SOME HOMOLOGICAL AND COHOMOLOGICAL NOTIONS ON T-LAU PRODUCT OF BANACH ALGEBRAS

MEHDI NEMATI[1] AND HOSSEIN JAVANSHIRI[*2]

Communicated by Y. Zhang

ABSTRACT. Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras and let $T : \mathcal{B} \to \mathcal{A}$ be a continuous homomorphism. Recently, we introduced a product $\mathcal{M} := \mathcal{A} \times_T \mathcal{B}$, which is a strongly splitting Banach algebra extension of $\mathcal{B}$ by $\mathcal{A}$. In the present paper, we characterize biprojectivity, approximate biprojectivity and biflatness of $\mathcal{A} \times_T \mathcal{B}$ in terms of $\mathcal{A}$ and $\mathcal{B}$. We also study some notions of amenability such as approximate amenability and pseudo-amenability of $\mathcal{A} \times_T \mathcal{B}$.

1. INTRODUCTION AND PRELIMINARIES

Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras and let $T : \mathcal{B} \to \mathcal{A}$ be a continuous homomorphism with $\|T\| \leq 1$. The T-Lau product $\mathcal{A} \times_T \mathcal{B}$ is defined as the Cartesian product $\mathcal{A} \times \mathcal{B}$ with the algebra product

$$(a, b)(a', b') = (aa' + T(b)a' + aT(b'), bb'),$$

and the norm $\|(a, b)\| = \|a\| + \|b\|$. This type of product was first introduced by Bhatt and Dabhi [5] for the case where $\mathcal{A}$ is commutative and was extended by the authors for the general case [12].

Here it should be noted that, our motivation of the study presented in this paper and previous paper [12] comes from some resent developments of the splitting of Banach algebra extensions which have been studied by many authors such as Bade et al. [1, 2, 3, 4], Feldman [7], Helemskii [11], Johnson [13, 14], Lau [16] and

Monfared [17, 18]. Moreover, splitting of Banach algebra extensions has been a major tool in the study of Banach algebras. For example, module extensions as generalizations of Banach algebra extensions were introduced and first studied by Gourdeau [10] were used to show that amenability of $\mathcal{A}^{**}$, the second dual space of $\mathcal{A}$, implies amenability of $\mathcal{A}$ and Zhang [22] used module extensions to answer an open problem regarding weak amenability, raised by Dales, Ghahramani and Grønbæk [6]. On the other hand, $\mathcal{A} \times_T \mathcal{B}$ is a strongly splitting Banach algebra extension of $\mathcal{B}$ by $\mathcal{A}$ that exhibits many properties that are not shared by arbitrary strongly splitting extensions. For example, commutativity is not preserved by a general strongly splitting extension whereas $\mathcal{A} \times_T \mathcal{B}$ is commutative if and only if the commutativity assumption are satisfied by $\mathcal{A}$ and $\mathcal{B}$. In addition to all these, the T-Lau product would provide not only new examples of Banach algebras by themselves, but also it provide a wealth of (counter) examples in different branches of functional analysis. These facts suggest that T-Lau products are worth to study.

In [12], apart from the study of many basic properties of $\mathcal{A} \times_T \mathcal{B}$ such as characterization of bounded approximate identity, ideal structure, n-weak amenability and Arens regularity, useful characterizations of derivations from $\mathcal{A} \times_T \mathcal{B}$ into $(\mathcal{A} \times_T \mathcal{B})^{(n)}$, the nth dual space of $\mathcal{A} \times_T \mathcal{B}$, were given. In continuation of our study on the T-Lau product of Banach algebras, in the present paper, we characterize some homological notions of $\mathcal{A} \times_T \mathcal{B}$ in terms of $\mathcal{A}$ and $\mathcal{B}$. Finally, some notions of amenability such as pseudo-amenability and approximate amenability, are investigated in detail for the T-Lau product of Banach algebras.

