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HYPERCYCLIC ABELIAN SEMIGROUPS OF AFFINE MAPS ON $\mathbb{C}^n$

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Abstract. We give a characterization of hypercyclic abelian semigroup $\mathcal{G}$ of affine maps on $\mathbb{C}^n$. If $\mathcal{G}$ is finitely generated, this characterization is explicit. We prove in particular that no abelian group generated by n affine maps on $\mathbb{C}^n$ has a dense orbit.

1. Introduction

Let $M_n(\mathbb{C})$ be the set of all square matrices of order $n \geq 1$ with entries in $\mathbb{C}$ and $GL(n,\ \mathbb{C})$ be the group of all invertible matrices of $M_n(\mathbb{C})$. A map $f:\ \mathbb{C}^n \longrightarrow \mathbb{C}^n$ is called an affine map if there exist $A \in M_n(\mathbb{C})$ and $a \in \mathbb{C}^n$ such that $f(x) = Ax + a$, $x \in \mathbb{C}^n$. We denote $f = (A, a)$, we call A the *linear part* of f. The map f is invertible if $A \in GL(n, \mathbb{C})$. Denote by $MA(n,\ \mathbb{C})$ the vector space of all affine maps on $\mathbb{C}^n$ and $GA(n,\ \mathbb{C})$ the group of all invertible affine maps of $MA(n, \mathbb{C})$.

Let $\mathcal{G}$ be an abelian affine sub-semigroup of $MA(n,\ \mathbb{C})$. For a vector $v \in \mathbb{C}^n$, we consider the orbit of $\mathcal{G}$ through v: $\mathcal{G}(v) = \{f(v) :\ f \in \mathcal{G}\} \subset \mathbb{C}^n$. Denote by $\overline{E}$ the closure of a subset $E \subset \mathbb{C}^n$. The group $\mathcal{G}$ is called *hypercyclic* if there exists a vector $v \in \mathbb{C}^n$ such that $\overline{\mathcal{G}(v)} = \mathbb{C}^n$. For an account of results and bibliography on hypercyclicity, we refer to the books [7] and [13].

The notion of hypercyclicity was investigated by many authors. More specific questions which arise naturally is to characterize this property for special types of matrices. N.S. Feldman proves in [12], that there are hypercyclic semigroup generated by $(n + 1)$ diagonal matrices on $\mathbb{C}^n$ and that there are no hypercyclic

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semigroup generated by n diagonalizable matrices on $\mathbb{C}^n$. In [9], G. Costakis, D. Hadjiloucas and A. Manoussos prove that for every positive integer n = 2 there exist $A_1, \dots, A_n$ of $n \times n$ non-(simultaneously) diagonalizable matrices over $\mathbb{R}$ generating an abelian hypercyclic semigroup. C.Costakis and I.Parissis prove in [11], that the minimum number of $n \times n$ matrices in Jordan form over $\mathbb{R}$ which generating an abelian hypercyclic semigroup is $n+1$. In [15], M.Javaheri constructs an explicit example of a 2-generator dense subsemigroup of 2×2 real matrices, and in [16], he proves that in both real and complex cases, there exists a pair of matrices that generates a dense subsemigroup of the set of $n \times n$ matrices. Moreover, he gives in [14], some examples of $n \times n$ matrices A and B over the filed $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ such that for almost every $x \in \mathbb{K}^n$, the orbit of x under the action of the semigroup generated by A and B is dense in $\mathbb{K}^n$. S.Shkarin proves in [19], that the minimal number of matrices generating an abelian hypercyclic semigroup on $\mathbb{C}^n$ (respectively, on $\mathbb{R}^n$) is $n+1$ (respectively, $\frac{n}{2}+\frac{5+(-1)^n}{4}$). H.Abel and A.Manoussos bring together in [2], some results about the density of subsemigroups of abelian Lie groups, the minimal number of topological generators of abelian Lie groups and a result about actions of algebraic groups. In [10], G. Costakis, D. Hadjiloucas and A. Manoussos give some results of locally hypercyclic abelian semigroup.

In this paper we will explore these notions to abelian semigroup affine.

We let $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$, $\mathbb{R}^* = \mathbb{R}\backslash\{0\}$ and $\mathbb{N}_0 = \mathbb{N}\backslash\{0\}$. Let $n \in \mathbb{N}_0$ be fixed, denote by:

• $\mathcal{B}_0 = (e_1, \dots, e_{n+1})$ the canonical basis of $\mathbb{C}^{n+1}$ and I_{n+1} the identity matrix of $GL(n+1, \mathbb{C})$.

For each $m = 1, 2, \dots, n+1$, denote by:

• $\mathbb{T}_m(\mathbb{C})$ the set of matrices over $\mathbb{C}$ of the form

$$\begin{bmatrix} \mu & & & 0 \\ a_{2,1} & \mu & & \\ \vdots & \ddots & \ddots & \\ a_{m,1} & \dots & a_{m,m-1} & \mu \end{bmatrix} \tag{1.1}$$

Let $r \in \mathbb{N}$ and $\eta = (n_1, \dots, n_r) \in \mathbb{N}_0^r$ such that $n_1 + \dots + n_r = n+1$. In particular, $r \leq n+1$. Write

• $\mathcal{K}_{\eta,r}(\mathbb{C}) := \mathbb{T}_{n_1}(\mathbb{C}) \oplus \dots \oplus \mathbb{T}_{n_r}(\mathbb{C})$. In particular if $r = 1$, then $\mathcal{K}_{\eta,1}(\mathbb{C}) = \mathbb{T}_{n+1}(\mathbb{C})$ and $\eta = (n+1)$.

• $\exp : \mathbb{M}_{n+1}(\mathbb{C}) \longrightarrow \mathrm{GL}(n+1, \mathbb{C})$ is the matrix exponential map; set $\exp(M) = e^M$, $M \in M_{n+1}(\mathbb{C})$.

