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LOCALLY PSEUDOCONVEX INDUCTIVE LIMIT OF SEQUENCES OF LOCALLY PSEUDOCONVEX ALGEBRAS

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Abstract. Conditions such that a locally k-convex inductive limit of a sequence of k_n-normed algebras is a locally m-(k-convex) algebra, are given. It is shown that every locally pseudoconvex inductive limit E of a sequence of commutative locally m-pseudoconvex algebras is a commutative locally m-pseudoconvex algebra if the multiplication in E is jointly continuous.

1. Introduction

A. Arosio asked in [4, p. 349] whether any locally convex inductive limit of normed algebras is a locally m-convex algebra. An answer to this question has been given in [3, p. 114] (see also [11, Theorem 15.4]), by showing that every locally convex inductive limit of a countable family of normed algebras is a locally m-convex algebra (another proof for this fact was given in [6, Theorem 1]). In this paper we give an analogous result in case of locally k-convex inductive limit of k_n-normed algebras. Moreover, it is shown that a locally pseudoconvex inductive limit E of commutative locally m-pseudoconvex algebras is a topological algebra of the same type as the factors if the multiplication in E is jointly continuous. In the locally convex case a similar result has been proved in [7].

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2. Preliminaries

Let E be a unital topological algebra over $\mathbb{K}$, the field of real numbers $\mathbb{R}$ or complex numbers $\mathbb{C}$, with separately continuous multiplication (in short, a *topological algebra*). If the underlying topological linear space of E is locally pseudoconvex (see [12, p. 4] or [13, p. 4]), then E is called a *locally pseudoconvex algebra*. In this case, E has a base $\mathcal{U} = \{U_\lambda : \lambda \in \Lambda\}$ of neighborhoods of zero consisting of balanced ($\mu U_\lambda \subset U_\lambda$ when $|\mu| \leqslant 1$) and pseudoconvex ($U_\lambda + U_\lambda \subset \mu U_\lambda$ for $\mu \geqslant 2$) sets. This base defines a set of numbers $\{k_\lambda : \lambda \in \Lambda\}$ in $(0,1]$ (see, for instance, [13, pp. 3–6] or [9, pp. 161–162]) such that

$$U_\lambda + U_\lambda \subset 2^{\frac{1}{k_\lambda}} U_\lambda$$

and

$$\Gamma_{k_\lambda}(U_\lambda) \subset 2^{\frac{1}{k_\lambda}} U_\lambda \text{ for each } \lambda \in \Lambda,$$

where

$$\Gamma_k(U) = \Big\{ \sum_{\nu=1}^{n} \mu_\nu u_\nu : n \in \mathbb{N}, u_1, \cdots, u_n \in U, \mu_1, \cdots, \mu_n \in \mathbb{K} \text{ with } \sum_{\nu=1}^{n} | \mu_\nu |^k \leqslant 1 \Big\}$$

for any subset U of E and $k \in (0,1]$. The set $\Gamma_k(U)$ is the *absolutely k-convex hull* of U in E. A subset $U \subset E$ is called *absolutely k-convex* if $U = \Gamma_k(U)$ and *absolutely pseudoconvex* if $U = \Gamma_k(U)$ for some $k \in (0,1]$. In case when

$$\inf\{k_\lambda : \lambda \in \Lambda\} = k > 0,$$

E is a *locally k-convex algebra* and when $k = 1$, then E is a *locally convex algebra*. A *locally m-pseudoconvex* (multiplicative pseudoconvex) *algebra* is a topological algebra which has a base of neighborhoods of zero which consist of m-pseudoconvex (that is, idempotent and absolutely pseudoconvex) sets. A *locally m-(k-convex)* algebra is a topological algebra which has a base of neighborhoods of zero, which are m-(k-convex) (that is, idempotent and absolutely k-convex). In case when $k = 1$, E is a *locally m-convex algebra*.

The topology on a locally pseudoconvex algebra E can be defined by a family $\mathcal{P} = \{p_\lambda : \lambda \in \Lambda\}$ of k_λ-homogeneous seminorms p_λ (that is, $p_\lambda(\mu a) = | \mu |^{k_\lambda} p_\lambda(a)$ for each $\mu \in \mathbb{K}$ and $a \in E$), defined by the base neighborhood U_λ of zero, where $k_\lambda \in (0,1]$ is the power of nonhomogeneity of p_λ for each $\lambda \in \Lambda$ and p_λ has been defined by

$$p_\lambda(a) = \inf\{| \mu |^{k_\lambda} : a \in \mu \Gamma_{k_\lambda}(U_\lambda)\}$$

for each $a \in E$ and $\lambda \in \Lambda$ (see [13, pp. 3–6], [5, pp. 189 and 195] or [1, pp. 15–16]).

When the topology an of algebra E is defined by a k-homogeneous submultiplicative norm $\| \cdot \|$ for some $k \in (0,1]$, then E is called a *k-normed algebra* and $\|e\| = 1$ whenever E has a unit e.