2. Biprojectivity and biflatness of $\mathcal{A} \times_T \mathcal{B}$

In what follows, $\mathcal{A}$ and $\mathcal{B}$ are Banach algebras and $T : \mathcal{B} \to \mathcal{A}$ is a homomorphism with $\|T\| \leq 1$. The notation $\mathcal{M}$ is used to denote the T-Lau product of the Banach algebras $\mathcal{A}$ and $\mathcal{B}$, $r_{\mathcal{A}}$ denotes the mapping defined by $r_{\mathcal{A}}((a, b)) = a + T(b)$ from $\mathcal{M}$ into $\mathcal{A}$, and $s_{\mathcal{B}} : \mathcal{B} \to \mathcal{M}$ denotes the map defined by the equality $s_{\mathcal{B}}(b) = (-T(b), b)$ for all $b \in \mathcal{B}$. Moreover, $\iota_{\mathcal{A}}$ denotes the usual injection map from $\mathcal{A}$ into $\mathcal{M}$ and $q_{\mathcal{B}}$ denotes the usual projection map from $\mathcal{M}$ into $\mathcal{B}$. Let also, for each Banach algebra $\mathcal{A}$, $\pi_{\mathcal{A}}$ denote the product morphism from the projective tensor product $\mathcal{A} \widehat{\otimes} \mathcal{A}$ into $\mathcal{A}$. A routine computations show that $\mathcal{A} \widehat{\otimes} \mathcal{A}$ becomes a Banach $\mathcal{A}$-bimodule with the following actions:

$$a \cdot (b \otimes c) = ab \otimes c, \qquad (b \otimes c) \cdot a = b \otimes ca$$

for all $a, b, c \in \mathcal{A}$. With the above actions $\pi_{\mathcal{A}}$ becomes an $\mathcal{A}$-bimodule homomorphism. In what follows the notation $B(\mathcal{A}, \mathcal{A} \widehat{\otimes} \mathcal{A})$ is used into denote the space of all bounded linear operators from $\mathcal{A}$ into $\mathcal{A} \widehat{\otimes} \mathcal{A}$.

Recall that a Banach algebra $\mathcal{A}$ is said to be *biprojective* if there is a bounded $\mathcal{A}$-bimodule homomorphism $\rho_{\mathcal{A}} : \mathcal{A} \to \mathcal{A} \widehat{\otimes} \mathcal{A}$, such that $\pi_{\mathcal{A}} \circ \rho_{\mathcal{A}} = id_{\mathcal{A}}$. Also, $\mathcal{A}$ is called *biflat* if there is a bounded $\mathcal{A}$-bimodule homomorphism $\theta_{\mathcal{A}} : (\mathcal{A} \widehat{\otimes} \mathcal{A})^* \to \mathcal{A}^*$ such that $\theta_{\mathcal{A}} \circ (\pi_{\mathcal{A}})^* = id_{\mathcal{A}^*}$. We also recall from [21] that $\mathcal{A}$ is *approximately biprojective* if there is a net $(\rho_{\alpha}^{\mathcal{A}})$ of bounded $\mathcal{A}$-bimodule homomorphisms from $\mathcal{A}$ into $\mathcal{A} \widehat{\otimes} \mathcal{A}$ such that $\pi_{\mathcal{A}} \circ \rho_{\alpha}^{\mathcal{A}}(a) \to a$ for all $a \in \mathcal{A}$. Following Pirkovskii [19]

for two Banach $\mathcal{A}$-bimodules X and Y, a net (ρ_α) of bounded linear maps from X into Y, satisfying

$$\rho_\alpha(a \cdot x) - a \cdot \rho_\alpha(x) \to 0, \quad \rho_\alpha(x \cdot a) - \rho_\alpha(x) \cdot a \to 0$$

for all $a \in \mathcal{A}$, will be called an *approximate $\mathcal{A}$-bimodule morphism* from X into Y. Recently, a new notion of approximate biprojectivity introduced and studied by Pourmahmoud-Aghababa [20], which is based on approximate $\mathcal{A}$-bimodule morphisms. Precisely, in the sense of [20], a Banach algebra $\mathcal{A}$ is called approximately biprojective if there exists an approximate $\mathcal{A}$-bimodule morphism $(\rho_\alpha^\mathcal{A})$ from $\mathcal{A}$ into $\mathcal{A} \widehat{\otimes} \mathcal{A}$ such that $\pi_\mathcal{A} \circ \rho_\alpha^\mathcal{A}(a) \longrightarrow a$ for all $a \in \mathcal{A}$.

