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HARDY-TYPE INEQUALITIES ON THE WEIGHTED CONES OF QUASI-CONCAVE FUNCTIONS

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Abstract. The complete characterization of the Hardy-type $L^p - L^q$ inequalities on the weighted cones of quasi-concave functions for all $0 < p, q < \infty$ is given.

1. Introduction

Let $\mathbb{R}_+ := [0, \infty)$. Denote $\mathfrak{M}$ the set of all measurable functions on $\mathbb{R}_+$, $\mathfrak{M}^+ \subset \mathfrak{M}$ the subset of all non-negative functions and $\mathfrak{M}^\downarrow \subset \mathfrak{M}^+(\mathfrak{M}^\uparrow \subset \mathfrak{M}^+)$ is the cone of all non-increasing (non-decreasing) functions. Let $\varphi \in \mathfrak{M}^+$ be a smooth strictly increasing function such that $\varphi(0) = 0, \varphi(\infty) = \infty$. Then we say, that φ is *admissible*. The cone of $\varphi-$*quasi-concave functions* is defined by

$$\Omega_\varphi := \left\{ f \in \mathfrak{M}^\uparrow : \frac{f}{\varphi} \in \mathfrak{M}^\downarrow \right\}.$$

Numerous papers were recently devoted to analysis of classical operators on the weighted cones of quasi-monotone and quasi-concave functions (see, for instance, recent survey [7], the paper [11], and literature given there). In particular, it plays an important role in the Lorentz spaces (see [1], [13], [3]).

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Let $u, v, w \in \mathfrak{M}^+$, $0 < p, q < \infty$. One of the challenging problems is to characterize the inequalities

$$\left(\int_0^\infty [Af(x)]^q u(x)dx\right)^{\frac{1}{q}} \le C_A \left(\int_0^\infty [f(x)]^p v(x)dx\right)^{\frac{1}{p}}, \ f \in \Omega_\varphi \tag{1.1}$$

and

$$\left(\int_0^\infty [Bf(x)]^q u(x)dx\right)^{\frac{1}{q}} \le C_B \left(\int_0^\infty [f(x)]^p v(x)dx\right)^{\frac{1}{p}}, \ f \in \Omega_\varphi, \tag{1.2}$$

where

$$Af(x) := \int_0^x f(y)w(y)dy$$

and

$$Bf(x) := \int_x^\infty f(y)w(y)dy.$$

In a perfect form characterization of (1.1) and (1.2) means sharp two-sided estimates of the (least possible) constants C_A and C_B, respectively, by integral functionals depending on weights and parameters of summation, so that finitness of the functional implies the validity of the inequality and vice versa. Also, this problem is closely related to the boundedness of the Hardy-Littlewood maximal operator in weighted Lorentz $\Gamma-$spaces and during last two decades it was solved for all parameters $0 < p, q < \infty$ except $0 < q < 1$ (see [4], [14],[11]). In the present paper we fill up the gap. Moreover, we use alternative method, which brings explicit criteria rather different from [14] and [11] and from implicit results of [4] and [5].

Section 2 is devoted to preliminaries. The main results are given in the Section 3.

We use signs := and =: for determining new quantities and $\mathbb{Z}$ for the set of all integers. For positive functionals F and G we write $F \lesssim G$, if $F \le cG$ with some positive constant c, which depends only on irrelevant parameters. $F \approx G$ means $F \lesssim G \lesssim F$ or $F = cG$. χ_E denotes the characteristic function (indicator) of a set E. Uncertainties of the form $0 \cdot \infty$, $\frac{\infty}{\infty}$ and $\frac{0}{0}$ are taken to be zero. $\square$ stands for the end of a proof.

2. Preliminaries

Denote

$$\Omega_{0,p} := \left\{ f \in \mathfrak{M}^\uparrow : \frac{f(t)}{t^p} \in \mathfrak{M}^\downarrow \right\}, \ 0 < p < \infty.$$

It is well known ([2], Proposition 2.5.10), that for any $f \in \Omega_{0,1}$ there is a concave function $\tilde{f}$ such that

$$\frac{1}{2}\tilde{f} \le f \le \tilde{f}$$

and it was shown in ([14], Lemma 2.3), that for every concave function there exists a sequence $\{h_n\} \in \mathfrak{M}^+$ such that

$$g_n(x) := \int_0^x dy \int_y^\infty \frac{h_n(s)ds}{s} \uparrow \tilde{f}(x).$$

Also, observe that

$$f \in \Omega_\varphi \Leftrightarrow f(\varphi^{-1}) \in \Omega_{0,1}$$

and for $0 < p < \infty$

$$f \in \Omega_\varphi \Leftrightarrow f^p \in \Omega_{\varphi^p}.$$

Thus, repeating the argument of ([14], Lemma 2.3) we obtain the following.