Let $(E_n)_{n\in\mathbb{N}}$ be a sequence of locally pseudoconvex algebras and for every $m, n \in \mathbb{N}$ with $m \leqslant n$ let

$$f_{nm} : E_m \to E_n$$

be a homomorphism such that

1) $f_{nn} = id_{E_n}$ for every $n \in \mathbb{N}$

and

2) $f_{on} = f_{om} \circ f_{mn}$ for any $m, n, o \in \mathbb{N}$ such that $n \leqslant m \leqslant o$.

The sequence of locally pseudoconvex algebras $(E_n)_{n\in\mathbb{N}}$ with the maps f_{nm} defined above is called an *inductive system of locally pseudoconvex algebras* and it is denoted by (E_n, f_{mn}).

Let E_0 be the disjoint union of algebras E_α. That is,

$$E_0 = \bigcup_{n\in\mathbb{N}} \{(a, n) : a \in E_n\}.$$

Then, $x, y \in E_0$ (that is, $x = (x_0, n)$ with $x_0 \in E_n$ and $y = (y_0, m)$ with $y_0 \in E_m$ for some n and m in $\mathbb{N}$) are *equivalent* (in short $x \sim y$) if there exists $o \in \mathbb{N}$ such that $n \leqslant o$, $m \leqslant o$ and

$$f_{on}(x_0) = f_{om}(y_0).$$

The quotient set $E_0/\sim$ is called the *inductive* (or *direct*) *limit* of the inductive system (E_n, f_{mn}). We shall denote this by $\underrightarrow{\lim}(E_n, f_{mn})$ or simply by $\underrightarrow{\lim} E_n$.

For every $n \in \mathbb{N}$, let $i_n : E_n \to E_0$ be the *canonical injection* or *natural injection* (that is, $i_n(x) = (x, n)$ for each $x \in E_n$) and $\pi : E_0 \to E_0/\sim$ the quotient map. Then,

$$f_n = \pi \circ i_n : E_n \to E = \underrightarrow{\lim} E_n \text{ for every } n \in \mathbb{N}$$

is the *canonical map* from E_n to E.

We endow E_0 with the *disjoint union topology* (that is, with the topology

$$\{U \subset E_0 : i_n^{-1}(U) \in \tau_n \text{ for every } n \in \mathbb{N}\},$$

where τ_n denotes the topology of E_n. Here i_n is an open and closed continuous map. When all algebras E_n are subalgebras of some algebra E, then every i_n is an inclusion $E_n \to E$. In this case, we endow E_0 with the *coherent topology*

$$\{U \subset E_0 : U \cap E_n \in \tau_n \text{ for every } n \in \mathbb{N}\})$$

and the inductive limit E we endow with the *final topology* $\tau_{\underrightarrow{\lim} E_n}$ (the *inductive limit topology*), defined by the homomorphisms f_n (that is

$$\tau_{\underrightarrow{\lim} E_n} = \{U \subset E : f_n^{-1}(U) \in \tau_n \text{ for every } n \in \mathbb{N}\}).$$

A base of neighborhoods of zero in this topology is

$$\{O \subset E : O \text{ is balanced and } f_n^{-1}(O) \in \mathcal{N}_n \text{ for every n} \in \mathbb{N}\},$$

(in particular, when every E_n is a subalgebra of E, then

$$\{O \subset E : O \text{ is balanced and } O \cap E_n \in \mathcal{N}_n \text{ for every n} \in \mathbb{N}\}),$$

where $\mathcal{N}_n$ denotes the set of all neighborhoods of zero in E_n. Then, f_n is a continuous (open) map for every $n \in \mathbb{N}$. Since

$$E = \bigcup_{n\in\mathbb{N}} f_n(E_n)$$

and $f_m \circ f_{mn} = f_n$ when $n \leqslant m$ (because $i_n(x_n) \sim i_m(f_{mn}(x_n))$) we get $f_n(E_n) \subseteq f_m(E_m)$ for any $m, n \in \mathbb{N}$ with $n \leqslant m$.

The algebraic operations in $\underrightarrow{\lim} E_n$ are defined as usual (see [10, p. 110]): for every $x, y \in E$ (then $x \in f_n(E_n)$ and $y \in f_m(E_m)$ for some $m, n \in \mathbb{N}$) there exists $o \in \mathbb{N}$ such that $m \leqslant o$, $n \leqslant o$, $x = f_o(x_o)$ and $y = f_o(y_o)$ for some $x_o, y_o \in E_o$. So, the algebraic operations in E are defined by

$$x + y = f_o(x_o + y_o), \qquad \lambda x = f_o(\lambda x_o), \quad xy = f_o(x_o y_o)$$

for every $\lambda \in \mathbb{K}$. With respect to such algebraic operations, $(E, \tau_{\underrightarrow{\lim} E_n})$ is a topological algebra (see [10, p. 115]).

Since the topology $\tau_{\underrightarrow{\lim} E_n}$ on E is not necessarily locally pseudoconvex, we consider on E the final locally pseudoconvex topology τ (see [2, pp. 1952–1953]) defined by the base of neighborhoods at $x \in E_n$ in the form

(1) $\quad \mathcal{L}_x = \{x + U : U \text{ is absolutely pseudoconvex in } E \text{ and } f_n^{-1}(U) \in \mathcal{N}_n\}$

where $\mathcal{N}_n$ denotes again the set of all neighborhoods of zero in E_n. Similarly as in [10, pp. 115–116], it is easy to show that (E, τ) is a locally pseudoconvex algebra.