Lemma 2.1. *With the above notations, the following statements hold.*
(i) $r_\mathcal{A} \circ \pi_\mathcal{M} = \pi_\mathcal{A} \circ (r_\mathcal{A} \otimes r_\mathcal{A})$,
(ii) $\iota_\mathcal{A} \circ \pi_\mathcal{A} = \pi_\mathcal{M} \circ (\iota_\mathcal{A} \otimes \iota_\mathcal{A})$,
(iii) $s_\mathcal{B} \circ \pi_\mathcal{B} = \pi_\mathcal{M} \circ (s_\mathcal{B} \otimes s_\mathcal{B})$,
(iv) $q_\mathcal{B} \circ \pi_\mathcal{M} = \pi_\mathcal{B} \circ (q_\mathcal{B} \otimes q_\mathcal{B})$.
Moreover, for all $a, a', a'' \in \mathcal{A}$ and $b, b', b'' \in \mathcal{B}$ we have
(v) $(a, b) \cdot (s_\mathcal{B} \otimes s_\mathcal{B})(b' \otimes b'') = (s_\mathcal{B} \otimes s_\mathcal{B})(bb' \otimes b'')$,
(vi) $(s_\mathcal{B} \otimes s_\mathcal{B})(b' \otimes b'') \cdot (a, b) = (s_\mathcal{B} \otimes s_\mathcal{B})(b' \otimes b''b)$,
(vii) $(a, b) \cdot (\iota_\mathcal{A} \otimes \iota_\mathcal{A})(a' \otimes a'') = (\iota_\mathcal{A} \otimes \iota_\mathcal{A})([a + T(b)]a' \otimes a'')$,
(viii) $(\iota_\mathcal{A} \otimes \iota_\mathcal{A})(a' \otimes a'') \cdot (a, b) = (\iota_\mathcal{A} \otimes \iota_\mathcal{A})(a' \otimes a''[a + T(b)])$.

Proof. We prove the assertions (i), (v) and (vii). The proof of the other assertions are similar.

(i). Let (a, b) and (a', b') are arbitrary elements of $\mathcal{M}$. On the one hand, observe that

$$
\begin{aligned}
r_\mathcal{A} \circ \pi_\mathcal{M}\big((a, b) \otimes (a', b')\big) &= r_\mathcal{A}\big(aa' + T(b)a' + aT(b'), bb'\big) \\
&= aa' + T(b)a' + aT(b') + T(bb'),
\end{aligned}
$$

and, on the other hand

$$
\begin{aligned}
\pi_\mathcal{A} \circ (r_\mathcal{A} \otimes r_\mathcal{A})\big((a, b) \otimes (a', b')\big) &= \pi_\mathcal{A}\big([a + T(b)] \otimes [a' + T(b')]\big) \\
&= aa' + T(b)a' + aT(b') + T(bb').
\end{aligned}
$$

This completes the proof of (i).

(v). This is because of

$$(a, b) \cdot [(-T(b'), b') \otimes (-T(b''), b'')] = (-T(bb'), bb') \otimes (-T(b''), b''),$$

for all $a \in \mathcal{A}$ and $b, b', b'' \in \mathcal{B}$.

(vi). This is because of

$$
\begin{aligned}
(a, b) \cdot [(a', 0) \otimes (a'', 0)] &= (a, b)(a', 0) \otimes (a'', 0) \\
&= (a + T(b)a', 0) \otimes (a'', 0),
\end{aligned}
$$

for all $a, a', a'' \in \mathcal{A}$ and $b \in \mathcal{B}$. $\qquad\square$

In the following result we give the proof for the approximate biprojectivity in the sense of [20]. The proof for the other cases are similar.

