LOCALLY PSEUDOCONVEX INDUCTIVE LIMIT OF SEQUENCES OF LOCALLY PSEUDOCONVEX ALGEBRAS

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Abstract. Conditions such that a locally $k$-convex inductive limit of a sequence of $k_n$-normed algebras is a locally $m$-($k$-convex) algebra, are given. It is shown that every locally pseudoconvex inductive limit $E$ of a sequence of commutative locally $m$-pseudoconvex algebras is a commutative locally $m$-pseudoconvex algebra if the multiplication in $E$ is jointly continuous.

1. Introduction

A. Arosio asked in [4, p. 349] whether any locally convex inductive limit of normed algebras is a locally $m$-convex algebra. An answer to this question has been given in [3, p. 114] (see also [11, Theorem 15.4]), by showing that every locally convex inductive limit of a countable family of normed algebras is a locally $m$-convex algebra (another proof for this fact was given in [6, Theorem 1]). In this paper we give an analogous result in case of locally $k$-convex inductive limit of $k_n$-normed algebras. Moreover, it is shown that a locally pseudoconvex inductive limit $E$ of commutative locally $m$-pseudoconvex algebras is a topological algebra of the same type as the factors if the multiplication in $E$ is jointly continuous. In the locally convex case a similar result has been proved in [7].

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2. Preliminaries

Let $E$ be a unital topological algebra over $\mathbb{K}$, the field of real numbers $\mathbb{R}$ or complex numbers $\mathbb{C}$, with separately continuous multiplication (in short, a topological algebra). If the underlying topological linear space of $E$ is locally pseudoconvex (see [12, p. 4] or [13, p. 4]), then $E$ is called a locally pseudoconvex algebra. In this case, $E$ has a base $\mathcal{U} = \{U_\lambda : \lambda \in \Lambda\}$ of neighborhoods of zero consisting of balanced ($\mu U_\lambda \subset U_\lambda$ when $|\mu| \leq 1$) and pseudoconvex ($U_\lambda + U_\lambda \subset \mu U_\lambda$ for $\mu \geq 2$) sets. This base defines a set of numbers $\{k_\lambda : \lambda \in \Lambda\}$ in $(0, 1]$ such that

$$U_\lambda + U_\lambda \subset 2^{\frac{1}{k_\lambda}} U_\lambda$$

and

$$\Gamma_{k_\lambda}(U_\lambda) \subset 2^{\frac{1}{k_\lambda}} U_\lambda$$

for each $\lambda \in \Lambda$,

where

$$\Gamma_k(U) = \left\{ \sum_{\nu=1}^n \mu_\nu u_\nu : n \in \mathbb{N}, u_1, \ldots, u_n \in U, \mu_1, \ldots, \mu_n \in \mathbb{K} \text{ with } \sum_{\nu=1}^n |\mu_\nu|^{k} \leq 1 \right\}$$

for any subset $U$ of $E$ and $k \in (0, 1]$. The set $\Gamma_k(U)$ is the absolutely $k$-convex hull of $U$ in $E$. A subset $U \subset E$ is called absolutely $k$-convex if $U = \Gamma_k(U)$ and absolutely pseudoconvex if $U = \Gamma_{k_\lambda}(U)$ for some $k \in (0, 1]$. In case when

$$\inf\{k_\lambda : \lambda \in \Lambda\} = k > 0,$$

$E$ is a locally $k$-convex algebra and when $k = 1$, then $E$ is a locally convex algebra. A locally $m$-pseudoconvex (multiplicative pseudoconvex) algebra is a topological algebra which has a base of neighborhoods of zero which consist of $m$-pseudoconvex (that is, idempotent and absolutely pseudoconvex) sets. A locally $m$-($k$-convex) algebra is a topological algebra which has a base of neighborhoods of zero, which are $m$-($k$-convex) (that is, idempotent and absolutely $k$-convex). In case when $k = 1$, $E$ is a locally $m$-convex algebra.

The topology on a locally pseudoconvex algebra $E$ can be defined by a family $\mathcal{P} = \{p_\lambda : \lambda \in \Lambda\}$ of $k_\lambda$-homogeneous seminorms $p_\lambda$ (that is, $p_\lambda(\mu a) = |\mu|^{k_\lambda} p_\lambda(a)$ for each $\mu \in \mathbb{K}$ and $a \in E$), defined by the base neighborhood $U_\lambda$ of zero, where $k_\lambda \in (0, 1]$ is the power of nonhomogeneity of $p_\lambda$ for each $\lambda \in \Lambda$ and $p_\lambda$ has been defined by

$$p_\lambda(a) = \inf\{|\mu|^{k_\lambda} : a \in \mu \Gamma_{k_\lambda}(U_\lambda)\}$$

for each $a \in E$ and $\lambda \in \Lambda$ (see [13, pp. 3–6], [5, pp. 189 and 195] or [1, pp. 15–16]).

When the topology an of algebra $E$ is defined by a $k$-homogeneous submultiplicative norm $\|\cdot\|$ for some $k \in (0, 1]$, then $E$ is called a $k$-normed algebra and $\|e\| = 1$ whenever $E$ has a unit $e$. 

Let \((E_n)_{n \in \mathbb{N}}\) be a sequence of locally pseudoconvex algebras and for every \(m, n \in \mathbb{N}\) with \(m \leq n\) let
\[f_{nm} : E_m \to E_n\]
be a homomorphism such that
1) \(f_{nn} = \text{id}_{E_n}\) for every \(n \in \mathbb{N}\)
and
2) \(f_{on} = f_{om} \circ f_{mn}\) for any \(m, n, o \in \mathbb{N}\) such that \(n \leq m \leq o\).