In this paper, we consider inductive limits of sequences $(E_n)_{n\in\mathbb{N}}$ of locally pseudoconvex algebras such that E_n is a subalgebra of E_{n+1} with continuous inclusion and the locally pseudoconvex inductive limit topology τ induces a topology coarser than the initial topology of E_n for each $n \in \mathbb{N}$.

3. On locally k-convex inductive limit of a sequence of locally k_n-convex algebras

It was shown in [4, Proposition 12] that any commutative locally convex inductive limit E of a countable family of normed algebras is locally m-convex. Later on, in [3, Theorem 2.1], it was shown that the commutativity of E in this result can be omitted (another proof of this fact has been given in [6, Theorem 1]). To show a similar result in the case when E is a locally pseudoconvex inductive limit of a sequence of k_n-normed algebras $(E_n, \|\cdot\|_n)$ with $k_n \in (0, 1]$ for each $n \in \mathbb{N}$, we need the next.

Lemma 3.1. *Let B, C be two subsets of an algebra and $k \in (0, 1]$. Then, $\Gamma_k(B)\Gamma_k(C) \subset \Gamma_k(BC)$. In particular, if U is an idempotent set, then $\Gamma_k(U)$ is also idempotent.*

Proof. Take $x \in \Gamma_k(B)$ and $y \in \Gamma_k(C)$. Then,

$$x = \sum_{n=1}^{p} a_n x_n \quad \text{and} \quad y = \sum_{m=1}^{q} b_m y_m,$$

where $x_1, \ldots, x_p \in B,\ y_1, \ldots, y_q \in C$,

$$\sum_{n=1}^{p} | a_n |^k \leq 1 \quad \text{and} \quad \sum_{m=1}^{q} | b_m |^k \leq 1.$$

Hence

$$xy = \Big(\sum_{n=1}^{p} a_n x_n\Big)\Big(\sum_{m=1}^{q} b_m y_m\Big) = \sum_{n=1}^{p}\sum_{m=1}^{q} a_n b_m x_n y_m,$$

where $x_n y_m \in BC$ and

$$\sum_{n=1}^{p}\sum_{m=1}^{q} | a_n b_m |^k = \sum_{n=1}^{p}\sum_{m=1}^{q} | a_n |^k | b_m |^k = \Big(\sum_{n=1}^{p} | a_n |^k\Big)\Big(\sum_{m=1}^{q} | b_m |^k\Big) \leq 1. \qquad \square$$

Theorem 3.2. *Let (E, τ) be a locally k-convex inductive limit of a sequence of k_n-normed algebras $(E_n, \| \cdot \|_n)$ with continuous inclusions. If $k, k_n \in (0,1]$ and $k \leqslant k_n$ for each $n \in \mathbb{N}$, then (E, τ) is a locally m-(k-convex) algebra.*

Proof. For any $n \in \mathbb{N}$, let $B_n = \{x \in E_n : \|x\|_n \leqslant 1\}$ (the unit ball in E_n), and let $k \in (0,1]$ be a number such that $k \leqslant k_n$ for each $n \in \mathbb{N}$. Then, B_n is an idempotent and absolutely k-convex set for each $n \in \mathbb{N}$. Indeed, if $a, b \in B_n$ and

$$| \lambda |^k + | \mu |^k \leqslant 1,$$

then

$$\|\lambda a + \mu b\|_n \leqslant | \lambda |^{k_n} \|a\|_n + | \mu |^{k_n} \|b\|_n \leqslant | \lambda |^{k_n} + | \mu |^{k_n} \leqslant | \lambda |^k + | \mu |^k \leqslant 1.$$

Hence, $\lambda a + \mu b \in B_n$. Taking this into account, we can assume that every norm $\| \cdot \|_n$ is k-homogeneous otherwise, instead of $\| \cdot \|_n$, we consider the new norm

$$\| \cdot \|_n^{\frac{k}{k_n}}$$

which is k-homogeneous.

Moreover, we can assume that $B_{n-1} \subseteq B_n$ for each $n > 1$. Otherwise, we replace k-norm $\| \cdot \|_n$ of the algebra E_n with equivalent k-norm $\| \cdot \|'_n$ such that $B'_{n-1} \subseteq B'_n$ for each $n > 1$ where $B'_n = \{a \in E_n : \|a\|'_n \leqslant 1\}$. Because the injection $E_{n-1} \to E_n$ is a continuous linear map, there exists $M_n \geqslant 1$ such that $\|a\|_n \leqslant M_n \|a\|_{n-1}$ for each $a \in E_{n-1}$ (see [5, Proposition 4.3.11], both norms here are k-homogeneous). We consider first the case when E_{n-1} and E_n have the same unit element e_n. Let $\|a\|'_1 = \|a\|_1$ (then $\|a\|_2 \leqslant M'_2 \|a\|'_1$ where $M'_2 = M_2$) and