Theorem 2.2. *Let $\mathcal{M}$ be biprojective/approximately biprojective. Then $\mathcal{A}$ and $\mathcal{B}$ are biprojective/approximately biprojective.*

Proof. Denote the approximate right inverse maps given in the hypothesis by $(\rho_\alpha^{\mathcal{M}})$. Now, for each α set

$$\rho_\alpha^{\mathcal{B}} := (q_{\mathcal{B}} \otimes q_{\mathcal{B}}) \circ \rho_\alpha^{\mathcal{M}} \circ \iota_{\mathcal{B}} \qquad \text{and} \quad \rho_\alpha^{\mathcal{A}} := (r_{\mathcal{A}} \otimes r_{\mathcal{A}}) \circ \rho_\alpha^{\mathcal{M}} \circ \iota_{\mathcal{A}}.$$

Using Lemma 2.1 (i) and (iv) we can see that

$$\pi_{\mathcal{A}} \circ \rho_\alpha^{\mathcal{A}}(a) \longrightarrow a \tag{2.1}$$

and

$$\pi_{\mathcal{B}} \circ \rho_\alpha^{\mathcal{B}}(b) \longrightarrow b. \tag{2.2}$$

On the other hand, a direct computation shows that

$$\lim_\alpha [\rho_\alpha^{\mathcal{A}}(aa') - a \cdot \rho_\alpha^{\mathcal{A}}(a')] = (r_{\mathcal{A}} \otimes r_{\mathcal{A}}) \left(\lim_\alpha \rho_\alpha^{\mathcal{M}}((a,0)(a',0)) - (a,0) \cdot \rho_\alpha^{\mathcal{A}}((a',0)) \right) \tag{2.3}$$

$$\lim_\alpha [\rho_\alpha^{\mathcal{A}}(aa') - \rho_\alpha^{\mathcal{A}}(a) \cdot a'] = (r_{\mathcal{A}} \otimes r_{\mathcal{A}}) \left(\lim_\alpha \rho_\alpha^{\mathcal{M}}((a,0)(a',0)) - \rho_\alpha^{\mathcal{A}}((a,0)) \cdot (a',0) \right), \tag{2.4}$$

$$\lim_\alpha [\rho_\alpha^{\mathcal{B}}(bb') - b \cdot \rho_\alpha^{\mathcal{B}}(b')] = (q_{\mathcal{B}} \otimes q_{\mathcal{B}}) \left(\lim_\alpha \rho_\alpha^{\mathcal{M}}((0,b)(0,b')) - (0,b) \cdot \rho_\alpha^{\mathcal{B}}((0,b')) \right), \tag{2.5}$$

$$\lim_\alpha [\rho_\alpha^{\mathcal{B}}(bb') - \rho_\alpha^{\mathcal{B}}(b) \cdot b'] = (q_{\mathcal{B}} \otimes q_{\mathcal{B}}) \left(\lim_\alpha \rho_\alpha^{\mathcal{M}}((0,b)(0,b')) - \rho_\alpha^{\mathcal{B}}((0,b)) \cdot (0,b') \right). \tag{2.6}$$

Now, invoke (2.1), (2.3) and (2.4) to conclude that $\mathcal{A}$ is approximately biprojective. Also (2.2), (2.5) and (2.6) imply that $\mathcal{B}$ is approximately biprojective. $\square$

Now we give sufficient conditions for the biprojectivity/approximate biprojectivity of T-Lau product of Banach algebras.

Theorem 2.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be biprojective/approximately biprojective Banach algebras. Then $\mathcal{M}$ is biprojective/approximately biprojective.*

Proof. Let $(\rho_\alpha^{\mathcal{A}})_{\alpha \in I} \subseteq B(\mathcal{A}, \mathcal{A} \widehat{\otimes} \mathcal{A})$ and $(\rho_\beta^{\mathcal{B}})_{\beta \in J} \subseteq B(\mathcal{B}, \mathcal{B} \widehat{\otimes} \mathcal{B})$ respectively be approximate $\mathcal{A}$-bimodule and approximate $\mathcal{B}$-bimodule morphisms such that

$$\pi_{\mathcal{A}} \circ \rho_\alpha^{\mathcal{A}}(a) \longrightarrow a, \quad \pi_{\mathcal{B}} \circ \rho_\beta^{\mathcal{B}}(b) \longrightarrow b$$

for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$. Now, consider the directed set

$$\Gamma := \{\gamma = (\alpha, \beta) : \alpha \in I, \beta \in J\},$$

with the order

$$(\alpha, \beta) \preceq (\alpha', \beta') \iff \alpha \le \alpha', \beta \le \beta'.$$