The sequence of locally pseudoconvex algebras \((E_n)_{n \in \mathbb{N}}\) with the maps \(f_{nm}\) defined above is called an \textit{inductive system of locally pseudoconvex algebras\) and it is denoted by \((E_n, f_{mn})\).

Let \(E_0\) be the disjoint union of algebras \(E_a\). That is,
\[E_0 = \bigcup_{n \in \mathbb{N}} \{(a, n) : a \in E_n\}.\]

Then, \(x, y \in E_0\) (that is, \(x = (x_0, n)\) with \(x_0 \in E_n\) and \(y = (y_0, m)\) with \(y_0 \in E_m\) for some \(n\) and \(m\) in \(\mathbb{N}\)) are equivalent (in short \(x \sim y\)) if there exists \(o \in \mathbb{N}\) such that \(n \leq o, m \leq o\) and
\[f_{on}(x_0) = f_{om}(y_0).\]

The quotient set \(E_0/\sim\) is called the \textit{inductive (or direct) limit\) of the inductive system \((E_n, f_{mn})\). We shall denote this by \(\lim(E_n, f_{mn})\) or simply by \(\lim E_n\).

For every \(n \in \mathbb{N}\), let \(i_n : E_n \to E_0\) be the \textit{canonical injection\) or \textit{natural injection\) (that is, \(i_n(x) = (x, n)\) for each \(x \in E_n\)) and \(\pi : E_0 \to E_0/\sim\) the quotient map. Then,
\[f_n = \pi \circ i_n : E_n \to E = \lim E_n\]
is the \textit{canonical map\) from \(E_n\) to \(E\).

We endow \(E_0\) with the \textit{disjoint union topology\) (that is, with the topology
\[\{U \subset E_0 : i_n^{-1}(U) \in \tau_n\text{ for every }n \in \mathbb{N}\},\]
where \(\tau_n\) denotes the topology of \(E_n\). Here \(i_n\) is an open and closed continuous map. When all algebras \(E_n\) are subalgebras of some algebra \(E\), then every \(i_n\) is an inclusion \(E_n \to E\). In this case, we endow \(E_0\) with the \textit{coherent topology\)
\[\{U \subset E_0 : U \cap E_n \in \tau_n\text{ for every }n \in \mathbb{N}\}\]
and the inductive limit \(E\) we endow with the \textit{final topology\) \(\tau_{\lim E_n}\) (the \textit{inductive limit topology\)}, defined by the homomorphisms \(f_n\) (that is
\[\tau_{\lim E_n} = \{U \subset E : f_n^{-1}(U) \in \tau_n\text{ for every }n \in \mathbb{N}\}\]).

A base of neighborhoods of zero in this topology is
\[\{O \subset E : O\text{ is balanced and }f_n^{-1}(O) \in \mathcal{N}_n\text{ for every }n \in \mathbb{N}\},\]
(in particular, when every \(E_n\) is a subalgebra of \(E\), then
\[\{O \subset E : O\text{ is balanced and }O \cap E_n \in \mathcal{N}_n\text{ for every }n \in \mathbb{N}\}\),
where \( \mathcal{N}_n \) denotes the set of all neighborhoods of zero in \( E_n \). Then, \( f_n \) is a
continuous (open) map for every \( n \in \mathbb{N} \). Since

\[
E = \bigcup_{n \in \mathbb{N}} f_n(E_n)
\]

and \( f_m \circ f_{mn} = f_n \) when \( n \leq m \) (because \( i_n(x_n) \sim i_m(f_{mn}(x_n)) \)) we get \( f_n(E_n) \subseteq f_m(E_m) \) for any \( m, n \in \mathbb{N} \) with \( n \leq m \).

The algebraic operations in \( \lim E_n \) are defined as usual (see [10, p. 110]): for

every \( x, y \in E \) (then \( x \in f_n(E_n) \) and \( y \in f_m(E_m) \) for some \( m, n \in \mathbb{N} \) there exists
\( o \in \mathbb{N} \) such that \( m \leq o, n \leq o \), \( x = f_o(x_o) \) and \( y = f_o(y_o) \) for some \( x_o, y_o \in E_o \).
So, the algebraic operations in \( E \) are defined by

\[
x + y = f_o(x_o + y_o), \quad \lambda x = f_o(\lambda x_o), \quad xy = f_o(x_oy_o)
\]

for every \( \lambda \in \mathbb{K} \). With respect to such algebraic operations, \((E, \tau_{\lim E_n})\) is a
topological algebra (see [10, p. 115]).

Since the topology \( \tau_{\lim E_n} \) on \( E \) is not necessarily locally pseudoconvex, we
consider on \( E \) the final locally pseudoconvex topology \( \tau \) (see [2, pp. 1952–1953])
defined by the base of neighborhoods at \( x \in E_n \) in the form

\[
\mathcal{L}_x = \{ x + U : U \text{ is absolutely pseudoconvex in } E \text{ and } f_n^{-1}(U) \in \mathcal{N}_n \}
\]

where \( \mathcal{N}_n \) denotes again the set of all neighborhoods of zero in \( E_n \). Similarly as in
[10, pp. 115–116], it is easy to show that \((E, \tau)\) is a locally pseudoconvex algebra.