$$\|a\|'_2 = \sup_{c \in E_2, q_2(c) \leqslant 1} q_2(ac)$$

where

$$q_2(a) = \sup_{s \in B'_1} \|sa\|_2$$

for all $a \in E_2$. Then

$$q_2(\lambda a) = |\lambda|^k q_2(a), \quad q_2(a+b) \leqslant q_2(a) + q_2(b),$$

$$\|a\|_2 \leqslant q_2(a) \leqslant \sup_{s \in B'_1} \|s\|_2 \|a\|_2 \leqslant M'_2 \|a\|_2$$

and

$$q_2(ab) = \sup_{s\in B_1'} \|s(ab)\|_2 \leqslant \sup_{s\in B_1'} \|sa\|_2\|b\|_2 = q_2(a)\|b\|_2 \leqslant q_2(a)q_2(b)$$

for each $\lambda \in \mathbb{K}$ and $a, b \in E_2$. Hence, q_2 is a k-norm on E_2 which is equivalent to $\|\cdot\|_2$. Taking this into account, $\|\cdot\|_2'$ is a k-homogeneous norm on E_2. Moreover,

$$\|ab\|_2' = (\|l_{ab}\|_2)_{op} = (\|l_a \circ l_b\|_2)_{op} \leqslant (\|l_a\|_2)_{op}(\|l_b\|_2)_{op} = \|a\|_2'\|b\|_2'$$

(here $\|a\|_2'$ is the operator norm $(\|l_a\|_2)_{op}$ of the left regular representation l_a of a on (E_2, q_2)),

$$q_2(a) = q_2(ae_2) = M_2'\, q_2(a\frac{e_2}{M_2'^{\frac{1}{k}}}) \leqslant M_2' \sup_{c\in E_2, q_2(c)\leqslant 1} q_2(ac) = M_2'\|a\|_2'$$

because $q_2(e_2) \leqslant M_2'$ and

$$\frac{1}{M_2'}\|a\|_2 \leqslant \frac{1}{M_2'}q_2(a) \leqslant \|a\|_2' \leqslant \sup_{c\in E_2, q_2(c)\leqslant 1} q_2(a)q_2(e_2c) = q_2(a)\|e_2\|_2' \leqslant M_2'\|a\|_2$$

for each $a, b \in E_2$. Since

$$\|t\|_2' = \sup_{c\in E_2, q_2(c)\leqslant 1} q_2(tc) = \sup_{c\in E_2, q_2(c)\leqslant 1} \sup_{s\in B_1'} \|(st)c\|_2 \leqslant \sup_{c\in E_2, q_2(c)\leqslant 1} q_2(c) = 1$$

for each $t \in B_1'$ (because $B_1' = B_1$ and $B_1t \subset B_1$), then $\|\cdot\|_2'$ is a k-norm on E_2, which is equivalent to $\|\cdot\|_2$, and satisfies the condition $B_1' \subseteq B_2'$.

The norm $\|\cdot\|_3'$ we define similarly, that is, we put

$$\|a\|_3' = \sup_{c\in E_3, q_3(c)\leqslant 1} q_3(ac)$$

where

$$q_3(a) = \sup_{s\in B_2'} \|sa\|_3$$

for all $a \in E_3$. Now, similarly as above, we have $\|a\|_3 \leqslant M_3'\|a\|_2'$ for $M_3' = M_3M_2$, $\frac{1}{M_3'}\|a\|_3 \leqslant \|a\|_3' \leqslant M_3'\|a\|_3$ for each $a \in E_3$ and $\|a\|_3' \leqslant 1$ for each $a \in B_2'$. Hence, $B_2' \subseteq B_3'$. Continuing in the same way, for every fixed $n \geqslant 4$ we define

$$\|a\|_n' = \sup_{c\in E_n, q_n(c)\leqslant 1} q_n(ac)$$

where

$$q_n(a) = \sup_{s\in B_{n-1}'} \|sa\|_n$$

for all $a \in E_n$ and show that $B_{n-1}' \subseteq B_n'$.

Let now E_{n-1} and E_n be arbitrary k-normed algebras. Instead of these algebras, we consider direct products $E_{n-1} \times \mathbb{K}$ and $E_n \times \mathbb{K}$ which are k-normed algebras with respect to the algebraic operations (similarly as in case of the unitization) and norm $\|(a, \lambda)\|_k = \|a\|_k + |\lambda|$ for each $(a, \lambda) \in E_k \times \mathbb{K}$ (here k is $n-1$ or n). Then $E_{n-1} \times \mathbb{K}$ and $E_n \times \mathbb{K}$ have the same unit element $(\theta, 1)$, where θ is the zero element in E_{n-1} and E_n. Moreover, $E_{n-1} \times \mathbb{K}$ is a subalgebra of $E_n \times \mathbb{K}$. Hence, there are equivalent k-norms $\|(\cdot,\cdot)\|_{n-1}'$ and $\|(\cdot,\cdot)\|_n'$ such that $\|(a, \lambda)\|_n' \leqslant \|(a, \lambda)\|_{n-1}'$ if $\|(a, \lambda)\|_{n-1}' \leqslant 1$. Thus