• Define the map $\Phi : GA(n, \mathbb{C}) \longrightarrow GL(n+1, \mathbb{C})$

$$f = (A, a) \longmapsto \begin{bmatrix} 1 & 0 \\ a & A \end{bmatrix}$$

We have the following composition formula

$$\begin{bmatrix} 1 & 0 \\ a & A \end{bmatrix} \begin{bmatrix} 1 & 0 \\ b & B \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ Ab + a & AB \end{bmatrix}.$$

Then Φ is an injective homomorphism of groups. It is continuous and it forms a bijection unto its image. Write

- $G = \Phi(\mathcal{G})$, it is an abelian sub-semigroup of $GL(n+1, \mathbb{C})$.
- Define the map $\Psi : MA(n, \mathbb{C}) \longrightarrow M_{n+1}(\mathbb{C})$

$$f = (A, a) \longmapsto \begin{bmatrix} 0 & 0 \\ a & A \end{bmatrix}$$

We can see that Ψ is injective and linear. Hence $\Psi(MA(n, \mathbb{C}))$ is a vector subspace of $M_{n+1}(\mathbb{C})$. We prove (see Lemma 2.5) that Φ and Ψ are related by the following property

$$\exp(\Psi(MA(n, \mathbb{C}))) = \Phi(GA(n, \mathbb{C})).$$

Let consider the normal form of $\mathcal{G}$: By Proposition 2.2, there exists a $P \in \Phi(\mathrm{GA}(n, \mathbb{C}))$ and a partition η of $(n+1)$ such that $G' = P^{-1}GP \subset \mathcal{K}_{\eta,r}(\mathbb{C}) \cap \Phi(MA(n, \mathbb{C}))$. For such a choice of matrix P, we let

- $\mathrm{g} = \exp^{-1}(G) \cap (P(\mathcal{K}_{\eta,r}(\mathbb{C}))P^{-1})$. If $G \subset \mathcal{K}^*_{\eta,r}(\mathbb{C})$, we have $P = I_{n+1}$ and $\mathrm{g} = \exp^{-1}(G) \cap \mathcal{K}_{\eta,r}(\mathbb{C})$.
- $\mathfrak{q} = \Psi^{-1}(\mathrm{g} \cap \Psi(MA(n, \mathbb{C}))) \subset MA(n, \mathbb{C})$. Then $\mathfrak{q}$ is an additive sub-semigroup of $MA(n, \mathbb{C})$ and we have $\Psi(\mathfrak{q}) = \mathrm{g}^1$. By Corollary 2.9, we have $exp(\Psi(\mathfrak{q})) = \Phi(\mathcal{G})$.
- $\mathfrak{q}_v = \{f(v), \quad f \in \mathfrak{q}\} \subset \mathbb{C}^n, \quad v \in \mathbb{C}^n$.

For groups of affine maps on $\mathbb{K}^n$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), their dynamics were recently initiated for some classes in different point of view, (see for instance, [17], [18], [8], [6]). The purpose here is to give analogous results as for linear abelian sub-semigroup of $GL(n, \mathbb{C})$ [4, Theorem 1.1].

Our main results are the following:

Theorem 1.1. *Let $\mathcal{G}$ be an abelian sub-semigroup of $MA(n, \mathbb{C})$. Then the following are equivalent:*

(i) *$\mathcal{G}$ is hypercyclic.*
(ii) *the orbit $\mathcal{G}(w_0)$ is dense in $\mathbb{C}^n$.*
(iii) *$\mathfrak{q}_{w_0}$ is an additive sub-semigroup dense in $\mathbb{C}^n$.*

Where w_0 is a particular point in $\mathbb{C}^n$, defined in section 3 and has a form related to $\mathcal{G}$.

For a vector $v \in \mathbb{C}^n$, we write $v = \mathrm{Re}(v) + i\mathrm{Im}(v)$ where $\mathrm{Re}(v)$ and $\mathrm{Im}(v) \in \mathbb{R}^n$. The next result can be stated as follows:

Theorem 1.2. *Let $\mathcal{G}$ be an abelian sub-semigroup of $MA(n, \mathbb{C})$ and let $f_1, \dots, f_p \in \mathcal{G}$ generating $\mathcal{G}^*$ and let $f'_1, \dots, f'_p \in \mathfrak{q}$ be such that $e^{\Psi(f'_1)} = \Phi(f_1), \dots, e^{\Psi(f'_p)} = \Phi(f_p)$. Then the following are equivalent:*

(i) $\mathcal{G}$ *is hypercyclic.*

(ii) $\mathfrak{q}_{w_0} = \begin{cases} \sum_{k=1}^{p} \mathbb{N} f'_k(w_0) + 2i\pi \sum_{k=2}^{r} \mathbb{Z}(p_2(Pe^{(k)})), & if\ r \geq 2 \\ \sum_{k=1}^{p} \mathbb{N} f'_k(w_0), & if\ r = 1 \end{cases}$

is an additive sub-semigroup dense in $\mathbb{C}^n$. *(The projection* p_2 *and the vectors* $e^{(k)}$ *are defined in the section 3).*

Corollary 1.3. *Let* $\mathcal{G}$ *be an abelian sub-semigroup of* $MA(n, \mathbb{C})$ *and* $G = \Phi(\mathcal{G})$. *Let* $P \in \Phi(GA(n, \mathbb{C}))$ *such that* $P^{-1}GP \subset \mathcal{K}_{\eta,r}(\mathbb{C})$ *where* $1 \leq r \leq n+1$ *and* $\eta = (n_1, \dots, n_r) \in \mathbb{N}_0^r$. *If* $\mathcal{G}$ *is generated by* $2n - r + 1$ *commuting invertible affine maps, then it has no dense orbit.*

Corollary 1.4. *Let* $\mathcal{G}$ *be an abelian sub-semigroup of* $MA(n, \mathbb{C})$. *If* $\mathcal{G}$ *is generated by* n *commuting invertible affine maps, then it has no dense orbit.*

2. Normal form of abelian affine groups

The concept of a normal form of linear abelian groups was introduced in [4], by A.Ayadi and H.Marzougui which was generalized in [5], to the abelian linear semigroups. In [3], A.Ayadi gave the following normal form for any abelian group of affine maps of $\mathbb{C}^n$. Let $r \in \mathbb{N}$ and $\eta = (n_1, \dots, n_r) \in \mathbb{N}_0^r$ such that $n_1 + \cdots + n_r = n + 1$.

Proposition 2.1. [5] *Let* G' *be an abelian sub-semigroup of* $M_m(\mathbb{C})$, $m \geq 1$. *Then there exists* $P \in GL(m, \mathbb{C})$ *such that* $P^{-1}G'P$ *is a sub-semigroup of* $\mathcal{K}_{\eta',r'}(\mathbb{C})$, *for some* $r' \leq m$ *and* $\eta' = (n'_1, \dots, n'_{r'}) \in \mathbb{N}_0^{r'}$.