Then the net $(\rho_\gamma^{\mathcal{M}}) \subseteq B(\mathcal{M}, \mathcal{M} \widehat{\otimes} \mathcal{M})$ defined by

$$\rho_\gamma^{\mathcal{M}}((a,b)) = \left((\iota_{\mathcal{A}} \otimes \iota_{\mathcal{A}}) \circ \rho_\alpha^{\mathcal{A}} \circ r_{\mathcal{A}} + (s_{\mathcal{B}} \otimes s_{\mathcal{B}}) \circ \rho_\beta^{\mathcal{B}} \circ q_{\mathcal{B}} \right)((a,b)), \qquad ((a,b) \in \mathcal{M})$$

is as required for the approximate biprojectivity of $\mathcal{M}$. Indeed, for each $(a, b) \in \mathcal{M}$ we have

$$
\begin{aligned}
& \pi_{\mathcal{M}} \circ \rho_\gamma^{\mathcal{M}}((a, b)) \\
&= \left(\pi_{\mathcal{M}} \circ (\iota_{\mathcal{A}} \otimes \iota_{\mathcal{A}}) \circ \rho_\alpha^{\mathcal{A}} \circ r_{\mathcal{A}} + \pi_{\mathcal{M}} \circ (s_{\mathcal{B}} \otimes s_{\mathcal{B}}) \circ \rho_\beta^{\mathcal{B}} \circ q_{\mathcal{B}}\right)((a, b)) \\
&= \left(\iota_{\mathcal{A}} \circ \pi_{\mathcal{A}} \circ \rho_\alpha^{\mathcal{A}} \circ r_{\mathcal{A}} + s_{\mathcal{B}} \circ \pi_{\mathcal{B}} \circ \rho_\beta^{\mathcal{B}} \circ q_{\mathcal{B}}\right)((a, b)) \\
&\longrightarrow \left(\iota_{\mathcal{A}} \circ r_{\mathcal{A}} + s_{\mathcal{B}} \circ q_{\mathcal{B}}\right)((a, b)) \\
&= (a + T(b), 0) + (-T(b), b) = (a, b),
\end{aligned}
$$

where the second equality follows from the Lemma 2.1 (ii) and (iii). Thus, for each $(a, b) \in \mathcal{M}$ we have

$$
\pi_{\mathcal{M}} \circ \rho_\gamma^{\mathcal{M}}((a, b)) \longrightarrow (a, b).
$$

Moreover, using Lemma 2.1 (v) and (vii) we have

$$
\rho_\gamma^{\mathcal{M}}((a, b)(a', b')) - (a, b) \cdot \rho_{(\alpha, \beta)}^{\mathcal{M}}((a', b')) \longrightarrow 0.
$$

By a same argument as above we have

$$
\rho_\gamma^{\mathcal{M}}((a, b)(a', b')) - \rho_\gamma^{\mathcal{M}}((a, b)) \cdot (a', b') \longrightarrow 0,
$$

and this completes the proof. $\qquad\square$

Remark 2.4. All of our results in Theorems 2.2 and 2.3 can be proved, by slight modifications of the arguments, with "approximate biprojectivity in the sense of [20]" replaced by "approximate biprojectivity in the sense of [21]".

In the following our aim is to study the biflatness of T-Lau product of Banach algebras. In fact, we will show that $\mathcal{A}$ and $\mathcal{B}$ are biflat if and only if $\mathcal{M}$ is biflat.

Theorem 2.5. *Let $\mathcal{M}$ be biflat. Then $\mathcal{A}$ and $\mathcal{B}$ are biflat.*

Proof. Let $\theta_{\mathcal{M}}$ be a bounded left inverse for $(\pi_{\mathcal{M}})^*$ which is a $\mathcal{M}$-bimodule homomorphism. Define $\theta_{\mathcal{B}} : (\mathcal{B} \widehat{\otimes} \mathcal{B})^* \to \mathcal{B}^*$ and $\theta_{\mathcal{A}} : (\mathcal{A} \widehat{\otimes} \mathcal{A})^* \to \mathcal{A}^*$ by $\theta_{\mathcal{B}} = (\iota_B)^* \circ \theta_M \circ (q_{\mathcal{B}} \otimes q_{\mathcal{B}})^*$ and $\theta_{\mathcal{A}} = (\iota_{\mathcal{A}})^* \circ \theta_M \circ (r_{\mathcal{A}} \otimes r_{\mathcal{A}})^*$, respectively. Obviously $\theta_{\mathcal{B}}$ and $\theta_{\mathcal{A}}$ are bounded $\mathcal{B}$-bimodule and $\mathcal{A}$-bimodule homomorphisms, respectively. Moreover,