$$\|a\|_n' = \|(a, 0)\|_n' \leqslant \|(a, 0)\|_{n-1}' = \|a\|_{n-1}'$$

for each $a \in B'_{n-1}$. Hence, $B'_{n-1} \subseteq B'_n$ for each fixed $n > 1$. Consequently, we can assume that

$$B_1 \subseteq B_2 \subseteq \cdots \subseteq B_n \subseteq \cdots$$

Since B_1 is bounded in B_2 (because $\|a\|_2 \leqslant M_2\|a\|_1$ for each $a \in E_1$) and B_2 is a neighborhood of zero in E_2, then there is a number $t_1 \geqslant 1$ such that $B_1 \subset t_1 B_2$. We put $B'_1 = B_1$ and

$$B'_n = \Gamma_k\Big(I\Big(B_{n-1}\bigcup \frac{1}{t_{n-1}}B_n\Big)\Big) = \Gamma_k\Big(\bigcup_{j\in\mathbb{N}}\Big(B_{n-1}\bigcup \frac{1}{t_{n-1}}B_n\Big)^j\Big)$$

for $n > 1$, where $I(U)$ is the idempotent hull (see [8, pp. 26 and 27]) of $U \subset E$. Then, B'_2 is an idempotent (by Lemma 3.1) and absolutely k-convex set. Because

$$B_1 \cup \frac{1}{t_1}B_2 \subset B_2 \subset B_2 \cup \frac{1}{t_2}B_3,$$

then $B'_2 \subset B'_3$ (it is clear that $I(U) \subset I(V)$ and $\Gamma_k(U) \subset \Gamma_k(V)$ if $U \subset V$). Since

$$\frac{1}{t_1}B_2 \subset \Big(B_1 \cup \frac{1}{t_1}B_2\Big) \subseteq I\Big(B_1 \cup \frac{1}{t_1}B_2\Big) \subseteq \Gamma_k\Big(I\Big(B_1 \cup \frac{1}{t_1}B_2\Big)\Big) = B'_2,$$

then, continuing in the same way, we have an increasing sequence $\{B'_n : n \geqslant 2\}$ of idempotent and absolutely k-convex sets B'_n such that

$$\frac{1}{t_{n-1}}B_n \subset B'_n. \tag{2}$$

Moreover, $B'_2 \subset t_1 B_2$. Indeed, for $x \in \frac{1}{t_1}\bigcup_{j\in\mathbb{N}}(B_1 \cup \frac{1}{t_1}B_2)^j$ we have $t_1 x \in (B_1 \cup \frac{1}{t_1}B_2)^{j_0}$ for some $j_0 \in \mathbb{N}$. Hence there is an element $y \in B_1 \cup \frac{1}{t_1}B_2$ such that $t_1 x = y^{j_0}$. If $y \in B_1$, then from $t_1 x \in B_1^{j_0} \subset B_1 \subset t_1 B_2$ follows that $x \in B_2$, otherwise $y \in \frac{1}{t_1}B_2$. Then, from $t_1 x \in \frac{1}{t_1^{j_0}}B_2^{j_0} \subset \frac{1}{t_1^{j_0}}B_2$ follows that $x \in \frac{1}{t_1^{j_0+1}}B_2 \subset B_2$ provided that B_2 is balanced. Arguing similarly, we have

$$B'_n \subset t_{n-1}B_n \tag{3}$$

where $t_n \geqslant 1$ for each $n \in \mathbb{N}$. Thus,

$$\frac{1}{t_{n-1}}B_n \subset B'_n \subset t_{n-1}B_n$$

for all $n \in \mathbb{N}$.

Now, we shall prove that

$$\mathcal{L}_\theta = \{\Gamma_k(\bigcup_{n\in\mathbb{N}} \varepsilon_n B'_n) : \varepsilon_n \in (0,1]\}$$

is a base of neighborhoods of zero in E which consists of idempotent absolutely k-convex sets. Clearly, every element of $\mathcal{L}_\theta$ is absolutely k-convex, to prove that every element $V = \Gamma_k(\bigcup_{n\in\mathbb{N}} \varepsilon_n B'_n)$ in $\mathcal{L}_\theta$ is idempotent, we consider

$x, y \in \bigcup_{n\in\mathbb{N}} \varepsilon_n B'_n$. Then, $x \in \varepsilon_n B'_n$ and $y \in \varepsilon'_m B'_m$ for some $m, n \in \mathbb{N}$. If $B'_n \subseteq B'_m$ (the case $B'_n \supset B'_m$ is similar), then

$$xy \in \varepsilon_n B'_m \varepsilon'_m B'_m \subseteq \varepsilon_n \varepsilon'_m B'_m B'_m \subseteq \varepsilon_n B'_m \subset \bigcup_{n\in\mathbb{N}} \varepsilon_n B'_n,$$

that is, $\bigcup_{n\in\mathbb{N}} \varepsilon_n B'_n$ is idempotent and hence V is idempotent by Lemma 3.1.

