mathcal{C}_{2n}$. This implies that $k(x)$ belongs to $\mathcal{C}_{2n}$ by Remark 1.2 (v). It is routine to check that $af(x) + bg(x)$ belongs to $\mathcal{C}_{2n}$.

In order to show that $f(x)g[h(x)/f(x)]$ belongs to $\mathcal{C}_{2n}$, by Theorem 1.3, we have only to show that this function is equal to an operator monotone function on any $2n$-point subset S of $(0, \infty)$. Since f, g, h belong to $\mathcal{C}_{2n}$, they are identified with operator monotone functions on S, without confusing let us still assume that these monotone functions are f, g, h respectively. Therefore, in order to complete the proof, we will show that the function $f(x)g[h(x)/f(x)]$ is operator monotone whenever f, g, h are operator monotone. Indeed, the function $f(x)g[h(x)/f(x)]$ was taken up as an issue of practice to be operator monotone due to [12, Theorem 3.2 and Lemma 4.1]. However, we can give here a more elementary proof by using the fact that a positive function F, which is strictly positive on $\mathbb{R}^+$ is operator monotone if and only if $0 < \arg F(z) \leq \arg z$ for any z in the upper half plane. This comes from [7, V(53)] and from the fact that $0 < \arg(z + a) < \arg(z)$ for $a > 0$ and z in the upper half plane. Note that $-\pi < \arg \frac{h(z)}{f(z)} < \pi$ if $0 < \arg z < \pi$.

When $0 < \arg \frac{h(z)}{f(z)} < \pi$, we have

$$\begin{aligned} 0 < \arg f(z) g\left(\frac{h(z)}{f(z)}\right) &= \arg f(z) + \arg g\left(\frac{h(z)}{f(z)}\right) \\ &\leq \arg f(z) + \arg \frac{h(z)}{f(z)} \\ &\leq \arg f(z) + \arg h(z) - \arg f(z) \\ &\leq \arg h(z) \\ &\leq \arg(z). \end{aligned}$$

When $-\pi < \arg \frac{h(z)}{f(z)} < 0$, we have

$$\begin{aligned} 0 < \arg h(z) &= \arg f(z) + \arg \frac{h(z)}{f(z)} \\ &\leq \arg f(z) + \arg g\left(\frac{h(z)}{f(z)}\right) \\ &= \arg f(z) g\left(\frac{h(z)}{f(z)}\right) \\ &< \arg f(z) < \pi. \end{aligned}$$

Hence $f(x)g(\frac{h(x)}{f(x)})$ belongs to $\mathcal{C}_{2n}$.

(ii): If $f(x) \in \mathcal{C}_{2n}$, by (i), $k(x) \in \mathcal{C}_{2n}$ and hence $f(x) + k(x)$ belongs to $\mathcal{C}_{2n}^{sym}$. To show that $\frac{f(x)k(x)}{f(x)+k(x)}$ belongs to $\mathcal{C}_{2n}^{sym}$, we apply the fact from (i) that $f(x)g[h(x)/f(x)]$ belongs to $\mathcal{C}_{2n}$ with $g(x) = x/(1+x)$ and $h(x) = k(x)$. □

Note that Proposition 3.1 still holds true in the space $\mathcal{C}_n$.

We have the application of Proposition 3.1 to the following well-known result (see [7, Exercise V. 4.15]).

Corollary 3.2. *If a polynomial of degree* m

$$p(x) = \sum_{i=0}^{m} a_i x^i, \quad a_m \neq 0$$

belongs to P', *then* $m \leq 1$.

Proof. Since p is monotone, $a_m > 0$. A function in P' belongs to $\mathcal{C}_{2n}$ for every n, so by Proposition 3.1, $xp(x^{-1})$ also belongs to $\mathcal{C}_{2n}$ for every n. Hence, $xp(x^{-1})$ belongs to P'. This implies that $xp(x^{-1})$ is monotone and this property holds only when the degree of $p(x)$ is not more than 1. □

3.1. **Matrix means of order one.** We recall the results in [4] for the sets $\mathcal{C}_1, \mathcal{C}_2$ as follows.

- $\mathcal{C}_1$ is the set of all positive functions on $(0, \infty)$.
- $\mathcal{C}_2$ consists of all quasi-concave functions (i.e., $f(s) \leq f(t) \max\{1, \frac{s}{t}\}$ for all $s, t > 0$).

For any connection σ of order 1, then the corresponding function f belongs to $\mathcal{C}_2$. Indeed, for any numbers $0 < t \le s$, we have

$$\begin{aligned} f(t)\max\{1, \frac{s}{t}\} &= (1\sigma t)\frac{s}{t} = \frac{s}{t}\sigma s \\ &\ge 1\sigma s = f(s), \text{ and,} \\ f(s)\max\{1, \frac{t}{s}\} &= (1\sigma s) \\ &\ge 1\sigma t = f(t). \end{aligned}$$

Combining this property with Theorem 2.4, we obtain:

Corollary 3.3.

(1) *Every connection σ of order 1 can be determined uniquely by*

$$x\sigma y = xf(\frac{y}{x}) \quad \forall x, y > 0,$$

where f is an interpolation function in $\mathcal{C}_2$.

(2) *Every function f in $\mathcal{C}_2$ can be represented uniquely by*

$$f(x) = 1\sigma x \quad \forall x > 0,$$

where σ is a connection of order 1*.*

From this corollary, we can easily get the functions in $\mathcal{C}_2$ from the corresponding connections and vise versa. For example, the functions in $\mathcal{C}_2$ which correspond to arithmetic mean, harmonic mean and the geometric mean are $\frac{1+x}{2}$, $\frac{2}{1+x}$ and $x^{\frac{1}{2}}$; and any (positive) linear combination of these functions also belongs to $\mathcal{C}_2$.

If we take the function $f(x) = 2\frac{x}{1+x} + \left(\frac{x}{1+x}\right)^2 \in \mathcal{C}_2 \setminus \mathcal{C}_3$ in [4, Example 3.13], we have a connection σ_f of order 1 which is not of order 2 as follows:

$$\begin{aligned} x\sigma_f y &= xf(\frac{y}{x}) \\ &= 2\frac{xy}{x+y} + \frac{xy^2}{(x+y)^2} \end{aligned}$$

for $x, y \in \mathbb{R}^+$.

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