In this paper, we consider inductive limits of sequences \((E_n)_{n \in \mathbb{N}}\) of locally pseudo-
convex algebras such that \( E_n \) is a subalgebra of \( E_{n+1} \) with continuous inclusion
and the locally pseudoconvex inductive limit topology \( \tau \) induces a topology
coarser than the initial topology of \( E_n \) for each \( n \in \mathbb{N} \).

3. On locally \( k \)-convex inductive limit of a sequence of locally
\( k_n \)-convex algebras

It was shown in [4, Proposition 12] that any commutative locally convex inductive
limit \( E \) of a countable family of normed algebras is locally \( m \)-convex. Later
on, in [3, Theorem 2.1], it was shown that the commutativity of \( E \) in this result

can be omitted (another proof of this fact has been given in [6, Theorem 1]). To
show a similar result in the case when \( E \) is a locally pseudoconvex inductive limit
of a sequence of \( k_n \)-normed algebras \((E_n, \| \cdot \|_n)\) with \( k_n \in (0, 1] \) for each \( n \in \mathbb{N} \),
we need the next.

Lemma 3.1. Let \( B, C \) be two subsets of an algebra and \( k \in (0, 1] \). Then,
\( \Gamma_k(B) \Gamma_k(C) \subseteq \Gamma_k(BC) \). In particular, if \( U \) is an idempotent set, then \( \Gamma_k(U) \)
is also idempotent.

Proof. Take \( x \in \Gamma_k(B) \) and \( y \in \Gamma_k(C) \). Then,

\[
x = \sum_{n=1}^{p} a_n x_n \quad \text{and} \quad y = \sum_{m=1}^{q} b_m y_m,
\]

where \( a_n, b_m \) are elements of \( B \) and \( C \), respectively.

Theorem 3.2. Let \( (E_n)_{n \in \mathbb{N}} \) be a sequence of locally pseudoconvex algebras
such that \( E_n \) is a subalgebra of \( E_{n+1} \) with continuous inclusion
and the locally pseudoconvex inductive limit topology \( \tau \) induces a topology
coarser than the initial topology of \( E_n \) for each \( n \in \mathbb{N} \).

Then, \( \lim E_n \) is also locally pseudoconvex.

Proof. The proof follows from the definition of the topology \( \tau_{\lim E_n} \) and the
above lemma.

Corollary 3.3. Let \( (E_n)_{n \in \mathbb{N}} \) be a sequence of locally pseudoconvex algebras
such that \( E_n \) is a subalgebra of \( E_{n+1} \) with continuous inclusion
and the locally pseudoconvex inductive limit topology \( \tau \) induces a topology
coarser than the initial topology of \( E_n \) for each \( n \in \mathbb{N} \).

Then, \( \lim E_n \) is also locally pseudoconvex.

Proof. The proof follows from the definition of the topology \( \tau_{\lim E_n} \) and the
above lemma.

Remark 3.4. The above theorem and corollary can be extended to the case
when \( \mathcal{N}_n \) is an arbitrary family of neighborhoods of zero in \( E_n \) that
is not necessarily locally pseudoconvex.

Proof. The proof follows from the definition of the topology \( \tau_{\lim E_n} \) and the
above lemma.
where \( x_1, \ldots, x_p \in B, \ y_1, \ldots, y_q \in C \),
\[
\sum_{n=1}^{p} |a_n|^k \leq 1 \quad \text{and} \quad \sum_{m=1}^{q} |b_m|^k \leq 1.
\]
Hence
\[
xy = \left( \sum_{n=1}^{p} a_n x_n \right) \left( \sum_{m=1}^{q} b_m y_m \right) = \sum_{n=1}^{p} \sum_{m=1}^{q} a_n b_m x_n y_m,
\]
where \( x_n y_m \in BC \) and
\[
\sum_{n=1}^{p} \sum_{m=1}^{q} |a_n b_m|^k \leq \left( \sum_{n=1}^{p} |a_n|^k \right) \left( \sum_{m=1}^{q} |b_m|^k \right) \leq 1.
\]

**Theorem 3.2.** Let \((E, \tau)\) be a locally \( k \)-convex inductive limit of a sequence of \( k_n \)-normed algebras \((E_n, \| \cdot \|_n)\) with continuous inclusions. If \( k, k_n \in (0, 1] \) and \( k \leq k_n \) for each \( n \in \mathbb{N} \), then \((E, \tau)\) is a locally \( m \)-\((k\)-convex\) algebra.

**Proof.** For any \( n \in \mathbb{N} \), let \( B_n = \{ x \in E_n : \| x \|_n \leq 1 \} \) (the unit ball in \( E_n \)), and let \( k \in (0, 1] \) be a number such that \( k \leq k_n \) for each \( n \in \mathbb{N} \). Then, \( B_n \) is an idempotent and absolutely \( k \)-convex set for each \( n \in \mathbb{N} \). Indeed, if \( a, b \in B_n \) and
\[
| \lambda |^k + | \mu |^k \leq 1,
\]
then
\[
\| \lambda a + \mu b \|_n = | \lambda |^{k_n} \| a \|_n + | \mu |^{k_n} \| b \|_n \leq | \lambda |^k + | \mu |^k \leq 1.
\]
Hence, \( \lambda a + \mu b \in B_n \). Taking this into account, we can assume that every norm \( \| \cdot \|_n \) is \( k \)-homogeneous otherwise, instead of \( \| \cdot \|_n \), we consider the new norm
\[
\| \cdot \|_{k_n}^k
\]
which is \( k \)-homogeneous.