$$
\begin{aligned}
\theta_{\mathcal{A}} \circ (\pi_{\mathcal{A}})^* &= \left((\iota_{\mathcal{A}})^* \circ \theta_M \circ (r_{\mathcal{A}} \otimes r_{\mathcal{A}})^*\right) \circ (\pi_{\mathcal{A}})^* \\
&= (\iota_{\mathcal{A}})^* \circ \theta_M \circ \left((r_{\mathcal{A}} \otimes r_{\mathcal{A}})^* \circ (\pi_{\mathcal{A}})^*\right) \\
&= (\iota_{\mathcal{A}})^* \circ \theta_M \circ \left((\pi_{\mathcal{M}})^* \circ (r_{\mathcal{A}})^*\right) \\
&= (\iota_{\mathcal{A}})^* \circ \left(\theta_M \circ (\pi_{\mathcal{M}})^*\right) \circ (r_{\mathcal{A}})^* \\
&= (\iota_{\mathcal{A}})^* \circ \left(\theta_M \circ (\pi_{\mathcal{M}})^*\right) \circ (r_{\mathcal{A}})^* \\
&= (\iota_{\mathcal{A}})^* \circ id_{\mathcal{M}^*} \circ (r_{\mathcal{A}})^* \\
&= (r_{\mathcal{A}} \circ \iota_{\mathcal{A}})^* \\
&= id_{\mathcal{A}^*},
\end{aligned}
$$

where the third equality follows from Lemma 2.1. Using the same proof as above, we can show that $\theta_{\mathcal{B}} \circ (\pi_{\mathcal{B}})^* = id_{\mathcal{B}^*}$, as required. $\qquad\square$

Now we consider the converse of Theorem 2.5. Before proceeding further, let us mention that $(\mathcal{A} \times_T \mathcal{B})^*$, the first dual space of $\mathcal{A} \times_T \mathcal{B}$, can be identified with $\mathcal{A}^* \times \mathcal{B}^*$ with the following definition

$$\langle (f, g), (a, b) \rangle = \langle f, a \rangle + \langle g, b \rangle,$$

for all $a \in \mathcal{A}$, $b \in \mathcal{B}$, $f \in \mathcal{A}^*$ and $g \in \mathcal{B}^*$. Moreover, we can find that the $(A \times_T B)$-bimodule actions on $(A \times_T B)^*$ are formulated as follows

$$(a, b) \cdot (f, g) = \big([a + T(b)] \cdot f, T^*(a \cdot f) + b \cdot g\big)$$

and

$$(f, g) \cdot (a, b) = \big(f \cdot [a + T(b)], T^*(f \cdot a) + g \cdot b\big)$$

for all $(a, b) \in A \times_T B$ and $(f, g) \in A^* \times B^*$, where T^* is the adjoint of the operator T.

Theorem 2.6. *Let $\mathcal{A}$ and $\mathcal{B}$ be biflat. Then $\mathcal{M}$ is biflat.*

Proof. Let $\theta_{\mathcal{A}} : (\mathcal{A} \widehat{\otimes} \mathcal{A})^* \to \mathcal{A}^*$ be a bounded $\mathcal{A}$-bimodule homomorphism and let $\theta_{\mathcal{B}} : (\mathcal{B} \widehat{\otimes} \mathcal{B})^* \to \mathcal{B}^*$ be a bounded $\mathcal{B}$-bimodule homomorphism such that $\theta_{\mathcal{A}} \circ (\pi_{\mathcal{A}})^* = id_{\mathcal{A}^*}$ and $\theta_{\mathcal{B}} \circ (\pi_{\mathcal{B}})^* = id_{\mathcal{B}^*}$. Define the map $\theta_{\mathcal{M}} : (\mathcal{M} \widehat{\otimes} \mathcal{M})^* \to \mathcal{M}^*$ by