Lemma 2.1. *Let $0 < p < \infty$ and φ is admissible. Then for any $f \in \Omega_\varphi$ there are $\tilde{f}(x) \approx [f(x)]^p$ and a sequence $\{h_n\} \in \mathfrak{M}^+$ such that*

$$g_n(x) := \int_0^{\varphi^p(x)} dy \int_y^\infty \frac{h_n(s)ds}{s} \uparrow \tilde{f}(x).$$

If $0 < p < \infty$ and $v \in \mathfrak{M}^+$ we define

$$L_v^p := \left\{ h \in \mathfrak{M} : \|h\|_{L_v^p} := \left(\int_0^\infty |h(x)|^p v(x) dx \right)^{\frac{1}{p}} < \infty \right\}$$

and we write L^p when $v \equiv 1$. Let $0 < q < \infty$ and $w \in \mathfrak{M}^+$. Consider operators T and S of the form

$$Th(x) := \left(\int_0^x w(y) \left(\int_y^\infty h \right)^q dy \right)^{\frac{1}{q}}$$

and

$$Sh(x) = \left(\int_0^x w(y) \left(\int_0^y h \right)^q dy \right)^{\frac{1}{q}}.$$

Let $u, v, w \in \mathfrak{M}^+$ be weights, $1 \le p < \infty$, $0 < r < \infty$. The following inequalities

$$\|Th\|_{L_u^r} \le C_T \|h\|_{L_v^p}, \ h \in \mathfrak{M}^+, \tag{2.1}$$

and

$$\|Sh\|_{L_u^r} \le C_S \|h\|_{L_v^p}, \ h \in \mathfrak{M}^+ \tag{2.2}$$

have been characterized in [8] and [9] for the operator T and in [12] for the both. The criteria in [8] and [9] were found by the discretization method [6] in a more complicated form than in [12]. Below we give the related results from [12] in a slightly improved form.

We suppose for simplicity that $0 < \int_t^\infty u < \infty$ for all $t > 0$ and define the functions $\zeta : [0, \infty) \to [0, \infty)$, $\zeta^{-1} : [0, \infty) \to [0, \infty)$ by

$$\zeta(x) := \sup \left\{ y > 0 : \int_y^\infty u \ge \frac{1}{2} \int_x^\infty u \right\},$$

$$\zeta^{-1}(x) := \sup \left\{ y > 0 : \int_y^\infty u \ge 2 \int_x^\infty u \right\}.$$

Here $\sup \varnothing = 0$. Let $\zeta^{-2} := \zeta^{-1}(\zeta^{-1})$. For $0 \le c < d < \infty$ and $h \in \mathfrak{M}^+$ we put

$$(H_{c,d}f)(x) := \chi_{(c,d]}(x) \int_x^{\zeta(d)} h,$$
$$(H_d f)(x) := \chi_{(0,d]}(x) \int_x^{\infty} h,$$
$$(H^*_{c,d}f)(x) := \chi_{(c,d]}(x) \int_{\zeta^{-1}(c)}^{x} h,$$
$$(H^*_d f)(x) := \chi_{(0,d]}(x) \int_0^{x} h.$$

Theorem 2.2. ([12], Theorems 4 and 5.) *Let $1 \le p < \infty$, $0 < r < \infty$, $0 < q < \infty$ and $\frac{1}{s} := \frac{1}{r} - \frac{1}{p}$ for $r < p$.*

(a) *For validity of the inequality (2.1) it is necessary and sufficient that the inequality*

$$\left(\int_0^\infty u(x) \left(\int_0^x w \right)^{\frac{r}{q}} \left(\int_x^\infty h \right)^r dx \right)^{\frac{1}{r}} \le A_0 \|h\|_{L^p_v},$$

holds for all $h \in \mathfrak{M}^+$ and the constant

$$A_1 := \begin{cases} \sup\limits_{t \in (0,\infty)} \left(\int_t^\infty u \right)^{\frac{1}{r}} \|H_t\|_{L^p_v \to L^q_w}, & p \le r, \\ \left(\int_0^\infty u(x) \left(\int_x^\infty u \right)^{\frac{s}{p}} \|H_{\zeta^{-1}(x),\zeta(x)}\|^s_{L^p_v \to L^q_w} \, dx \right)^{\frac{1}{s}}, & r < p, \end{cases}$$

is finite. Moreover, $C_T \approx A_0 + A_1$.

(b) *The inequality (2.2) is true if and only if the inequality*

$$\left(\int_0^\infty u(x) \left(\int_{\zeta^{-2}(x)}^x w \right)^{\frac{r}{q}} \left(\int_0^{\zeta^{-2}(x)} h \right)^r dx \right)^{\frac{1}{r}} \le \boldsymbol{A}_0 \|h\|_{L^p_v}$$

holds for all $h \in \mathfrak{M}^+$ and the constant

$$\boldsymbol{A}_1 := \begin{cases} \sup\limits_{t \in (0,\infty)} \left(\int_t^\infty u \right)^{\frac{1}{r}} \|H^*_t\|_{L^p_v \to L^q_w}, & p \le r, \\ \left(\int_0^\infty u(x) \left(\int_x^\infty u \right)^{\frac{s}{p}} \|H^*_{\zeta^{-1}(x),\zeta(x)}\|^s_{L^p_v \to L^q_w} \, dx \right)^{\frac{1}{s}}, & r < p \end{cases}$$

is finite. Moreover, $C_S \approx \boldsymbol{A}_0 + \boldsymbol{A}_1$.