Moreover, we can assume that \( B_{n-1} \subseteq B_n \) for each \( n > 1 \). Otherwise, we replace \( k \)-norm \( \| \cdot \|_n \) of the algebra \( E_n \) with equivalent \( k \)-norm \( \| \cdot \|_n' \) such that \( B_{n-1}' \subseteq B_n' \) for each \( n > 1 \) where \( B_n' = \{ a \in E_n : \| a \|_n' \leq 1 \} \). Because the injection \( E_{n-1} \to E_n \) is a continuous linear map, there exists \( M_n \geq 1 \) such that \( \| a \|_n \leq M_n \| a \|_{n-1} \) for each \( a \in E_{n-1} \) (see [5, Proposition 4.3.11], both norms here are \( k \)-homogeneous). We consider first the case when \( E_{n-1} \) and \( E_n \) have the same unit element \( e_n \). Let \( \| a \|_1 = \| a \|_1 \) (then \( \| a \|_2 \leq M_2' \| a \|_1 \) where \( M_2' = M_2 \)) and
\[
\| a \|_2' = \sup_{c \in E_2, q_2(c) \leq 1} q_2(ac)
\]
where
\[
q_2(a) = \sup_{s \in B_1'} \| sa \|_2
\]
for all \( a \in E_2 \). Then
\[
q_2(\lambda a) = | \lambda |^k q_2(a), \quad q_2(a + b) \leq q_2(a) + q_2(b),
\]
\[
\| a \|_2 \leq q_2(a) \leq \sup_{s \in B_1'} \| s \|_2 \| a \|_2 \leq M_2' \| a \|_2
\]
and 

\[ q_2(ab) = \sup_{s \in B_1'} \|s(ab)\|_2 \leq \sup_{s \in B_1'} \|sa\|_2 \|b\|_2 = q_2(a)\|b\|_2 \leq q_2(a)q_2(b) \]

for each \( \lambda \in \mathbb{K} \) and \( a, b \in E_2 \). Hence, \( q_2 \) is a \( k \)-norm on \( E_2 \) which is equivalent to \( \| \cdot \|_2 \). Taking this into account, \( \| \cdot \|_2 \) is a \( k \)-homogeneous norm on \( E_2 \). Moreover,

\[
\|ab\|_2 = (\|l_\lambda a\|_2)_{op} = (\|l_\lambda a\|_2)_{op} \leq (\|l_\lambda a\|_2)_{op} = \|a\|_2 \|b\|_2
\]

(here \( \|a\|_2 \) is the operator norm \( (\|l_\lambda a\|_2)_{op} \) of the left regular representation \( l_\lambda a \) of \( a \) on \((E_2, q_2)\),

\[
q_2(a) = q_2(ae_2) = M'_2 q_2(a \cdot \frac{e_2}{M'_2}) \leq M'_2 \sup_{c \in E_2, q_2(c) \leq 1} q_2(ac) = M'_2 \|a\|_2
\]

because \( q_2(e_2) \leq M'_2 \) and

\[
\frac{1}{M'_2} \|a\|_2 \leq \frac{1}{M'_2} q_2(a) \leq \|a\|_2 \leq \sup_{c \in E_2, q_2(c) \leq 1} q_2(a)q_2(e_2c) = q_2(a)\|e_2\|_2 \leq M'_2 \|a\|_2
\]

for each \( a, b \in E_2 \). Since

\[
\|t\|_2 = \sup_{c \in E_2, q_2(c) \leq 1} q_2(tc) = \sup_{c \in E_2, q_2(c) \leq 1} \| (st)c \|_2 \leq \sup_{c \in E_2, q_2(c) \leq 1} q_2(c) = 1
\]

for each \( t \in B_1' \) (because \( B_1' = B_1 \) and \( B_1 t \subseteq B_1 \)), then \( \| \cdot \|'_2 \) is a \( k \)-norm on \( E_2 \), which is equivalent to \( \| \cdot \|_2 \), and satisfies the condition \( B'_1 \subseteq B'_2 \).

The norm \( \| \cdot \|'_3 \) we define similarly, that is, we put

\[
\|a\|'_3 = \sup_{c \in E_2, q_2(c) \leq 1} q_3(ac)
\]

where

\[
q_3(a) = \sup_{s \in B_2'} \|sa\|_3
\]

for all \( a \in E_3 \). Now, similarly as above, we have \( \|a\|_3 \leq M'_3 \|a\|'_3 \) for \( M'_3 = M_3M'_2 \),

\[
\|a\|_3 \leq \|a\|'_3 \leq M'_3 \|a\|_3
\]

for each \( a \in E_3 \) and \( \|a\|'_3 \leq 1 \) for each \( a \in B'_2 \). Hence, \( B'_2 \subseteq B'_3 \). Continuing in the same way, for every fixed \( n \geq 4 \) we define

\[
\|a\|'_n = \sup_{c \in E_n, q_n(c) \leq 1} q_n(ac)
\]

where

\[
q_n(a) = \sup_{s \in B'_n} \|sa\|_n
\]

for all \( a \in E_n \) and show that \( B'_{n-1} \subseteq B'_n \).