Denote by $\mathcal{K}^*_{\eta,r}(\mathbb{C}) := \mathcal{K}_{\eta,r}(\mathbb{C}) \cap \mathrm{GL}(n+1,\ \mathbb{C})$.

Proposition 2.2. [3, Proposition 2.1] *Let* $\mathcal{G}$ *be an abelian subgroup of* $GA(n, \mathbb{C})$ *and* $G = \Phi(\mathcal{G})$. *Then there exists* $P \in \Phi(GA(n, \mathbb{C}))$ *such that* $P^{-1}GP$ *is a subgroup of* $\mathcal{K}^*_{\eta,r}(\mathbb{C}) \cap \Phi(GA(n, \mathbb{C}))$, *for some* $r \leq n+1$ *and* $\eta = (n_1, \dots, n_r) \in \mathbb{N}_0^r$.

A more computational version of Proposition 2.2 for the semigroup case, is the following:

Proposition 2.3. *Let* $\mathcal{G}$ *be an abelian sub-semigroup of* $MA(n, \mathbb{C})$ *and* $G = \Phi(\mathcal{G})$. *Then there exists* $P \in \Phi(GA(n, \mathbb{C}))$ *such that* $P^{-1}GP$ *is a subgroup of* $\mathcal{K}^*_{\eta,r}(\mathbb{C}) \cap \Phi(GA(n, \mathbb{C}))$, *for some* $r \leq n+1$ *and* $\eta = (n_1, \dots, n_r) \in \mathbb{N}_0^r$.

Proof. Suppose first, $G \subset \mathrm{GL}(n+1, \mathbb{C})$. Let $\widehat{G}$ be the group generated by G. Then $\widehat{G}$ is abelian and by Proposition 2.2, there exists a $P \in \Phi(GA(n, \mathbb{C}))$ such that $P^{-1}\widehat{G}P$ is an abelian subgroup of $\mathcal{K}^*_{\eta,r}(\mathbb{C})$, for some $r \in \{1, \dots, n+1\}$ and $\eta \in (\mathbb{N}_0)^r$. In particular, $P^{-1}GP \subset \mathcal{K}^*_{\eta,r}(\mathbb{C})$.

Suppose now, $G \subset M_{n+1}(\mathbb{C})$. For every $A \in G$, there exists $\lambda_A \in \mathbb{C}$ such that $(A - \lambda_A I_{n+1}) \in \mathrm{GL}(n+1, \mathbb{C})$ (one can take λ_A non eigenvalue of A). Write $\widehat{L}$ be the group generated by $L := \{A - \lambda_A I_{n+1} : \ A \in G\}$. Then $\widehat{L}$ is an abelian sub-semigroup of $GL(n+1, \mathbb{C})$. Hence by above, there exists a $P \in \Phi(GA(n, \mathbb{C}))$ such that $P^{-1}\widehat{L}P \subset \mathcal{K}^*_{\eta,r}(\mathbb{C})$, for some $\eta \in (\mathbb{N}_0)^r$. As

$$P^{-1}LP = \left\{P^{-1}AP - \lambda_A I_{n+1} : \ A \in G\right\}$$

then $P^{-1}GP \subset \mathcal{K}_{\eta,r}(\mathbb{C})$. This proves the proposition. □

The group $G' = P^{-1}GP$ is called the *normal form* of G. Since $P \in \Phi(GA(n, \mathbb{C}))$ and $G \subset \Phi(MA(n, \mathbb{C}))$ then $G' \subset \Phi(MA(n, \mathbb{C}))$. As Φ is an injective homomorphism, $\mathcal{G}' := \Phi^{-1}(G')$ is an abelian semigroup of $MA(n, \mathbb{C})$ which is called the *normal form* of $\mathcal{G}$.

The proof of Theorem 1.1 is broken up into a series of lemmata.

Lemma 2.4. [4, Proposition 3.2] *$exp(\mathcal{K}_{\eta,r}(\mathbb{C})) = \mathcal{K}^*_{\eta,r}(\mathbb{C})$.*

Lemma 2.5. [3, Lemma 2.8] *$exp(\Psi(MA(n, \mathbb{C})) = GA(n, \mathbb{C})$.*

Lemma 2.6. [3, Lemma 2.9] *If $N \in P\mathcal{K}_{\eta,r}(\mathbb{C})P^{-1}$ such that $e^N \in \Phi(GA(n, \mathbb{C}))$, then there exists $k \in \mathbb{Z}$ such that $N - 2ik\pi I_{n+1} \in \Psi(MA(n, \mathbb{C}))$.*

Denote by $G^* = G \cap GL(n+1, \mathbb{C})$.

Lemma 2.7. [4, Lemma 4.2] *One has $exp(\mathrm{g}) = G^*$.*

Denote by:
- $\mathrm{g}^1 = \mathrm{g} \cap \Psi(MA(n, \mathbb{C}))$. It is an additive sub-semigroup of $M_{n+1}(\mathbb{C})$ (because by Lemma 3.2, g is an additive sub-semigroup of $M_{n+1}(\mathbb{C})$).
- $\mathrm{g}^1_u = \{Bu : \ B \in \mathrm{g}^1\} \subset \mathbb{C}^{n+1}, \quad u \in \mathbb{C}^{n+1}$.

Corollary 2.8. [3, Corollary 2.11] *Let $G = \Phi(\mathcal{G})$. We have $\mathrm{g} = \mathrm{g}^1 + 2i\pi\mathbb{Z}I_{n+1}$.*

We let $\mathcal{G}^* = \mathcal{G} \cap GA(n, \mathbb{C})$.