Analogously, to solve the inequality (1.2) we need sublinear positive operators $\mathcal{T}$ and $\mathcal{S}$ of the form

$$\mathcal{T}h(x) := \left(\int_x^\infty w(t) \left(\int_0^t h \right)^q dt \right)^{\frac{1}{q}}, h \in \mathfrak{M}^+,$$

$$\mathcal{S}h(x) := \left(\int_x^\infty w(t) \left(\int_t^\infty h \right)^q dt \right)^{\frac{1}{q}}, h \in \mathfrak{M}^+$$

and weighted inequalities

$$\|\mathcal{T}h\|_{L_u^r} \le C_{\mathcal{T}} \|h\|_{L_v^p}, h \in \mathfrak{M}^+, \tag{2.3}$$

$$\|\mathcal{S}h\|_{L_u^r} \le C_{\mathcal{S}} \|h\|_{L_v^p}, h \in \mathfrak{M}^+, \tag{2.4}$$

where $u, v \in \mathfrak{M}^+$, $1 \le p < \infty,\ 0 < r < \infty$, which have been characterized in [12]. Below we give the related results from [12].

We suppose for simplicity that $0 < \int_0^t u < \infty$ for all $t > 0$ and define the functions $\sigma : [0;\infty) \to [0;\infty)$, $\sigma^{-1} : [0;\infty) \to [0;\infty)$, by

$$\sigma(x) := \inf \left\{ y > 0 : \int_0^y u \ge 2 \int_0^x u \right\},$$

$$\sigma^{-1}(x) := \inf \left\{ y > 0 : \int_0^y u \ge \frac{1}{2} \int_0^x u \right\}.$$

Let $\sigma^2 := \sigma(\sigma)$. For $0 \le c < d < \infty$ and $h \in \mathfrak{M}^+$ we put

$$\mathcal{H}_c h(x) := \chi_{[c,\infty)}(x) \int_0^x h,$$

$$\mathcal{H}_{c,d} h(x) := \chi_{[c,d)}(x) \int_{\sigma^{-1}(c)}^x h,$$

$$\mathcal{H}_c^* h(x) := \chi_{[c,\infty)}(x) \int_x^\infty h,$$

$$\mathcal{H}_{c,d}^* h(x) := \chi_{[c,d)}(x) \int_x^{\sigma(d)} h.$$

Theorem 2.3. ([12], Theorems 2 and 3.) *Let $1 \le p < \infty, 0 < r < \infty, 0 < q < \infty$ and $\frac{1}{s} = \frac{1}{r} - \frac{1}{p}$ for $r < p$.*

(a) *For validity of the inequality (2.3) it is necessary and sufficient that the inequality*

$$\left(\int_0^\infty u(x) \left(\int_x^\infty w \right)^{\frac{r}{q}} \left(\int_0^x h \right)^r dx \right)^{\frac{1}{r}} \le B_0 \left(\int_0^\infty h^p v \right)^{\frac{1}{p}}, h \in \mathfrak{M}^+,$$

holds for all $h \in \mathfrak{M}^+$ and the constant

$$B_1 := \begin{cases} \sup_{t>0} \left(\int_0^t u \right)^{\frac{1}{r}} \|\mathcal{H}_t\|_{L_v^p \to L_w^q}, & p \le r; \\ \left(\int_0^\infty u(x) \left(\int_0^x u \right)^{\frac{s}{p}} \left\| \mathcal{H}_{[\sigma^{-1}(x), \sigma(x)]} \right\|_{L_v^p \to L_w^q}^s dx \right)^{\frac{1}{s}}, & r < p, \end{cases}$$

is finite. Moreover, $C_{\mathcal{T}} \approx B_0 + B_1$.

(b) *For validity of the inequality* (2.4) *it is necessary and sufficient that the inequality*

$$\left(\int_0^\infty u(x)\left(\int_x^{\sigma^2(x)} w\right)^{\frac{r}{q}}\left(\int_{\sigma^2(x)}^\infty h\right)^r dx\right)^{\frac{1}{r}} \le \mathbf{B}_0\left(\int_0^\infty h^p v\right)^{\frac{1}{p}}, h \in \mathfrak{M}^+,$$

holds for all $h \in \mathfrak{M}^+$ *and the constant*

$$\mathbf{B}_1 := \begin{cases} \sup_{t>0}\left(\int_0^t u\right)^{\frac{1}{r}} \|\mathcal{H}_t^*\|_{L_v^p \to L_w^q}, & p \le r; \\ \left(\int_0^\infty u(x)\left(\int_0^x u\right)^{\frac{s}{p}} \left\|\mathcal{H}^*_{[\sigma^{-1}(x),\sigma(x)]}\right\|^s_{L_v^p \to L_w^q} dx\right)^{\frac{1}{s}}, & r < p, \end{cases}$$

is finite. Moreover, $C_{\mathcal{S}} \approx \mathbf{B}_0 + \mathbf{B}_1$.