Let now \( E_{n-1} \) and \( E_n \) be arbitrary \( k \)-normed algebras. Instead of these algebras, we consider direct products \( E_{n-1} \times \mathbb{K} \) and \( E_n \times \mathbb{K} \) which are \( k \)-normed algebras with respect to the algebraic operations (similarly as in case of the unitization) and norm \( \|(a, \lambda)\|_k = \|a\|_k + |\lambda| \) for each \( (a, \lambda) \in E_k \times \mathbb{K} \) (here \( k \) is \( n - 1 \) or \( n \)). Then \( E_{n-1} \times \mathbb{K} \) and \( E_n \times \mathbb{K} \) have the same unit element \( (\theta, 1) \), where \( \theta \) is the zero element in \( E_{n-1} \) and \( E_n \). Moreover, \( E_{n-1} \times \mathbb{K} \) is a subalgebra of \( E_n \times \mathbb{K} \). Hence, there are equivalent \( k \)-norms \( \|(\cdot, \cdot)\|'_n \) and \( \|(\cdot, \cdot)\|'_n \) such that

\[
\|(a, \lambda)\|'_n \leq \|(a, \lambda)\|'_{n-1} \text{ if } \|(a, \lambda)\|'_n \leq 1.
\]

Thus

\[
\|a\|'_n = \|(a, 0)\|'_n \leq \|(a, 0)\|'_{n-1} = \|a\|'_{n-1}
\]
for each $a \in B'_{n-1}$. Hence, $B'_{n-1} \subseteq B'_n$ for each fixed $n > 1$. Consequently, we can assume that

$$B_1 \subseteq B_2 \subseteq \cdots \subseteq B_n \subseteq \cdots$$

Since $B_1$ is bounded in $B_2$ (because $\|a\|_2 \leq M_2\|a\|_1$ for each $a \in E_1$) and $B_2$ is a neighborhood of zero in $E_2$, then there is a number $t_1 \geq 1$ such that $B_1 \subseteq t_1B_2$. We put $B'_1 = B_1$ and

$$B'_n = \Gamma_k\left(I\left(B_{n-1} \bigcup \frac{1}{t_{n-1}}B_n\right)\right) = \Gamma_k\left(\bigcup_{j \in \mathbb{N}}\left(B_{n-1} \bigcup \frac{1}{t_{n-1}}B_n\right)^j\right)$$

for $n > 1$, where $I(U)$ is the idempotent hull (see [8, pp. 26 and 27]) of $U \subseteq E$. Then, $B'_2$ is an idempotent (by Lemma 3.1) and absolutely $k$-convex set. Because

$$B_1 \cup \frac{1}{t_1}B_2 \subseteq B_2 \subseteq B_2 \cup \frac{1}{t_2}B_3,$$

then $B'_2 \subseteq B'_3$ (it is clear that $I(U) \subseteq I(V)$ and $\Gamma_k(U) \subseteq \Gamma_k(V)$ if $U \subseteq V$). Since

$$\frac{1}{t_1}B_2 \subseteq \left(B_1 \cup \frac{1}{t_1}B_2\right) \subseteq I\left(B_1 \cup \frac{1}{t_1}B_2\right) \subseteq \Gamma_k\left(\left(B_1 \cup \frac{1}{t_1}B_2\right)\right) = B'_2,$$

then, continuing in the same way, we have an increasing sequence $\{B'_n : n \geq 2\}$ of idempotent and absolutely $k$-convex sets $B'_n$ such that

$$(2) \quad \frac{1}{t_{n-1}}B_n \subset B'_n.$$  

Moreover, $B'_2 \subseteq t_1B_2$. Indeed, for $x \in \frac{1}{t_1}\bigcup_{j \in \mathbb{N}}(B_1 \cup \frac{1}{t_1}B_2)^j$ we have $t_1x \in (B_1 \cup \frac{1}{t_1}B_2)^{j_0}$ for some $j_0 \in \mathbb{N}$. Hence there is an element $y \in B_1 \cup \frac{1}{t_1}B_2$ such that $t_1x = y^{j_0}$. If $y \in B_1$, then from $t_1x \in B_1^{j_0} \subseteq B_1 \subseteq t_1B_2$ follows that $x \in B_2$, otherwise $y \in \frac{1}{t_1}B_2$. Then, from $t_1x \in \frac{1}{t_1}B_2^{j_0} \subseteq \frac{1}{t_1}B_2$ follows that $x \in \frac{1}{t_1^{j_0+1}}B_2 \subseteq B_2$ provided that $B_2$ is balanced. Arguing similarly, we have

$$(3) \quad B'_n \subset t_{n-1}B_n$$

where $t_n \geq 1$ for each $n \in \mathbb{N}$. Thus,

$$\frac{1}{t_{n-1}}B_n \subset B'_n \subset t_{n-1}B_n$$

for all $n \in \mathbb{N}$.

Now, we shall prove that

$$\mathcal{L}_\theta = \{\Gamma_k\left(\bigcup_{n \in \mathbb{N}}\varepsilon_nB'_n\right) : \varepsilon_n \in (0, 1]\}$$

is a base of neighborhoods of zero in $E$ which consists of idempotent absolutely $k$-convex sets. Clearly, every element of $\mathcal{L}_\theta$ is absolutely $k$-convex, to prove that every element $V = \Gamma_k(\bigcup_{n \in \mathbb{N}}\varepsilon_nB'_n)$ in $\mathcal{L}_\theta$ is idempotent, we consider
Here, \( x, y \in \bigcup_{n \in \mathbb{N}} \varepsilon_n B'_{m_n} \). Then, \( x \in \varepsilon_n B'_{m_n} \) and \( y \in \varepsilon'_m B'_m \) for some \( m, n \in \mathbb{N} \). If \( B'_n \subseteq B'_m \) (the case \( B'_n \supset B'_m \) is similar), then

\[
xy \in \varepsilon_n B'_m \varepsilon'_m B'_m \subseteq \varepsilon_n \varepsilon'_m B'_m B'_m \subseteq \varepsilon_n B'_m \subseteq \bigcup_n \varepsilon_n B'_n,
\]

that is, \( \bigcup_{n \in \mathbb{N}} \varepsilon_n B'_n \) is idempotent and hence \( V \) is idempotent by Lemma 3.1.