Corollary 2.9. *We have $exp(\Psi(\mathfrak{q})) = \Phi(\mathcal{G}^*)$.*

Proof. By Lemmas 2.7 and 2.8, We have $G = exp(\mathrm{g}) = exp(\mathrm{g}^1 + 2i\pi\mathbb{Z}I_{n+1}) = exp(\mathrm{g}^1)$. Since $\mathrm{g}^1 = \Psi(\mathfrak{q})$, we get $exp(\Psi(\mathfrak{q})) = \Phi(\mathcal{G})$. □

3. Proof of Theorem 1.1

Let $\widetilde{G}$ be the semigroup generated by G and $\mathbb{C}I_{n+1} = \{\lambda I_{n+1} : \quad \lambda \in \mathbb{C}\}$. Then $\widetilde{G}$ is an abelian sub-semigroup of $GL(n+1, \mathbb{C})$. By Proposition 2.2, there exists $P \in \Phi(GA(n, \mathbb{C}))$ such that $P^{-1}GP$ is a sub-semigroup of $\mathcal{K}^*_{\eta,r}(\mathbb{C})$ for some $r \leq n+1$ and $\eta = (n_1, \dots, n_r) \in \mathbb{N}_0^r$ and this also implies that $P^{-1}\widetilde{G}P$ is a sub-semigroup of $\mathcal{K}^*_{\eta,r}(\mathbb{C})$. Set $\widetilde{\mathrm{g}} = exp^{-1}(\widetilde{G}) \cap (P\mathcal{K}_{\eta,r}(\mathbb{C})P^{-1})$ and $\widetilde{\mathrm{g}}_{v_0} = \{Bv_0 : B \in \widetilde{\mathrm{g}}\}$. Denote by:

- $u_0 = (e_{1,1}, \dots, e_{r,1}) \in \mathbb{C}^{n+1}$ where $e_{k,1} = (1, 0, \dots, 0) \in \mathbb{C}^{n_k}$, for $k = 1, \dots, r$. So $u_0 \in \{1\} \times \mathbb{C}^n$.
- $p_2 : \mathbb{C} \times \mathbb{C}^n \longrightarrow \mathbb{C}^n$ the second projection defined by $p_2(x_1, \dots, x_{n+1}) = (x_2, \dots, x_{n+1})$.
- $e^{(k)} = (e_1^{(k)}, \dots, e_r^{(k)}) \in \mathbb{C}^{n+1}$ where

$$e_j^{(k)} = \begin{cases} 0 \in \mathbb{C}^{n_j} & \text{if } \ j \neq k \\ e_{k,1} & \text{if } \ j = k \end{cases} \qquad for \ \ every \ \ 1 \leq j, \ k \leq r.$$

- $v_0 = Pu_0$. So $v_0 \in \{1\} \times \mathbb{C}^n$, since $P \in \Phi(GA(n, \mathbb{C}))$.
- $w_0 = p_2(v_0) \in \mathbb{C}^n$. We have $v_0 = (1, w_0)$.

Since $P \in \Phi(GA(n, \mathbb{C}))$, we have $Pu_0 = v_0 \in \{1\} \times \mathbb{C}^n$. Then we have the following theorem, applied to $\widetilde{G}$:

Theorem 3.1. [5, Theorem 1.1] *Under the notations above, the following properties are equivalent:*

(i) *$\widetilde{G}$ has a dense orbit in $\mathbb{C}^{n+1}$.*
(ii) *the orbit $\widetilde{G}(v_0)$ is dense in $\mathbb{C}^{n+1}$.*
(iii) *$\widetilde{\mathrm{g}}_{v_0}$ is an additive sub-semigroup dense in $\mathbb{C}^{n+1}$.*

Lemma 3.2. [4, Lemma 4.1] *The sets* g *and* $\widetilde{\mathrm{g}}$ *are additive subgroups of $M_{n+1}(\mathbb{C})$. In particular,* g_{v_0} *and* $\widetilde{\mathrm{g}}_{v_0}$ *are additive subgroups of $\mathbb{C}^{n+1}$.*

Recall that $\mathrm{g}^1 = \mathrm{g} \cap \Psi(MA(n, \mathbb{C}))$ and $\mathfrak{q} = \Psi^{-1}(\mathrm{g}^1) \subset MA(n, \mathbb{C})$.

Lemma 3.3. *Under the notations above, one has:*

(i) $\widetilde{\mathrm{g}} = \mathrm{g}^1 + \mathbb{C}I_{n+1}$.
(ii) $\{0\} \times \mathfrak{q}_{w_0} = \mathrm{g}^1_{v_0}$.

Proof. (i) Let $B \in \widetilde{\mathrm{g}}$, then $e^B \in \widetilde{G}$. One can write $e^B = \lambda A$ for some $\lambda \in \mathbb{C}^*$ and $A \in G$. Let $\mu \in \mathbb{C}$ such that $e^\mu = \lambda$, then $e^{B - \mu I_{n+1}} = A$. Since $B - \mu I_{n+1} \in P\mathcal{K}_{\eta,r}(\mathbb{C})P^{-1}$, so $B - \mu I_{n+1} \in exp^{-1}(G) \cap P\mathcal{K}_{\eta,r}(\mathbb{C})P^{-1} = \mathrm{g}$. By Corollary 2.8, there exists $k \in \mathbb{Z}$ such that $B' := B - \mu I_{n+1} + 2ik\pi I_{n+1} \in \mathrm{g}^1$. Then $B \in \mathrm{g}^1 + \mathbb{C}I_{n+1}$ and hence $\widetilde{\mathrm{g}} \subset \mathrm{g}^1 + \mathbb{C}I_{n+1}$. Since $\mathrm{g}^1 \subset \widetilde{\mathrm{g}}$ and $\mathbb{C}I_{n+1} \subset \widetilde{\mathrm{g}}$, it follows that $\mathrm{g}^1 + \mathbb{C}I_{n+1} \subset \widetilde{\mathrm{g}}$ (since $\widetilde{\mathrm{g}}$ is an additive group, by Lemma 3.2). This

proves (i).
(ii) Since $\Psi(\mathfrak{q}) = \mathrm{g}^1$ and $v_0 = (1, w_0)$, we obtain for every $f = (B, b) \in \mathfrak{q}$,

$$\begin{aligned}\Psi(f)v_0 &= \begin{bmatrix} 0 & 0 \\ b & B \end{bmatrix} \begin{bmatrix} 1 \\ w_0 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ b + Bw_0 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ f(w_0) \end{bmatrix}.\end{aligned}$$

Hence $\mathrm{g}^1_{v_0} = \{0\} \times \mathfrak{q}_{w_0}$.

□

Lemma 3.4. *The following assertions are equivalent:*

(i) $\overline{\mathfrak{q}_{w_0}} = \mathbb{C}^n$.
(ii) $\overline{\mathrm{g}^1_{v_0}} = \{0\} \times \mathbb{C}^n$.
(iii) $\overline{\widetilde{\mathrm{g}}_{v_0}} = \mathbb{C}^{n+1}$.