3. Main results

Suppose that $\mathfrak{S} : \mathfrak{M}^+ \to \mathfrak{M}^+$ is a positive monotone operator, that is for all $\{g_n\} \subset \mathfrak{M}^+, g \in \mathfrak{M}^+$ such that $g_n \uparrow g$ it implies $\mathfrak{S}g_n \uparrow \mathfrak{S}g$.

Lemma 3.1. *Let* $0 < p, q < \infty$; $u, v \in \mathfrak{M}^+$, φ *be admissible and let* $\mathfrak{S} : \mathfrak{M}^+ \to \mathfrak{M}^+$ *be a positive monotone operator. Then the inequality*

$$\left(\int_0^\infty [\mathfrak{S}f(x)]^q u(x)dx\right)^{\frac{1}{q}} \le C_{\mathfrak{S}}\left(\int_0^\infty [f(x)]^p v(x)dx\right)^{\frac{1}{p}}, \ f \in \Omega_\varphi \tag{3.1}$$

is equivalent to

$$\left(\int_0^\infty [\mathfrak{S}(B_\varphi h)^{\frac{1}{p}}(x)]^q u(x)dx\right)^{\frac{p}{q}} \le C^p \int_0^\infty h(x)V(x)dx, \ h \in \mathfrak{M}^+, \tag{3.2}$$

where

$$B_\varphi h(x) := [\varphi(x)]^p \int_0^\infty \frac{h(z)dz}{[\varphi(x)]^p + [\varphi(z)]^p},$$

$$V(x) := \int_0^\infty \frac{[\varphi(z)]^p v(z)dz}{[\varphi(x)]^p + [\varphi(z)]^p}$$

and $C_{\mathfrak{S}} \approx C$.

Proof. The proof follows by change $f^p \to f$ in (3.1), applying Lemma 2.1 and the equivalence

$$\int_0^{[\varphi(x)]^p} dy \int_y^\infty \frac{h(z)dz}{z} \approx [\varphi(x)]^p \int_0^\infty \frac{h([\varphi(z)]^p)d([\varphi(z)]^p)}{[\varphi(x)]^p + [\varphi(z)]^p}.$$

□

Theorem 3.2. *Let* $0 < p, q < \infty$ *and* $u, v, w \in \mathfrak{M}^+$. *Then for the best constant* C_A *of the inequality* (1.1) *we have*

$$C_A \approx A_0 + A_1 + \boldsymbol{A}_0 + \boldsymbol{A}_1,$$

where the constants on the right-hand side are given or estimated by the following:

$$A_0^p = \sup_{t>0} \frac{\left(\int_0^t u(x)\left(\int_0^x \varphi w\right)^q dx\right)^{\frac{p}{q}}}{[\varphi(t)]^p V(t)},\ p \le q \tag{3.3}$$

and

$$A_0^p \approx \left(\int_0^\infty [(\varphi(t))^p V(t)]^{\frac{q}{q-p}} \times \right.$$

$$\left. \times \left(\int_0^t u(s)\left(\int_0^s \varphi w\right)^q ds\right)^{\frac{q}{p-q}} u(t)\left(\int_0^t \varphi w\right)^q dt\right)^{\frac{p-q}{q}},\ q < p. \tag{3.4}$$

$$\boldsymbol{A}_0^p = \sup_{t>0} \frac{\left(\int_{\zeta^2(t)}^\infty u(x)\left(\int_{\zeta^{-2}(x)}^x w\right)^q dx\right)^{\frac{p}{q}}}{V(t)},\ p \le q \tag{3.5}$$

and for $q < p$

$$\boldsymbol{A}_0^p \approx \left(\int_0^\infty \left(\frac{\int_t^\infty u(s)\left(\int_{\zeta^{-2}(s)}^s w\right)^q ds}{V(\zeta^{-2}(t))}\right)^{\frac{q}{p-q}} u(t)\left(\int_{\zeta^{-2}(t)}^t w\right)^q dt\right)^{\frac{p-q}{q}}. \tag{3.6}$$

If $p \le q$, *then*

$$A_1 = \sup_{t\in(0,\infty)} \left(\int_t^\infty u\right)^{\frac{1}{q}} \frac{\int_0^t \varphi w}{\varphi(t)[V(t)]^{\frac{1}{p}}},\ 0 < p \le 1, \tag{3.7}$$

and

$$A_1 \approx \sup_{t\in(0,\infty)} \left(\int_t^\infty u\right)^{\frac{1}{q}} \left(\int_0^t \left(\frac{\int_0^s \varphi w}{[\varphi(s)]^p V(s)}\right)^{\frac{1}{p-1}} \varphi(s)w(s)ds\right)^{\frac{1}{p'}}, p > 1, \tag{3.8}$$

where $p' := \frac{p}{p-1}$.