To show that \( \mathcal{L}_\theta \) is a base of neighborhoods of zero for some topology \( \tau' \) on \( E \), we show that \( \mathcal{L}_\theta \) satisfies the following conditions:

1) if \( V \in \mathcal{L}_\theta \), then the zero element \( \theta \in V \);
2) if \( V_1, V_2 \in \mathcal{L}_\theta \), then there exists a set \( V_3 \in \mathcal{L}_\theta \) such that \( V_3 \subset V_1 \cap V_2 \);
3) if \( V \in \mathcal{L}_\theta \), then there exists a set \( V_0 \in \mathcal{L}_\theta \) and for every \( y \in V_0 \) a set \( W = y + V_0 \) such that \( W \subset V \).

Clearly 1) holds. To show that 2) holds, we put \( V_1 = \Gamma_k(\bigcup_{n \in \mathbb{N}} \varepsilon_n B'_n) \), \( V_2 = \Gamma_k(\bigcup_{n \in \mathbb{N}} \varepsilon'_n B'_n) \) and \( \varepsilon''_n = \inf\{\varepsilon_n, \varepsilon'_n\} \) for every \( n \in \mathbb{N} \). Since \( \frac{\varepsilon''_n}{\varepsilon_n} \leq 1 \) and \( \frac{\varepsilon''_n}{\varepsilon'_n} \leq 1 \), then

\[
\varepsilon''_n B'_n \subseteq \varepsilon_n B'_n, \quad \varepsilon''_n B'_n \subseteq \varepsilon'_n B'_n
\]

and hence

\[
\varepsilon''_n B'_n \subset \bigcup_{n \in \mathbb{N}} \varepsilon_n B'_n \cap \bigcup_{n \in \mathbb{N}} \varepsilon'_n B'_n \subset V_1 \cap V_2
\]

for every \( n \in \mathbb{N} \). Thus, we can put

\[
V_3 = \Gamma_k(\bigcup_{n \in \mathbb{N}} \varepsilon''_n B'_n).
\]

Then,

\[
V_3 \subset \Gamma_k(V_1 \cap V_2) \subset \Gamma_k(V_1) \cap \Gamma_k(V_2) = V_1 \cap V_2.
\]

3) If \( V \in \mathcal{L}_\theta \), then

\[
V = \Gamma_k(\bigcup_{n \in \mathbb{N}} \varepsilon_n B'_n)
\]

for some sequence \( (\varepsilon_n) \), where \( \varepsilon_n \in (0, 1] \) for each \( n \in \mathbb{N} \). Since \( V \) is \( k \)-convex, \( 2^{-\frac{1}{k}} V + 2^{-\frac{1}{k}} V \subset V \). Moreover, \( 2^{-\frac{1}{k}} V \in \mathcal{L}_\theta \) since \( 2^{-\frac{1}{k}} V = \Gamma_k(\bigcup_{n \in \mathbb{N}} 2^{-\frac{1}{k}} \varepsilon_n B'_n) \), where \( 2^{-\frac{1}{k}} \varepsilon_n \in (0, 1] \) for every \( n \in \mathbb{N} \). Thus \( V_0 = 2^{-\frac{1}{k}} V \in \mathcal{L}_\theta \) and \( W = y + V_0 \subset V \) for every \( y \in V_0 \). Consequently, by Theorem 4.5 from [14], \( \mathcal{L}_\theta \) is a base of neighborhoods of zero for a locally \( m \)-\((k \)-convex\) topology \( \tau' \) on \( E \).

Claim that \( \tau = \tau' \). For it, let \( O \) be a neighborhood of zero in the topology \( \tau' \). Then, there exists a neighborhood \( U \) of zero such that

\[
U = \Gamma_k\left( \bigcup_{n \in \mathbb{N}} \varepsilon_n B'_n \right)
\]

for some sequence \( (\varepsilon_n) \), where \( \varepsilon_n \in (0, 1] \) for each \( n \in \mathbb{N} \), and \( U \subseteq O \). Take \( n_0 \in \mathbb{N} \) and let \( f_{n_0} : E_{n_0} \to E \) be the canonical map \((f_{n_0} \) is the inclusion). Since
$$\frac{1}{t_{n_0}-1} B_{n_0} \subset B'_{n_0}$$ by (2), then

$$f_n^{-1}(U) = \Gamma_k(\bigcup_{n \in \mathbb{N}} \varepsilon_n B'_n) \cap E_{n_0} \supset \varepsilon_{n_0} B'_{n_0} \supset \frac{\varepsilon_{n_0}}{t_{n_0}-1} B_{n_0},$$

where $\varepsilon_{n_0} B_{n_0}$ is a neighborhood of zero in $E_{n_0}$. Thus, $f_n^{-1}(U)$ is a neighborhood of zero in $E_n$ for every $n \in \mathbb{N}$. Hence, by (1), $U$ is a neighborhood of zero in $E$ in the topology $\tau$. Thus $\tau' \subseteq \tau$.