Proof. *(i)* $\Longleftrightarrow$ *(ii)* follows from the fact that $\{0\} \times \mathfrak{q}_{w_0} = \mathrm{g}^1_{v_0}$ (Lemma 3.3,(ii)).
(ii) $\Longrightarrow$ *(iii)* : By Lemma 3.3,(ii), $\widetilde{\mathrm{g}}_{v_0} = \mathrm{g}^1_{v_0} + \mathbb{C}v_0$. Since $v_0 = (1, w_0) \notin \{0\} \times \mathbb{C}^n$ and $\mathbb{C}I_{n+1} \subset \widetilde{\mathrm{g}}$, we obtain $\mathbb{C}v_0 \subset \widetilde{\mathrm{g}}_{v_0}$ and so $\mathbb{C}v_0 \subset \overline{\widetilde{\mathrm{g}}_{v_0}}$. Therefore $\mathbb{C}^{n+1} = \{0\} \times \mathbb{C}^n \oplus \mathbb{C}v_0 = \overline{\mathrm{g}^1_{v_0}} \oplus \mathbb{C}v_0 \subset \overline{\widetilde{\mathrm{g}}_{v_0}}$ (since, by Lemma 3.2, $\widetilde{\mathrm{g}}_{v_0}$ is an additive subsemigroup of $\mathbb{C}^{n+1}$). Thus $\overline{\widetilde{\mathrm{g}}_{v_0}} = \mathbb{C}^{n+1}$.
(iii) $\Longrightarrow$ *(ii)* : Let $x \in \mathbb{C}^n$, then $(0, x) \in \overline{\widetilde{\mathrm{g}}_{v_0}}$ and there exists a sequence $(A_m)_{m\in\mathbb{N}} \subset \widetilde{\mathrm{g}}$ such that $\underset{m\to+\infty}{lim} A_m v_0 = (0, x)$. By Lemma 3.3, we can write $A_m v_0 = \lambda_m v_0 + B_m v_0$ with $\lambda_m \in \mathbb{C}$ and $B_m = \begin{bmatrix} 0 & 0 \\ b_m & B^1_m \end{bmatrix} \in \mathrm{g}^1$ for every $m \in \mathbb{N}$. Since $B_m v_0 \in \{0\} \times \mathbb{C}^n$ for every $m \in \mathbb{N}$ then $A_m v_0 = (\lambda_m,\ b_m + B^1_m w_0 + \lambda_m w_0)$. It follows that $\underset{m\to+\infty}{lim} \lambda_m = 0$ and $\underset{m\to+\infty}{lim} A_m v_0 = \underset{m\to+\infty}{lim} B_m v_0 = (0, x)$, thus $(0, x) \in \overline{\mathrm{g}^1_{v_0}}$. Hence $\{0\} \times \mathbb{C}^n \subset \overline{\mathrm{g}^1_{v_0}}$. Since $\mathrm{g}^1 \subset \Psi(MA(n, \mathbb{C}))$, $\mathrm{g}^1_{v_0} \subset \{0\} \times \mathbb{C}^n$ then we conclude that $\overline{\mathrm{g}^1_{v_0}} = \{0\} \times \mathbb{C}^n$. □

Lemma 3.5. *Let $x \in \mathbb{C}^n$ and $G = \Phi(\mathcal{G})$. The following are equivalent:*

(i) $\overline{\mathcal{G}(x)} = \mathbb{C}^n$.
(ii) $\overline{G(1, x)} = \{1\} \times \mathbb{C}^n$.
(iii) $\overline{\widetilde{G}(1, x)} = \mathbb{C}^{n+1}$.

Proof. *(i)* $\Longleftrightarrow$ *(ii)* : is obvious since $\{1\} \times \mathcal{G}(x) = G(1, x)$ by construction.
(iii) $\Longrightarrow$ *(ii)* : Let $y \in \mathbb{C}^n$ and $(B_m)_m$ be a sequence in $\widetilde{G}$ such that $\underset{m\to+\infty}{lim} B_m(1, x) = (1, y)$. One can write $B_m = \lambda_m \Phi(f_m)$, with $f_m \in \mathcal{G}$ and $\lambda_m \in \mathbb{C}^*$, thus $B_m(1, x) = (\lambda_m,\ \ \lambda_m f_m(x))$, so $\underset{m\to+\infty}{lim} \lambda_m = 1$. Therefore, $\underset{m\to+\infty}{lim} \Phi(f_m)(1, x) = \underset{m\to+\infty}{lim} \frac{1}{\lambda_m} B_m(1, x) = (1, y)$. Hence, $(1, y) \in \overline{G(1, x)}$.

$(ii) \Longrightarrow (iii)$: Since $\mathbb{C}^{n+1} \backslash (\{0\} \times \mathbb{C}^n) = \bigcup_{\lambda \in \mathbb{C}^*} \lambda (\{1\} \times \mathbb{C}^n)$ and for every $\lambda \in \mathbb{C}^*$, $\lambda G(1, x) \subset \widetilde{G}(1, x)$, we get

$$\begin{aligned} \mathbb{C}^{n+1} &= \overline{\mathbb{C}^{n+1} \backslash (\{0\} \times \mathbb{C}^n)} \\ &= \overline{\bigcup_{\lambda \in \mathbb{C}^*} \lambda (\{1\} \times \mathbb{C}^n)} \\ &= \overline{\bigcup_{\lambda \in \mathbb{C}^*} \lambda \overline{G(1, x)}} \subset \overline{\widetilde{G}(1, x)} \end{aligned}$$

Hence $\mathbb{C}^{n+1} = \overline{\widetilde{G}(1, x)}$. □

An orbit O of G is called somewhere dense orbit if the interior of its closure $\overset{\circ}{\overline{O}} \neq \emptyset$.

Proposition 3.6. *Let G be an abelian subsemigroup of $M_n(\mathbb{C})$ and $G^* = G \cap GL(n, \mathbb{C})$. Then G is hypercyclic (resp. has a somewhere dense orbit) if and only if so is (resp. has) G^* .*

Proof. Suppose that $\overset{\circ}{\overline{G^*(u)}} \neq \emptyset$, for some $u \in \mathbb{K}^n$. Then $\emptyset \neq \overset{\circ}{\overline{G^*(u)}} \subset \overset{\circ}{\overline{G(u)}}$ and so $\overset{\circ}{\overline{G(u)}} \neq \emptyset$. Conversely, suppose that $\overset{\circ}{\overline{G(u)}} \neq \emptyset$, for some $u \in \mathbb{C}^n$. By proposition 2.1, one can suppose that G is an abelian sub-semigroup of $\mathcal{K}_{\eta,r}(\mathbb{C})$. Write $G' := (G \backslash G^*) \cup \{I_n\}$. then G' is a sub-semigroup of G.