If $q < p$, *then*

$$A_1 = \left(\int_0^\infty u(x)\left(\int_x^\infty u\right)^{\frac{q}{p-q}} \left(\sup_{s\in(\zeta^{-1}(x),\zeta(x))} \frac{\left(\int_{\zeta^{-1}(x)}^s \varphi w\right)^p}{[\varphi(s)]^p V(s)}\right)^{\frac{q}{p-q}} dx\right)^{\frac{p-q}{pq}} \tag{3.9}$$

for $0 < p \le 1$ *and*

$$A_1 \approx \left(\int_0^\infty u(x)\left(\int_x^\infty u\right)^{\frac{q}{p-q}} \times \right.$$

$$\left. \times \left(\int_{\zeta^{-1}(x)}^{\zeta(x)} \left(\frac{\int_{\zeta^{-1}(x)}^t \varphi w}{[\varphi(t)]^p V(t)}\right)^{\frac{1}{p-1}} \varphi(t)w(t)dt\right)^{\frac{q(p-1)}{p-q}} dx\right)^{\frac{p-q}{pq}} \tag{3.10}$$

for $p > 1$.

When $p \le q$

$$\boldsymbol{A}_1 = \sup_{t\in(0,\infty)} \left(\int_t^\infty u\right)^{\frac{1}{q}} \sup_{s\in(0,t)} \frac{\int_s^t w}{[V(s)]^{\frac{1}{p}}},\ 0 < p \le 1 \tag{3.11}$$

and

$$\boldsymbol{A}_1 \approx \sup_{t\in(0,\infty)} \left(\int_t^\infty u\right)^{\frac{1}{q}} \left(\int_0^t \left(\frac{\int_s^t w}{V(s)}\right)^{\frac{1}{p-1}} w(s)ds\right)^{\frac{1}{p'}},\ p > 1. \tag{3.12}$$

Let $q < p$. Then

$$\boldsymbol{A}_1 = \left(\int_0^\infty u(x)\left(\int_x^\infty u\right)^{\frac{q}{p-q}} \left(\sup_{s\in(\zeta^{-1}(x),\zeta(x))} \frac{\left(\int_s^{\zeta(x)} w\right)^p}{V(s)}\right)^{\frac{q}{p-q}} dx\right)^{\frac{p-q}{pq}}, \tag{3.13}$$

if $0 < p \le 1$ and

$$\boldsymbol{A}_1 \approx \left(\int_0^\infty u(x)\left(\int_x^\infty u\right)^{\frac{q}{p-q}} \times\right.$$
$$\left.\times \left(\int_{\zeta^{-1}(x)}^{\zeta(x)} \left(\frac{\int_s^{\zeta(x)} w}{V(s)}\right)^{\frac{1}{p-1}} w(s)ds\right)^{\frac{q(p-1)}{p-q}} dx\right)^{\frac{p-q}{pq}} \tag{3.14}$$

for $p > 1$.

Proof. By Lemma 3.1 and applying

$$B_\varphi h(x) \approx [\varphi(x)]^p \int_x^\infty \frac{h}{[\varphi]^p} + \int_0^x h \tag{3.15}$$

we see, that (1.1) is equivalent to the pair of inequalities:

$$\left(\int_0^\infty \left[\int_0^x \left(\int_y^\infty h\right)^{\frac{1}{p}} \varphi(y)w(y)dy\right]^q u(x)dx\right)^{\frac{p}{q}} \lesssim C_1^p \int_0^\infty h[\varphi]^p V,\ h \in \mathfrak{M}^+,$$

and

$$\left(\int_0^\infty \left[\int_0^x \left(\int_0^y h\right)^{\frac{1}{p}} w(y)dy\right]^q u(x)dx\right)^{\frac{p}{q}} \lesssim C_2^p \int_0^\infty hV,\ h \in \mathfrak{M}^+,$$

governing by Theorem 2.2, so we have

$$C_1 \approx A_0 + A_1$$

and

$$C_2 \approx \mathbf{A}_0 + \mathbf{A}_1,$$

where A_0 is the least constant of the inequality

$$\left(\int_0^\infty u(x)\left(\int_0^x \varphi w\right)^q \left(\int_x^\infty h\right)^{\frac{q}{p}} dx\right)^{\frac{p}{q}} \le A_0^p \int_0^\infty h(z)[\varphi(z)]^p V(z)dz \tag{3.16}$$

for all $h \in \mathfrak{M}^+$ and A_1 is defined by

$$A_1^p := \begin{cases} \sup\limits_{t\in(0,\infty)} \left(\int_t^\infty u\right)^{\frac{p}{q}} \|H_t\|_{L^1_{[\varphi]^p V} \to L^{\frac{1}{p}}_{\varphi w}}, & p \le q, \\ \left(\int_0^\infty u(x) \left(\int_x^\infty u\right)^{\frac{q}{p-q}} \|H_{\zeta^{-1}(x),\zeta(x)}\|^{\frac{q}{p-q}}_{L^1_{[\varphi]^p V} \to L^{\frac{1}{p}}_{\varphi w}} dx\right)^{\frac{p-q}{q}}, & q < p. \end{cases} \tag{3.17}$$