To show that $\mathcal{L}_\theta$ is a base of neighborhoods of zero for some topology τ' on E, we show that $\mathcal{L}_\theta$ satisfies the following conditions:

1) if $V \in \mathcal{L}_\theta$, then the zero element $\theta \in V$;
2) if $V_1, V_2 \in \mathcal{L}_\theta$, then there exists a set $V_3 \in \mathcal{L}_\theta$ such that $V_3 \subset V_1 \cap V_2$;
3) if $V \in \mathcal{L}_\theta$, then there exists a set $V_0 \in \mathcal{L}_\theta$ and for every $y \in V_0$ a set $W = y + V_0$ such that $W \subset V$.

Clearly 1) holds. To show that 2) holds, we put $V_1 = \Gamma_k(\bigcup_{n\in\mathbb{N}} \varepsilon_n B'_n)$, $V_2 = \Gamma_k(\bigcup_{n\in\mathbb{N}} \varepsilon'_n B'_n)$ and $\varepsilon''_n = \inf\{\varepsilon_n, \varepsilon'_n\}$ for every $n \in \mathbb{N}$. Since $\frac{\varepsilon''_n}{\varepsilon_n} \leqslant 1$ and $\frac{\varepsilon''_n}{\varepsilon'_n} \leqslant 1$, then

$$\varepsilon''_n B'_n \subseteq \varepsilon_n B'_n, \quad \varepsilon''_n B'_n \subseteq \varepsilon'_n B'_n$$

and hence

$$\varepsilon''_n B'_n \subset (\bigcup_{n\in\mathbb{N}} \varepsilon_n B'_n) \cap (\bigcup_{n\in\mathbb{N}} \varepsilon'_n B'_n) \subset V_1 \cap V_2$$

for every $n \in \mathbb{N}$. Thus, we can put

$$V_3 = \Gamma_k(\bigcup_{n\in\mathbb{N}} \varepsilon''_n B'_n).$$

Then,

$$V_3 \subset \Gamma_k(V_1 \cap V_2) \subset \Gamma_k(V_1) \cap \Gamma_k(V_2) = V_1 \cap V_2.$$

3) If $V \in \mathcal{L}_\theta$, then

$$V = \Gamma_k(\bigcup_{n\in\mathbb{N}} \varepsilon_n B'_n)$$

for some sequence (ε_n), where $\varepsilon_n \in (0, 1]$ for each $n \in \mathbb{N}$. Since V is k-convex, $2^{-\frac{1}{k}}V + 2^{-\frac{1}{k}}V \subset V$. Moreover, $2^{-\frac{1}{k}}V \in \mathcal{L}_\theta$ since $2^{-\frac{1}{k}}V = \Gamma_k(\bigcup_{n\in\mathbb{N}} 2^{-\frac{1}{k}}\varepsilon_n B'_n)$, where $2^{-\frac{1}{k}}\varepsilon_n \in (0, 1]$ for every $n \in \mathbb{N}$. Thus $V_0 = 2^{-\frac{1}{k}}V \in \mathcal{L}_\theta$ and $W = y + V_0 \subset V$ for every $y \in V_0$. Consequently, by Theorem 4.5 from [14], $\mathcal{L}_\theta$ is a base of neighborhoods of zero for a locally m-(k-convex) topology τ' on E.

Claim that $\tau = \tau'$. For it, let O be a neighborhood of zero in the topology τ'. Then, there exists a neighborhood U of zero such that

$$U = \Gamma_k\big(\bigcup_{n\in\mathbb{N}} \varepsilon_n B'_n\big)$$

for some sequence (ε_n), where $\varepsilon_n \in (0, 1]$ for each $n \in \mathbb{N}$, and $U \subseteq O$. Take $n_0 \in \mathbb{N}$ and let $f_{n_0} : E_{n_0} \to E$ be the canonical map (f_{n_0} is the inclusion). Since

$\frac{1}{t_{n_0-1}}B_{n_0} \subset B'_{n_0}$ by (2), then

$$f_{n_0}^{-1}(U) = \Gamma_k(\bigcup_{n\in\mathbb{N}} \varepsilon_n B'_n) \cap E_{n_0} \supset \varepsilon_{n_0} B'_{n_0} \supset \frac{\varepsilon_{n_0}}{t_{n_0-1}} B_{n_0},$$

where $\frac{\varepsilon_{n_0}}{t_{n_0-1}}B_{n_0}$ is a neighborhood of zero in E_{n_0}. Thus, $f_n^{-1}(U)$ is a neighborhood of zero in E_n for every $n \in \mathbb{N}$. Hence, by (1), U is a neighborhood of zero in E in the topology τ. Thus $\tau' \subseteq \tau$.