- If $G' = \{I_n\}$ then $G = G^*$ and so G^* has a somewhere dense orbit.
- If $G' \neq \{I_n\}$ then

$$G(u) \subset \left(\bigcup_{A \in (G' \backslash \{I_n\})} Im(A) \right) \cup G^*(u).$$

As every $A \in (G' \backslash \{I_n\})$, is non invertible, then $Im(A) \subset \bigcup_{k=1}^{r} H_k$ where

$$H_k := \left\{ u = [u_1, \dots, u_r]^T \in \mathbb{C}^n, \ u_j \in \mathbb{C}^{n_j}, \ u_k \in \{0\} \times \mathbb{C}^{n_k - 1} \ \begin{array}{c} 1 \leq j \leq r, \\ j \neq k \end{array} \right\}.$$

It follows that

$$G(u) \subset \left(\bigcup_{k=1}^{r} H_k \right) \cup G^*(u),$$

and so

$$\overline{G(u)} \subset \left(\bigcup_{k=1}^{r} H_k \right) \cup \overline{G^*(u)}.$$

Since $\dim H_k = n - 1$, $\overset{\circ}{H_k} = \emptyset$, for every $1 \leq k \leq r$ and therefore $\overset{\circ}{\overline{G^*(u)}} \neq \emptyset$. □

Lemma 3.7. *Let G be an abelian subsemigroup of $\mathcal{K}_{\eta,r}(\mathbb{C})$, $G^* = G \cap GL(n,\mathbb{C})$ and $\mathrm{g}^* = exp^{-1}(G^*) \cap \mathcal{K}_{\eta,r}(\mathbb{C})$. Then $\mathrm{g} = \mathrm{g}^*$.*

Proof. Let $G' = G \backslash G^*$. Since $e^A \in GL(n,\mathbb{C})$ for every $A \in M_n(\mathbb{C})$ and $G' \subset M_n(\mathbb{C}) \backslash GL(n,\mathbb{C})$ then $exp^{-1}(G^*) = \emptyset$. As $\mathrm{g} = (exp^{-1}(G') \cap \mathcal{K}_{\eta,r}(\mathbb{C})) \cup \mathrm{g}^*$ then $\mathrm{g} = \mathrm{g}^*$. □

Proof of Theorem 1.1. $(ii) \Longrightarrow (i)$: is obvious.
$(i) \Longrightarrow (ii)$: Suppose that $\mathcal{G}$ is hypercyclic, so $\overline{\mathcal{G}(x)} = \mathbb{C}^n$ for some $x \in \mathbb{C}^n$. By Lemma 3.5,(iii), $\overline{\widetilde{G}(1,x)} = \mathbb{C}^{n+1}$ and by Theorem 3.1, $\overline{\widetilde{G}(v_0)} = \mathbb{C}^{n+1}$. Then by Lemma 3.5, $\overline{\mathcal{G}(w_0)} = \mathbb{C}^n$, since $v_0 = (1, w_0)$.
$(ii) \Longrightarrow (iii)$: Suppose that $\overline{\mathcal{G}(w_0)} = \mathbb{C}^n$. By Lemma 3.5, $\overline{\widetilde{G}(v_0)} = \mathbb{C}^{n+1}$ and by Theorem 3.1, $\overline{\widetilde{\mathrm{g}}_{v_0}} = \mathbb{C}^{n+1}$. Then by Lemma 3.4, $\overline{\mathfrak{q}_{w_0}} = \mathbb{C}^n$.
$(iii) \Longrightarrow (ii)$: Suppose that $\overline{\mathfrak{q}_{w_0}} = \mathbb{C}^n$. By Lemma 3.4, $\overline{\widetilde{\mathrm{g}}_{v_0}} = \mathbb{C}^{n+1}$ and by Theorem 3.1, $\overline{\widetilde{G}(v_0)} = \mathbb{C}^{n+1}$. Then by Lemma 3.5, $\overline{\mathcal{G}(w_0)} = \mathbb{C}^n$. □

4. Finitely generated subgroups

Recall the following result proved in [5] which applied to G can be stated as following:

Proposition 4.1. [5, Proposition 5.1] *Let G be an abelian sub-semigroup of $M_n(\mathbb{C})$ such that G^* is generated by $A_1, \dots, A_p$ and let $B_1, \dots, B_p \in \mathrm{g}$ such that $A_k = e^{B_k}$, $k = 1, \dots, p$ and $P \in GL(n+1,\mathbb{C})$ satisfying $P^{-1}GP \subset \mathcal{K}_{\eta,r}(\mathbb{C})$. Then:*

$$\mathrm{g} = \sum_{k=1}^{p} \mathbb{N}B_k + 2i\pi \sum_{k=1}^{r} \mathbb{Z}PJ_kP^{-1} \quad \text{and} \quad \mathrm{g}_{v_0} = \sum_{k=1}^{p} \mathbb{N}B_k v_0 + \sum_{k=1}^{r} 2i\pi\mathbb{Z}Pe^{(k)},$$

where $J_k = \mathrm{diag}(J_{k,1}, \dots, J_{k,r})$ with $J_{k,i} = 0 \in \mathbb{T}_{n_i}(\mathbb{C})$ if $i \neq k$ and $J_{k,k} = I_{n_k}$.

Proposition 4.2. *Let $\mathcal{G}$ be an abelian sub-semigroup of $GA(n,\mathbb{C})$ such that $\mathcal{G}^*$ is generated by $f_1, \dots, f_p$ and let $f'_1, \dots, f'_p \in \mathfrak{q}$ such that $e^{\Psi(f'_k)} = \Phi(f_k)$, $k = 1, .., p$. Let P be as in Proposition 2.2. Then:*

$$\mathfrak{q}_{w_0} = \begin{cases} \sum_{k=1}^{p} \mathbb{N}f'_k(w_0) + \sum_{k=2}^{r} 2i\pi\mathbb{Z}p_2(Pe^{(k)}), \ if \ r \geq 2 \\ \sum_{k=1}^{p} \mathbb{N}f'_k(w_0), \ if \ r = 1 \end{cases}$$