Analogously, $\mathbf{A}_0$ is the best possible constant in the inequality

$$\left(\int_0^\infty u(x) \left(\int_{\zeta^{-2}(x)}^x w\right)^q \left(\int_0^{\zeta^{-2}(x)} h\right)^{\frac{q}{p}} dx\right)^{\frac{p}{q}} \le \mathbf{A}_0^p \int_0^\infty hV, \;\; h \in \mathfrak{M}^+$$

and $\mathbf{A}_1$ is determined from

$$\mathbf{A}_1^p := \begin{cases} \sup\limits_{t\in(0,\infty)} \left(\int_t^\infty u\right)^{\frac{p}{q}} \|H_t^*\|_{L^1_V \to L^{\frac{1}{p}}_w}, & p \le q, \\ \left(\int_0^\infty u(x) \left(\int_x^\infty u\right)^{\frac{q}{p-q}} \|H^*_{\zeta^{-1}(x),\zeta(x)}\|^{\frac{q}{p-q}}_{L^1_V \to L^{\frac{1}{p}}_w} dx\right)^{\frac{p-q}{q}}, & q < p. \end{cases} \tag{3.18}$$

If $k(x,y) \ge 0$ is a measurable kernel on $\mathbb{R}_+ \times \mathbb{R}_+$ and

$$Kf(x) := \int_0^\infty k(x,y) f(y) dy,$$

then by well known results ([10], Chapter XI, § 1.5, Theorem 4, see also [7], Theorem 1.1)

$$\|K\|_{L^1 \to L^\lambda} = \sup_{t>0} \left(\int_0^\infty [k(x,t)]^\lambda dx\right)^{\frac{1}{\lambda}}, \;\; 1 \le \lambda < \infty.$$

Applying this result to the inequality (3.16) with $\lambda = \frac{q}{p}$ and

$$k(x,y) = \chi_{[x,\infty)}(y) \frac{[u(x)]^{\frac{p}{q}} \left(\int_0^x \varphi w\right)^p}{[\varphi(y)]^p V(y)},$$

we find (3.3). If

$$If(x) := \rho(x) \int_x^\infty f(y) \varkappa(y) dy$$

then by the dual version of ([15], Theorem 3.3) we have

$$\|I\|_{L^1 \to L^\lambda} \approx \left(\int_0^\infty [\operatorname*{ess\,sup}_{y>x} \varkappa(y)]^{\frac{\lambda}{1-\lambda}} \left(\int_0^x \rho\right)^{\frac{\lambda}{1-\lambda}} \rho(x) dx\right)^{\frac{1-\lambda}{\lambda}}.$$

By this result with $\lambda = \frac{q}{p}$ applying to (3.16) we obtain (3.4) for $q < p$. Analogously, we find (3.5) and (3.6). Again, applying ([10], Chapter XI, § 1.5, Theorem 4) and ([15], Theorem 3.3) we obtain

$$\|H_t\|_{L^1_{[\varphi(z)]^p V(z)} \to L^{\frac{1}{p}}_{\varphi w}} = \sup_{s\in(0,t)} \frac{\left(\int_0^s \varphi w\right)^p}{[\varphi(s)]^p V(s)}, \;\; 0 < p \le 1$$

and

$$\|H_t\|_{L^1_{[\varphi(z)]^pV(z)}\to L^{\frac{1}{p}}_{\varphi w}} \approx \left(\int_0^t \left(\frac{\int_0^s \varphi w}{[\varphi(s)]^p V(s)}\right)^{\frac{1}{p-1}} \varphi(s)w(s)ds\right)^{p-1}, \; p>1,$$

so that (3.17) brings (3.7) and (3.8). Similarly,

$$\|H_{\zeta^{-1}(x),\zeta(x)}\|_{L^1_{[\varphi(z)]^pV(z)}\to L^{\frac{1}{p}}_{\varphi w}} = \sup_{s\in(\zeta^{-1}(x),\zeta(x))} \frac{\left(\int_{\zeta^{-1}(x)}^s \varphi w\right)^p}{[\varphi(s)]^p V(s)}, \; 0<p\le 1$$

and

$$\|H_{\zeta^{-1}(x),\zeta(x)}\|_{L^1_{[\varphi(z)]^pV(z)}\to L^{\frac{1}{p}}_{\varphi w}} \approx \left(\int_{\zeta^{-1}(x)}^{\zeta(x)} \left(\frac{\int_{\zeta^{-1}(x)}^t \varphi w}{[\varphi(t)]^p V(t)}\right)^{\frac{1}{p-1}} \varphi(t)w(t)dt\right)^{p-1}, \; p>1.$$

Hence, (3.17) implies (3.9) and (3.10).