To prove that $\tau \subseteq \tau'$, let $U$ be a neighborhood of zero in the topology $\tau$. Then, there is in $E$ an absolutely $k$-convex neighborhood $V$ of zero such that $V \subset U$ and $f_n^{-1}(V) = V \cap E_n$ is a neighborhood of zero in $E_n$ for every $n \in \mathbb{N}$. Since $\{\varepsilon_n B_n : \varepsilon_n > 0\}$ is a base of neighborhoods of zero in $(E_n, \tau_n)$ (see [12, p. 14]), then $\varepsilon_n B_n \subset E_n \cap V \subset V$ for some $\varepsilon_n < 1$. As it has been shown in (3), $B'_n \subset t_{n-1} B_n$ with $t_{n-1} \geq 1$. Therefore $\frac{\varepsilon_n}{t_{n-1}} B'_n \subset V$, where $\frac{\varepsilon_n}{t_{n-1}} \in (0, 1]$ for every $n$. Hence, from

$$\bigcup_{n \in \mathbb{N}} \frac{\varepsilon_n}{t_{n-1}} B'_n \subset V$$

it follows

$$\Gamma_k\left(\bigcup_{n \in \mathbb{N}} \frac{\varepsilon_n}{t_{n-1}} B'_n\right) \subset \Gamma_k(V) = V \subset U.$$

Hence, $\tau \subseteq \tau'$. It means that $\tau = \tau'$.

**Corollary 3.3.** Locally $k$-convex inductive limit of a sequence of locally $k$-normed algebras with continuous inclusions is a locally $m$-(k-convex) algebra for every $k \in (0, 1]$. 

4. **LOCALLY PSEUDOCONVEX INDUCTIVE LIMIT OF LOCALLY $m$-PSEUDOCONVEX ALGEBRAS**

It is known that the inductive limit of locally $m$-convex algebras is not necessarily a locally $m$-convex algebra (see the example in [6]). It was shown in [7, Theorem, p. 150] that the locally convex inductive limit $E$ of a sequence of commutative locally $m$-convex algebras is a locally $m$-convex algebra if the multiplication in $E$ is jointly continuous. Next we prove an analogous result for the case of locally pseudoconvex inductive limit of a sequence of commutative locally $m$-pseudoconvex algebras.

**Theorem 4.1.** Let $E$ be a locally pseudoconvex inductive limit of a sequence of commutative locally $m$-pseudoconvex algebras $E_n$ with continuous inclusions. If the multiplication is jointly continuous in $E$, then $E$ is a commutative locally $m$-pseudoconvex algebra.

**Proof.** Let $U$ be a neighborhood of zero in $E$. Then, there is a neighborhood $V_1 \subset U$ of zero such that $\Gamma_k(V_1) = V_1$ for some $k \in (0, 1]$. By the jointly continuity of multiplication in $E$, there exists a neighborhood $O_1$ of zero such that $O_1 O_1 \subset V_1$. Now we put $V_2 = O_1 \cap V_1$. Then, by the jointly continuity
of multiplication, there exists a neighborhood $O_2$ of zero such that $O_2O_2 \subset V_2$. Inductively we define $V_{v+1} = O_v \cap V_v$ for each $v \geq 1$. Since,

$$V_1 \supset V_2 \supset \ldots \supset V_v \supset \ldots,$$

then $V_v \subset U$ for every $v \in \mathbb{N}$.

Since the canonical map (the inclusion) $f_n : E_n \to E$ is continuous for every $n \in \mathbb{N}$, there exists for every $v \in \mathbb{N}$ an $m$-pseudoconvex neighborhood $V_{n,v}$ of zero in $E_n$ such that $V_{n,v} \subset V_v$. Now, for every $n \in \mathbb{N}$, we put $V'_{n,1} = V_{n,1}$ and

$$V'_{n,v+1} = V'_{n,v} \cap V_{n,v+1}$$

for $v \geq 1$. Then,

$$V'_{n,v+1} \subset V'_{n,v} \text{ for all } n, v \in \mathbb{N}$$

and $(V'_{n,v})$ is a sequence of idempotent neighborhoods of zero in $E_n$ (since $V_{n,v}$ is an idempotent neighborhood of zero in $E_n$) for all $n, v \in \mathbb{N}$.

Let $n_0 \in \mathbb{N}$ and $1 \leq p < n_0$ be fixed. We define a new sequence $(V''_{n,v})$ of idempotent neighborhoods of zero in $E_n$ as follows: we put $V''_{p,1} = V_{p,1}$ and for $v \geq 1$ put

$$V''_{p,v+1} = V''_{n_0,v+1} \cap V''_{p,v}$$

and

$$V''_{n,v} = V'_{n,v} \text{ for } n \geq n_0 \text{ and } v \in \mathbb{N}.$$ 

So, by definition of $(V''_{n,v})$, (4), (5) and (6), we have that

$$V''_{n,v+1} \subset V''_{n,v} \text{ for all } v, n \in \mathbb{N}$$

and from

$$V''_{n_0,s} V''_{p,q} \subset V''_{n_0,s} V''_{p,s} \subset V''_{n_0,s} V''_{n_0,s} \subset V''_{n_0,s} \subset V''_{n_0,s} \subset V''_{n_0,s} \subset V_s \subset V_1$$

it follows that

$$V''_{n_0,s} V''_{p,q} \subset V''_{n_0,s} \subset V_1$$

for every natural number $p$ with $p \leq n_0$ and every natural numbers $s$ and $q$ with $s \leq q$.