Proof. Let $G = \Phi(\mathcal{G})$. Then G is generated by $\Phi(f_1), \dots, \Phi(f_p)$. Apply Proposition 4.1 to G, $A_k = \Phi(f_k)$, $B_k = \Psi(f'_k) \in \mathrm{g}^1$, then we have

$$\mathrm{g} = \sum_{k=1}^{p} \mathbb{Z}\Psi(f'_k) + 2i\pi\mathbb{Z}\sum_{k=1}^{r} PJ_kP^{-1}.$$

We have $\sum_{k=1}^{p} \mathbb{Z}\Psi(f'_k) \subset \Psi(MA(n,\mathbb{C}))$. Moreover, for every $k = 2, \dots, r$, $J_k \in \Psi(MA(n,\mathbb{C}))$, hence $PJ_kP^{-1} \in \Psi(MA(n,\mathbb{C}))$, since $P \in \Phi(GA(n,\mathbb{C}))$. However, $mPJ_1P^{-1} \notin \Psi(MA(n,\mathbb{C}))$ for every $m \in \mathbb{Z}\backslash\{0\}$, since J_1 has the form $J_1 = \mathrm{diag}(1, J')$ where $J' \in M_n(\mathbb{C})$. As $\mathrm{g}^1 = \mathrm{g} \cap \Psi(MA(n,\mathbb{C}))$, then $mPJ_1P^{-1} \notin \mathrm{g}^1$ for every $m \in \mathbb{Z}\backslash\{0\}$. Hence we obtain:

$$\mathrm{g}^1 = \begin{cases} \sum_{k=1}^{p} \mathbb{N}\Psi(f'_k) + \sum_{k=2}^{r} 2i\pi\mathbb{Z}PJ_kP^{-1}, & if\ r \geq 2 \\ \sum_{k=1}^{p} \mathbb{N}\Psi(f'_k), & if\ r = 1 \end{cases}$$

Since $J_k u_0 = e^{(k)}$, we get

$$\mathrm{g}^1_{v_0} = \begin{cases} \sum_{k=1}^{p} \mathbb{N}\Psi(f'_k)v_0 + \sum_{k=2}^{r} 2i\pi\mathbb{Z}Pe^{(k)}, & if\ r \geq 2 \\ \sum_{k=1}^{p} \mathbb{N}\Psi(f'_k)v_0, & if\ r = 1 \end{cases}$$

By Lemma 3.3,(iii), one has $\{0\} \times \mathfrak{q}_{w_0} = \mathrm{g}^1_{v_0}$ and $\Psi(f'_k)v_0 = (0,\ f'_k(w_0))$, so $\mathfrak{q}_{w_0} = p_2\left(\mathrm{g}^1_{v_0}\right)$. It follows that

$$\mathfrak{q}_{w_0} = \begin{cases} \sum_{k=1}^{p} \mathbb{N}f'_k(w_0) + \sum_{k=2}^{r} 2i\pi\mathbb{Z}p_2(Pe^{(k)}), & if\ r \geq 2 \\ \sum_{k=1}^{p} \mathbb{N}f'_k(w_0), & if\ r = 1 \end{cases}$$

The proof is completed. □

Proof of Theorem 1.2: This follows directly from Theorem 1.1, Proposition 4.2.

Proof of Corollary 1.3: First, by Proposition **??**, if $F = \mathbb{Z}u_1 + \dots + \mathbb{Z}u_m$, $u_k \in \mathbb{C}^n$ with $m \leq 2n$, then F cannot be dense in $\mathbb{C}^n$. Now, by the form of $\mathfrak{q}_{w_0}$ in Proposition 4.2, $\mathfrak{q}_{w_0}$ cannot be dense in $\mathbb{C}^n$ and so Corollary 1.3 follows by Theorem 1.2. □

Proof of Corollary 1.4: Since $n \leq 2n - r + 1$ (because $r \leq n + 1$), Corollary 1.4 follows from Corollary 1.3. □

5. Example

Lemma 5.1. [12, Lemma 2.2] *Let $\alpha_1, \dots, \alpha_n$ be n positive numbers linearly independent over $\mathbb{Q}$. Then $H = \{(m_1 - m_0\alpha_1, \dots, m_n - m_0\alpha_n) \ : \ m_0, \dots, m_n \in \mathbb{N}\}$ is dense in $\mathbb{R}^n$.*

We identify $\mathbb{C}^n$ to $\mathbb{R}^{2n}$ and by applying the Lemma 5.1, we obtain the following result:

Lemma 5.2. *Let $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n$ be $2n$ positive numbers linearly independent over $\mathbb{Q}$. Then $H = \mathbb{N}^n + i\mathbb{N}^n - \mathbb{N}(\alpha_1 + i\beta_1, \dots, \alpha_n + i\beta_n)$ is dense in $\mathbb{C}^n$.*

Lemma 5.3. [1] *The elements of the set $\{\sqrt{m}, \quad m \in \mathbb{N}$ and $\sqrt{m} \notin \mathbb{N}\}$ are linearly independent over $\mathbb{Q}$.*

Example 5.4. Let $\mathcal{G}$ the sub-semigroup of $GA(2, \mathbb{C})$ generated by $f_1 = (A_1, a_1)$, $f_2 = (A_2, a_2)$, $f_3 = (A_3, a_3)$ and $f_4 = (A_4, a_4)$ where $A_1 = I_2$, $\ a_1 = (2i\pi, 0)$,
$A_2 = diag\,(1, e^{2\pi})\,,\ a_2 = (0,\ 0)\,,\ A_3 = I_2$,
$a_3 = (2i\pi, 0)\,,\ A_4 = dig\left(1,\ e^{-2\sqrt{5}-2i\sqrt{7}}\right) \quad a_4 = (-2\sqrt{2} - 2i\sqrt{3},\ 0)$.
Then $\mathcal{G}$ is hypercyclic.