By the same way,

$$\|H_t^*\|_{L^1_V\to L^{\frac{1}{p}}_w} = \sup_{s\in(0,t)} \frac{\left(\int_s^t w\right)^p}{V(s)}, \; 0<p\le 1$$

and

$$\|H_t^*\|_{L^1_V\to L^{\frac{1}{p}}_w} \approx \left(\int_0^t \left(\frac{\int_s^t w}{V(s)}\right)^{\frac{1}{p-1}} w(s)ds\right)^{p-1}, \; p>1.$$

Now, (3.11) and (3.12) follow from (3.18). Moreover, we have

$$\|H^*_{\zeta^{-1}(x),\zeta(x)}\|_{L^1_V\to L^{\frac{1}{p}}_w} = \sup_{s\in(\zeta^{-1}(x),\zeta(x))} \frac{\left(\int_s^{\zeta(x)} w\right)^p}{V(s)}, \; 0<p\le 1$$

and

$$\|H^*_{\zeta^{-1}(x),\zeta(x)}\|_{L^1_V\to L^{\frac{1}{p}}_w} \approx \left(\int_{\zeta^{-1}(x)}^{\zeta(x)} \left(\frac{\int_s^{\zeta(x)} w}{V(s)}\right)^{\frac{1}{p-1}} w(s)ds\right)^{p-1}, \; p>1.$$

Thus, from (3.18) we find (3.13) and (3.14) and the relations (3.3)-(3.14) are proved. □

Remark 3.3. The diagonal case $\varphi(t)=t$, $u=v$, $1<p=q<\infty$ was solved in [16].

Theorem 3.4. *Let $0<p,q<\infty$ and $u,v,w\in\mathfrak{M}^+$. Then for the best constant C_B of the inequality (1.2) we have*

$$C_B \approx \mathbf{B_0}+\mathbf{B_1}+B_0+B_1,$$

where the constants on the right-hand side are determined or sandwiched by the following relations.

$$\mathbf{B}_0^p = \sup_{t>0}[V(t)]^{-1}\left(\int_0^{\sigma^{-2}(t)} u(x)\left(\int_x^{\sigma^2(x)}\varphi w\right)^q dx\right)^{\frac{p}{q}},\ p\le q \tag{3.19}$$

and

$$\mathbf{B}_0^p \approx \left(\int_0^\infty [V(\sigma^2(t))]^{\frac{q}{q-p}}\times\right.$$

$$\left.\times\left(\int_0^t u(x)\left(\int_x^{\sigma^2(x)}\varphi w\right)^q dx\right)^{\frac{q}{p-q}} u(t)\left(\int_t^{\sigma^2(t)}\varphi w\right)^q dt\right)^{\frac{p-q}{q}} \tag{3.20}$$

for $q<p$.

$$B_0^p = \sup_{t>0}[V(t)]^{-1}\left(\int_0^t u(x)\left(\int_x^\infty w\right)^q dx\right)^{\frac{p}{q}},\ p\le q, \tag{3.21}$$

$$B_0^p \approx \left(\int_0^\infty [V(t)]^{\frac{q}{q-p}}\times\right.$$

$$\left.\times\left(\int_t^\infty u(x)\left(\int_x^\infty w\right)^q dx\right)^{\frac{q}{p-q}} u(t)\left(\int_t^\infty w\right)^q dt\right)^{\frac{p-q}{q}},\ q<p. \tag{3.22}$$

For $p\le q$

$$\mathbf{B}_1 = \sup_{t>0}\left(\int_0^t u\right)^{\frac{1}{q}}\sup_{s>t}[\varphi^p(s)V(s)]^{\frac{-1}{p}}\left(\int_t^s \varphi w\right),\ 0<p\le 1, \tag{3.23}$$

$$\mathbf{B}_1 \approx \sup_{t>0}\left(\int_0^t u\right)^{\frac{1}{q}}\left(\int_t^\infty\left(\frac{\int_t^s \varphi w}{[\varphi^p(s)V(s)]}\right)^{\frac{1}{p-1}}\varphi(s)w(s)ds\right)^{\frac{1}{p'}},\ p>1 \tag{3.24}$$

and for $q<p$

$$\mathbf{B}_1 = \left(\int_0^\infty u(x)\left(\int_0^x u\right)^{\frac{q}{p-q}}\times\right.$$

$$\left.\times\left[\sup_{\sigma^{-1}(x)<s<\sigma(x)}\frac{\left(\int_{\sigma^{-1}(x)}^s \varphi w\right)^p}{\varphi^p(s)V(s)}\right]^{\frac{q}{p-q}} dx\right)^{\frac{p-q}{pq}},\ 0<p\le 1, \tag{3.25}$$

$$\mathbf{B}_1 \approx \left(\int_0^\infty u(x) \left(\int_0^x u \right)^{\frac{q}{p-q}} \times \right.$$

$$\left. \times \left(\int_{\sigma^{-1}(x)}^{\sigma(x)} \left(\frac{\int_{\sigma^{-1}(x)}^s \varphi w}{\varphi^p(s) V(s)} \right)^{\frac{1}{p-1}} \varphi(s) w(s) ds \right)^{\frac{(p-1)q}{p-q}} dx \right)^{\frac{p-q}{pq}}, \tag{3.26}$$