For any numbers $v(1), \ldots, v(r) \in \mathbb{N}$ with $1 = v(0) < v(1) < v(2) < \ldots < v(r)$ and $n(1), \ldots, n(r+1) \in \mathbb{N}$ (arbitrary $r+1$ (not necessarily different and ordered) numbers) we show by induction on $r \in \mathbb{N}$ that

$$V''_{n(1),1} V''_{n(2),v(1)} \cdots V''_{n(r+1),v(r)} \subset V_1$$

For $r = 1$, (9) holds by (8) (if $n(2) > n(1)$, we can rename these numbers).

Now, we suppose that (9) is true for $r - 1$ and prove that (9) is true for $r$ too. Again, we can assume that $n(r) \geq n(r+1)$ (otherwise we can rename the numbers). Then, using also (8), we get

$$(V''_{n(1),1} V''_{n(2),v(1)} \cdots V''_{n(r+1),v(r)} V''_{n(r),v(r-1)} V''_{n(r),v(r-1)} V''_{n(r+1),v(r) \subset V_1}$$

and by the induction hypothesis, we get the assertion.
Now, we put $W_n = V''_{n,n}$. Then, using (7)
\[ W_{v(r)} = V''_{v(r),v(r)} \subset V''_{v(r),v(r) - 1} \subset V_{v(r),v(r) - 1} \]
for each $r \in \mathbb{N}$. Therefore
\[ (10) \quad W_{v(1)} W_{v(2)} \cdots W_{v(r)} \subset V''_{v(1),1} V''_{v(2),v(1)} \cdots V''_{v(r),v(r) - 1} \subset V_1 \]
by (9) (which holds for any choice of $r + 1$ natural numbers $n(1), \ldots, n(r)$ and $n(r + 1)$).

Take $m(1), \ldots, m(s) \in \mathbb{N}$ (arbitrary fixed not necessarily different $s$ natural numbers). We can find $r \leq s$ natural numbers $v(1), \ldots, v(r)$ such that
\[ 1 < v(1) < v(2) < \cdots < v(r) \]
and the set
\[ \{m(1), \ldots, m(s)\} = \{v(1), \ldots, v(r)\}. \]
By commutativity of $E_n$ and idempotency of $W_n$, we have
\[ W_{m(1)} \cdots W_{m(s)} = \prod_{i=1}^{r} W_{v(i)}^{j:v(j)=v(i)} \subset \prod_{i=1}^{r} W_{v(i)} \subset V_1 \]
for every $r \in \mathbb{N}$, see also (10). Put
\[ W := \bigcup_{s \in \mathbb{N}} \left( \bigcup_{(m(1), \ldots, m(s)) \in \mathbb{N}^s} W_{m(1)} \cdots W_{m(s)} \right). \]
Then, $W$ is an idempotent subset of $V_1$. Indeed, if $x, y \in W$, then
\[ x \in \bigcup W_{m(1)} \cdots W_{m(s_0)}, \]
where the union is taken over all $(m(1), \ldots, m(s_0)) \in \mathbb{N}^s_0$ and
\[ y \in \bigcup W_{m(1)} \cdots W_{m(s_1)}, \]
where the union is taken over all $(m(1), \ldots, m(s_1)) \in \mathbb{N}^s_1$ for some $s_0$ and $s_1$. Therefore,
\[ x \in W_{m'(1)} \cdots W_{m'(s_0)} \quad \text{and} \quad y \in W_{m''(1)} \cdots W_{m''(s_1)} \]
for some $(m'(1), \ldots, m'(s_0)) \in \mathbb{N}^s_0$ and $(m''(1), \ldots, m''(s_1)) \in \mathbb{N}^s_1$. Thus,
\[ xy \in W_{m'(1)} \cdots W_{m'(s_0)} W_{m''(1)} \cdots W_{m''(s_1)} \subset \]
\[ \bigcup W_{m(1)} \cdots W_{m(s_0 + s_1)} \subset W, \]
where the union is taken over all $(m(1), \ldots, m(s_0 + s_1)) \in \mathbb{N}^{s_0 + s_1}$. By Lemma 3.1, the absolutely $k$-convex hull of any idempotent set is idempotent and $k$-convex. So,
\[ W' := \Gamma_k(W) \subset \Gamma_k(V_1) = V_1 \subset U \]
is an $m$-(k-convex) subset of $U$. Since
\[ W' \cap E_n = \Gamma_k(W) \cap E_n \supset W \cap E_n \supset W_n = V''_{n,n} \]
for each $n \in \mathbb{N}$ and $V''_{n,n}$ is an neighborhood of zero in $E_n$, then $W'$ in $E$ is an absolutely $m$-(k-convex) neighborhood of zero.

Thus, $E$ is a commutative locally $m$-pseudoconvex algebra in the locally pseudoconvex inductive limit topology on $E$. □
A topological algebra is locally idempotent if it has a base of idempotent neighborhoods of zero (see [1, p. 196]). Hence, every locally \( m \)-pseudoconvex (in particular, locally \( m \)-convex) algebra is a locally idempotent algebra.

**Theorem 4.2.** Let \( E \) be a topological inductive limit of a sequence of commutative locally idempotent algebras \( E_n \) with continuous inclusions. If the multiplication is jointly continuous in \( E \), then \( E \) is a commutative locally idempotent algebra.

**Proof.** The proof is similar that of Theorem 2. \( \square \)

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