To prove that $\tau \subseteq \tau'$, let U be a neighborhood of zero in the topology τ. Then, there is in E an absolutely k-convex neighborhood V of zero such that $V \subset U$ and $f_n^{-1}(V) = V \cap E_n$ is a neighborhood of zero in E_n for every $n \in \mathbb{N}$. Since $\{\varepsilon_n B_n : \varepsilon_n > 0\}$ is a base of neighborhoods of zero in (E_n, τ_n) (see [12, p. 14]), then $\varepsilon_n B_n \subset E_n \cap V \subset V$ for some $\varepsilon_n < 1$. As it has been shown in (3), $B'_n \subset t_{n-1}B_n$ with $t_{n-1} \geqslant 1$. Therefore $\frac{\varepsilon_n}{t_{n-1}}B'_n \subset V$, where $\frac{\varepsilon_n}{t_{n-1}} \in (0,1]$ for every n. Hence, from

$$\bigcup_{n\in\mathbb{N}} \frac{\varepsilon_n}{t_{n-1}} B'_n \subset V$$

it follows

$$\Gamma_k\Big(\bigcup_{n\in\mathbb{N}} \frac{\varepsilon_n}{t_{n-1}} B'_n\Big) \subset \Gamma_k(V) = V \subset U.$$

Hence, $\tau \subseteq \tau'$. It means that $\tau = \tau'$. □

Corollary 3.3. *Locally k-convex inductive limit of a sequence of locally k-normed algebras with continuous inclusions is a locally m-(k-convex) algebra for every $k \in (0,1]$.*

4. Locally pseudoconvex inductive limit of locally m-pseudoconvex algebras

It is known that the inductive limit of locally m-convex algebras is not necessarily a locally m-convex algebra (see the example in [6]). It was shown in [7, Theorem, p. 150] that the locally convex inductive limit E of a sequence of commutative locally m-convex algebras is a locally m-convex algebra if the multiplication in E is jointly continuous. Next we prove an analogous result for the case of locally pseudoconvex inductive limit of a sequence of commutative locally m-pseudoconvex algebras.

Theorem 4.1. *Let E be a locally pseudoconvex inductive limit of a sequence of commutative locally m-pseudoconvex algebras E_n with continuous inclusions. If the multiplication is jointly continuous in E, then E is a commutative locally m-pseudoconvex algebra.*

Proof. Let U be a neighborhood of zero in E. Then, there is a neighborhood $V_1 \subset U$ of zero such that $\Gamma_k(V_1) = V_1$ for some $k \in (0,1]$. By the jointly continuity of multiplication in E, there exists a neighborhood O_1 of zero such that $O_1O_1 \subset V_1$. Now we put $V_2 = O_1 \cap V_1$. Then, by the jointly continuity of multiplication, there exists a neighborhood O_2 of zero such that $O_2O_2 \subset V_2$. Inductively we define $V_{v+1} = O_v \cap V_v$ for each $v \geqslant 1$. Since,

$$V_1 \supset V_2 \supset \ldots \supset V_v \supset \ldots,$$

then $V_v \subset U$ for every $v \in \mathbb{N}$.

Since the canonical map (the inclusion) $f_n : E_n \to E$ is continuous for every $n \in \mathbb{N}$, there exists for every $v \in \mathbb{N}$ an m-pseudoconvex neighborhood $V_{n,v}$ of zero in E_n such that $V_{n,v} \subset V_v$. Now, for every $n \in \mathbb{N}$, we put $V'_{n,1} = V_{n,1}$ and

$$V'_{n,v+1} = V'_{n,v} \cap V_{n,v+1} \tag{4}$$

for $v \geqslant 1$. Then,

$$V'_{n,v+1} \subset V'_{n,v} \text{ for all } n, v \in \mathbb{N} \tag{5}$$

and $(V'_{n,v})$ is a sequence of idempotent neighborhoods of zero in E_n (since $V_{n,v}$ is an idempotent neighborhood of zero in E_n) for all $n, v \in \mathbb{N}$.

Let $n_0 \in \mathbb{N}$ and $1 \leqslant p < n_0$ be fixed. We define a new sequence $(V''_{n,v})$ of idempotent neighborhoods of zero in E_n as follows: we put $V''_{p,1} = V_{p,1}$ and for $v \geqslant 1$ put

$$V''_{p,v+1} = V'_{n_0,v+1} \cap V''_{p,v} \tag{6}$$

and

$$V''_{n,v} = V'_{n,v} \text{ for } n \geqslant n_0 \text{ and } v \in \mathbb{N}.$$

So, by definition of $(V''_{n,v})$, (4), (5) and (6), we have that

$$V''_{n,v+1} \subseteq V''_{n,v} \text{ for all } v, n \in \mathbb{N} \tag{7}$$

and from

$$V''_{n_0,s}V''_{p,q} \subset V''_{n_0,s}V''_{p,s} \subset V'_{n_0,s}V'_{n_0,s} \subset V'_{n_0,s} \subset V_{n_0,s} \subset V_s \subset V_1$$

it follows that

$$V''_{n_0,s}V''_{p,q} \subset V''_{n_0,s} \subset V_1 \tag{8}$$

for every natural number p with $p \leqslant n_0$ and every natural numbers s and q with $s \leqslant q$.