if $p > 1$. *For* $p \le q$

$$B_1 = \sup_{t>0} \left(\int_0^t u \right)^{\frac{1}{q}} \sup_{s>t} [V(s)]^{\frac{-1}{p}} \left(\int_s^\infty w \right), \ 0 < p \le 1, \tag{3.27}$$

$$B_1 \approx \sup_{t>0} \left(\int_0^t u \right)^{\frac{1}{q}} \left(\int_t^\infty \left(\frac{\int_s^\infty w}{V(s)} \right)^{\frac{1}{p-1}} w(s) ds \right)^{\frac{1}{p'}}, \ p > 1, \tag{3.28}$$

and for $q < p$

$$B_1 = \left(\int_0^\infty u(x) \left(\int_0^x u \right)^{\frac{q}{p-q}} \left[\sup_{\sigma^{-1}(x) < s < \sigma(x)} \frac{\left(\int_s^{\sigma(x)} w \right)^p}{V(s)} \right]^{\frac{q}{p-q}} dx \right)^{\frac{p-q}{pq}}, \tag{3.29}$$

when $0 < p \le 1$ *and*

$$B_1 \approx \left(\int_0^\infty u(x) \left(\int_0^x u \right)^{\frac{q}{p-q}} \times \right.$$

$$\left. \times \left(\int_{\sigma^{-1}(x)}^{\sigma(x)} \left(\frac{\int_s^{\sigma(x)} w}{V(s)} \right)^{\frac{1}{p-1}} w(s) ds \right)^{\frac{(p-1)q}{p-q}} dx \right)^{\frac{p-q}{pq}}, \ p > 1. \tag{3.30}$$

Proof. By Lemma 3.1 and applying (3.15) we see, that (1.2) is equivalent to validity of the pair of inequalities

$$\left(\int_0^\infty \left[\int_x^\infty \left(\int_y^\infty h \right)^{\frac{1}{p}} \varphi(y) w(y) dy \right]^q u(x) dx \right)^{\frac{p}{q}} \le \mathbf{C}_1^p \int_0^\infty h [\varphi]^p V,$$

$$\left(\int_0^\infty \left[\int_x^\infty \left(\int_0^y h \right)^{\frac{1}{p}} w(y) dy \right]^q u(x) dx \right)^{\frac{p}{q}} \le \mathbf{C}_2^p \int_0^\infty hV$$

for all $h \in \mathfrak{M}^+$, described by Theorem 2.3, so we have

$$\mathbf{C}_1 \approx \mathbf{B_0} + \mathbf{B_1}$$

and

$$\mathbf{C}_2 \approx B_0 + B_1,$$

where $\mathbf{B_0}$ is the least constant of the inequality

$$\left(\int_0^\infty u(x)\left(\int_x^{\sigma^2(x)}\varphi w\right)^q\left(\int_{\sigma^2(x)}^\infty h\right)^{\frac{q}{p}}dx\right)^{\frac{p}{q}} \le \mathbf{B_0^p}\int_0^\infty h\varphi^p V,\ h\in\mathfrak{M}^+.$$

and $\mathbf{B_1}$ is defined by

$$\mathbf{B}_1^p := \begin{cases} \sup_{t>0}\left(\int_0^t u\right)^{\frac{p}{q}}\|\mathcal{H}_t^*\|_{L^1_{\varphi^p V}\to L^{\frac{1}{p}}_{\varphi w}}, & p\le q;\\ \left(\int_0^\infty u(x)\left(\int_0^x u\right)^{\frac{q}{p-q}}\left\|\mathcal{H}^*_{\sigma^{-1}(x),\sigma(x)}\right\|^{\frac{q}{p-q}}_{L^1_{\varphi^p V}\to L^{\frac{1}{p}}_{\varphi w}}dx\right)^{\frac{p-q}{q}}, & q<p.\end{cases}$$

Analogously, B_0 is the least constant of the inequality

$$\left(\int_0^\infty u(x)\left(\int_x^\infty w\right)^q\left(\int_0^x h\right)^{\frac{q}{p}}dx\right)^{\frac{p}{q}} \le B_0^p\int_0^\infty hV,\ h\in\mathfrak{M}^+$$

and B_1 is defined by

$$B_1^p := \begin{cases} \sup_{t>0}\left(\int_0^t u\right)^{\frac{p}{q}}\|\mathcal{H}_t\|_{L^1_V\to L^{\frac{1}{p}}_w}, & p\le q;\\ \left(\int_0^\infty u(x)\left(\int_0^x u\right)^{\frac{q}{p-q}}\left\|\mathcal{H}_{\sigma^{-1}(x),\sigma(x)}\right\|^{\frac{q}{p-q}}_{L^1_V\to L^{\frac{1}{p}}_w}dx\right)^{\frac{1}{s}}, & q<p.\end{cases}$$

Again, arguing as in the proof of Theorem 3.2 we obtain (3.19)–(3.30) and theorem is proved. □

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