Proof. First one can check that $\mathcal{G}$ is abelian: $f_if_j = f_jf_i$ for every $i, j = 1, 2, 3, 4$. Denote by $G = \Phi(\mathcal{G})$. Then G is generated by

$$\Phi(f_1) = \begin{bmatrix} 1 & 0 & 0 \\ 2\pi & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ \Phi(f_2) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{2\pi} \end{bmatrix},$$

$$\Phi(f_3) = \begin{bmatrix} 1 & 0 & 0 \\ 2i\pi & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ \Phi(f_4) = \begin{bmatrix} 1 & 0 & 0 \\ -2\sqrt{2} - 2i\sqrt{3} & 1 & 0 \\ 0 & 0 & e^{-2\sqrt{5}-2i\sqrt{7}} \end{bmatrix}.$$

Let $f'_i = (B_i, b_i)$, $i = 1, 2, 3, 4$ where

$$B_1 = \text{diag}(0, 0) = 0, \quad b_1 = (2\pi,\ 0),$$
$$B_2 = \text{diag}(0,\ 2\pi), \qquad b_2 = (0,\ 0),$$
$$B_3 = \text{diag}(0,\ 0), \qquad b_3 = (2i\pi,\ 0),$$
$$B_4 = \text{diag}(0,\ -2\sqrt{5} - 2i\sqrt{7}), \quad b_4 = (-2\sqrt{2} - 2i\sqrt{3},\ 0).$$

Then we have $e^{\Psi(f'_i)} = \Phi(f_i)$, $i = 1, 2, 3, 4$.
Here $r = 2$, $\eta = (2, 1)$, G is an abelian sub-semigroup of $\mathcal{K}^*_{(2,1),2}(\mathbb{C})$. We have $P = I_2$, $u_0 = v_0 = (1, 0, 1)$, $e^{(2)} = (0, 0, 1)$ and $w_0 = (0, 1)$. By Proposition 4.2, $\mathfrak{q}_{w_0} = \sum_{k=1}^{4} \mathbb{N} f'_k(w_0) + 2i\pi\mathbb{Z}p_2(e^{(2)})$. Then $H \subset \mathfrak{q}_{w_0}$, where

$$H = \mathbb{N}(2\pi, 0) + \mathbb{N}(0, 2\pi) + \mathbb{N}(2i\pi, 0) - 2\mathbb{N}(\sqrt{2} + i\sqrt{3},\ \sqrt{5} + i\sqrt{7}) + \mathbb{N}(0, 2i\pi).$$

By Lemma 5.3, one has $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$ and $\sqrt{7}$ are rationally independent, then by Lemma 5.2, H is dense in $\mathbb{C}^2$, so is $\mathfrak{q}_{w_0}$. By Theorem 1.2, $\mathcal{G}$ is hypercyclic. □

Example 5.5. Let $\mathcal{G}$ the sub-semigroup of $GA(3,\mathbb{C})$ generated by $f_1 = (A_1, a_1)$, $f_2 = (A_2, a_2)$, $f_3 = (A_3, a_3)$, $f_4 = (A_4, a_4)$ and $f_5 = (A_5, a_5)$ where $A_1 = I_3$, $a_1 = (2\pi,\ 0,\ 0)$, $A_2 = \mathrm{diag}(1,\ e^{2\pi},\ 1)$, $a_2 = (0,\ 2\pi)$, $A_3 = \mathrm{diag}(1,\ 1,\ e^{2i\pi})$, $a_3 = (2i\pi,\ 0,\ 0)$, $A_4 = \mathrm{diag}(1,\ 1,\ e^{2\pi})$, $a_4 = (0,\ 0,\ 0)$ and

$$A_5 = \mathrm{diag}(1,\ e^{-2\sqrt{5}-2i\sqrt{7}},\ e^{-2\sqrt{11}-2i\sqrt{13}}),\quad a_5 = (-2\sqrt{2} - 2i\sqrt{3},\ 0,\ 0).$$

Then $\mathcal{G}$ is hypercyclic.

Proof. First one can check that $\mathcal{G}$ is abelian: $f_i f_j = f_j f_i$ for every $i, j = 1, 2, 3, 4, 5$. Denote by $G = \Phi(\mathcal{G})$. Then G is generated by

$$\Phi(f_1) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2\pi & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},\ \Phi(f_2) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^{2\pi} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\Phi(f_3) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2i\pi & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},\quad \Phi(f_4) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{2\pi} \end{bmatrix},$$

and

$$\Phi(f_5) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2\sqrt{2} - 2i\sqrt{3} & 1 & 0 & 0 \\ 0 & 0 & e^{-2\sqrt{5}-2i\sqrt{7}} & 0 \\ 0 & 0 & 0 & e^{-2\sqrt{11}-2i\sqrt{13}} \end{bmatrix},$$

Let $f'_i = (B_i, b_i)$, $i = 1, 2, 3, 4, 5$ where

$$B_1 = 0,\quad b_1 = (2\pi,\ 0,\ 0),$$

$$B_2 = \mathrm{diag}(0,\ 2\pi,\ 0),\quad b_2 = (0,\ 0,\ 0),$$

$$B_3 = \mathrm{diag}(0,\ 0,\ 2i\pi),\quad b_3 = (2i\pi,\ 0,\ 0),$$

$$B_4 = \mathrm{diag}(0,\ 0,\ 2\pi),\quad b_4 = (0,\ 0,\ 0),$$

$$B_5 = \mathrm{diag}(0,\ -2\sqrt{5} - 2i\sqrt{7},\ -2\sqrt{11} - 2i\sqrt{13}),\quad b_5 = (-2\sqrt{2} - 2i\sqrt{3},\ 0,\ 0).$$

Then we have $e^{\Psi(f'_i)} = \Phi(f_i)$, $i = 1, 2, 3, 4, 5$.
Here $r = 2$, $\eta = (2, 1, 1)$, G is an abelian sub-semigroup of $\mathcal{K}^*_{(2,1,1),3}(\mathbb{C})$. We have $P = I_4$, $u_0 = v_0 = (1, 0, 1, 1)$, $e^{(2)} = (0, 0, 1, 0)$, $e^{(3)} = (0, 0, 0, 1)$ and $w_0 = (0, 1, 1)$. By Proposition 4.2, $\mathfrak{q}_{w_0} = \sum_{k=1}^{4} \mathbb{N} f'_k(w_0) + 2i\pi\mathbb{Z}p_2(e^{(2)}) + 2i\pi\mathbb{Z}p_2(e^{(3)})$. Then $H \subset \mathfrak{q}_{w_0}$, where

$$H = 2\pi\mathbb{N}^3 + 2i\pi\mathbb{N}^3 - 2\mathbb{N}(\sqrt{2} + i\sqrt{3},\ \sqrt{5} + i\sqrt{7},\ \sqrt{11} + i\sqrt{13}).$$

By Lemma 5.3, one has $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, $\sqrt{7}$, $\sqrt{11}$ and $\sqrt{13}$ are rationally independent then by Lemma 5.2, H is dense in $\mathbb{C}^3$, so is $\mathfrak{q}_{w_0}$. By Theorem 1.2, $\mathcal{G}$ is hypercyclic. □

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