For any numbers $v(1), \ldots, v(r) \in \mathbb{N}$ with $1 = v(0) < v(1) < v(2) < \ldots < v(r)$ and $n(1), \ldots, n(r+1) \in \mathbb{N}$ (arbitrary $r+1$ (not necessarily different and ordered) numbers) we show by induction on $r \in \mathbb{N}$ that

$$V''_{n(1),1}V''_{n(2),v(1)} \cdots V''_{n(r+1),v(r)} \subset V_1 \tag{9}$$

For $r = 1$, (9) holds by (8) (if $n(2) > n(1)$, we can rename these numbers).

Now, we suppose that (9) is true for $r - 1$ and prove that (9) is true for r too. Again, we can assume that $n(r) \geq n(r+1)$ (otherwise we can rename the numbers). Then, using also (8), we get

$$(V''_{n(1),1}V''_{n(2),v(1)} \cdots V''_{n(r-1),v(r-2)})V''_{n(r),v(r-1)}V''_{n(r+1),v(r)} \subset$$
$$(V''_{n(1),1}V''_{n(2),v(1)} \cdots V''_{n(r-1),v(r-2)})V''_{n(r),v(r-1)}$$

and by the induction hypothesis, we get the assertion.

Now, we put $W_n = V''_{n,n}$. Then, using (7)

$$W_{v(r)} = V''_{v(r),v(r)} \subset V''_{v(r),v(r)-1} \subset V''_{v(r),v(r-1)}$$

for each $r \in \mathbb{N}$. Therefore

$$W_{v(1)}W_{v(2)} \cdots W_{v(r)} \subset V''_{v(1),1}V''_{v(2),v(1)} \cdots V''_{v(r),v(r-1)} \subset V_1 \tag{10}$$

by (9) (which holds for any choice of $r+1$ natural numbers $n(1), \ldots, n(r)$ and $n(r+1)$).

Take $m(1), \ldots, m(s) \in \mathbb{N}$ (arbitrary fixed not necessarily different s natural numbers). We can find $r \leq s$ natural numbers $v(1), \ldots v(r)$ such that

$$1 < v(1) < v(2) < \ldots < v(r)$$

and the set

$$\{m(1), \ldots, m(s)\} = \{v(1), \ldots, v(r)\}.$$

By commutativity of E_n and idempotency of W_n, we have

$$W_{m(1)} \cdots W_{m(s)} = \prod_{i=1}^{r} W_{v(i)}^{|j\,:\,m(j)=v(i)|} \subset \prod_{i=1}^{r} W_{v(i)} \subset V_1$$

for every $r \in \mathbb{N}$, see also (10). Put

$$W := \bigcup_{s\in\mathbb{N}} \Big(\bigcup_{(m(1),\ldots,m(s))\in\mathbb{N}^s} W_{m(1)} \cdots W_{m(s)} \Big).$$

Then, W is an idempotent subset of V_1. Indeed, if $x, y \in W$, then

$$x \in \bigcup W_{m(1)} \cdots W_{m(s_0)},$$

where the union is taken over all $(m(1), \ldots, m(s_0)) \in \mathbb{N}^{s_0}$ and

$$y \in \bigcup W_{m(1)} \cdots W_{m(s_1)},$$

where the union is taken over all $(m(1), \ldots, m(s_1)) \in \mathbb{N}^{s_1}$ for some s_0 and s_1. Therefore,

$$x \in W_{m'(1)} \cdots W_{m'(s_0)} \text{ and } y \in W_{m''(1)} \cdots W_{m''(s_1)}$$

for some $(m'(1), \ldots, m'(s_0)) \in \mathbb{N}^{s_0}$ and $(m''(1), \ldots, m''(s_1)) \in \mathbb{N}^{s_1}$. Thus,

$$xy \in W_{m'(1)} \cdots W_{m'(s_0)}W_{m''(1)} \cdots W_{m''(s_1)} \subset$$

$$\bigcup W_{m(1)} \cdots W_{m(s_0+s_1)} \subset W,$$

where the union is taken over all $(m(1), \ldots, m(s_0+s_1)) \in \mathbb{N}^{s_0+s_1}$. By Lemma 3.1, the absolutely k-convex hull of any idempotent set is idempotent and k-convex. So,

$$W' := \Gamma_k(W) \subset \Gamma_k(V_1) = V_1 \subset U$$

is an m-(k-convex) subset of U. Since

$$W' \cap E_n = \Gamma_k(W) \cap E_n \supset W \cap E_n \supset W_n = V''_{n,n}$$

for each $n \in \mathbb{N}$ and $V''_{n,n}$ is an neighborhood of zero in E_n, then W' in E is an absolutely m-(k-convex) neighborhood of zero.

Thus, E is a commutative locally m-pseudoconvex algebra in the locally pseudoconvex inductive limit topology on E. □

A topological algebra is *locally idempotent* if it has a base of idempotent neighborhoods of zero (see [1, p. 196]). Hence, every locally m-pseudoconvex (in particular, locally m-convex) algebra is a locally idempotent algebra.

Theorem 4.2. *Let E be a topological inductive limit of a sequence of commutative locally idempotent algebras E_n with continuous inclusions. If the multiplication is jointly continuous in E, then E is a commutative locally idempotent algebra.*

Proof. The proof is similar that of Theorem 2. □

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