$$\begin{aligned} \theta_{\mathcal{M}}(\Phi) &= \big(\theta_{\mathcal{A}} \circ (\iota_{\mathcal{A}} \otimes \iota_{\mathcal{A}})^*(\Phi), \theta_{\mathcal{B}} \circ (s_{\mathcal{B}} \otimes s_{\mathcal{B}})^*(\Phi) \\ &\quad + T^*(\theta_{\mathcal{A}} \circ (\iota_{\mathcal{A}} \otimes \iota_{\mathcal{A}})^*(\Phi))\big) \end{aligned}$$

for all $\Phi \in (\mathcal{M} \widehat{\otimes} \mathcal{M})^*$. Then for each $(f, g) \in \mathcal{A}^* \times \mathcal{B}^*$ we have

$$\begin{aligned} &\langle \theta_{\mathcal{M}} \circ (\pi_{\mathcal{M}})^*, (f, g) \rangle \\ &= \big(\theta_{\mathcal{A}} \circ (\iota_{\mathcal{A}} \otimes \iota_{\mathcal{A}})^* \circ (\pi_{\mathcal{M}})^*((f, g)), \theta_{\mathcal{B}} \circ (s_{\mathcal{B}} \otimes s_{\mathcal{B}})^* \circ (\pi_{\mathcal{M}})^*((f, g)) \\ &\quad + T^*(\theta_{\mathcal{A}} \circ (\iota_{\mathcal{A}} \otimes \iota_{\mathcal{A}})^* \circ (\pi_{\mathcal{M}})^*((f, g)))\big) \\ &= \big(\theta_{\mathcal{A}} \circ (\pi_{\mathcal{A}})^* \circ (\iota_{\mathcal{A}})^*((f, g)), \theta_{\mathcal{B}} \circ (\pi_{\mathcal{B}})^* \circ (s_{\mathcal{B}})^*((f, g)) \\ &\quad + T^*(\theta_{\mathcal{A}} \circ (\pi_{\mathcal{A}})^* \circ (\iota_{\mathcal{A}})^*((f, g)))\big) \\ &= \big(id_{\mathcal{A}^*} \circ (\iota_{\mathcal{A}})^*(\Phi), id_{\mathcal{B}^*} \circ (s_{\mathcal{B}})^*((f, g)) \\ &\quad + T^*(id_{\mathcal{A}^*} \circ (\iota_{\mathcal{A}})^*((f, g))))\big) \\ &= \big(f, (-T^*(f) + g) + T^*(f)\big) \\ &= (f, g), \end{aligned}$$

where the second equality follows from Lemma 2.1. Thus $\theta_{\mathcal{M}} \circ (\pi_{\mathcal{M}})^* = id_{\mathcal{M}^*}$. Now, we show that $\theta_{\mathcal{M}}$ is a $\mathcal{M}$-bimodule homomorphism. To see this, first note that it is easy to check that the following equalities hold.

$$\begin{aligned} (\iota_{\mathcal{A}} \otimes \iota_{\mathcal{A}})^*(\Phi \cdot (a, b)) &= (\iota_{\mathcal{A}} \otimes \iota_{\mathcal{A}})^*(\Phi) \cdot [a + T(b)], \\ (\iota_{\mathcal{A}} \otimes \iota_{\mathcal{A}})^*((a, b) \cdot \Phi) &= [a + T(b)] \cdot (\iota_{\mathcal{A}} \otimes \iota_{\mathcal{A}})^*(\Phi), \\ (s_{\mathcal{B}} \otimes s_{\mathcal{B}})^*(\Phi \cdot (a, b)) &= (s_{\mathcal{B}} \otimes s_{\mathcal{B}})^*(\Phi) \cdot b, \\ (s_{\mathcal{B}} \otimes s_{\mathcal{B}})^*((a, b) \cdot \Phi) &= b \cdot (s_{\mathcal{B}} \otimes s_{\mathcal{B}})^*(\Phi). \end{aligned}$$

for all $\Phi \in (\mathcal{M} \widehat{\otimes} \mathcal{M})^*$ and $(a,b) \in \mathcal{M}$. Thus, if we set $f := \theta_{\mathcal{A}} \circ (\iota_{\mathcal{A}} \otimes \iota_{\mathcal{A}})^*(\Phi)$ and $g := \theta_{\mathcal{B}} \circ (s_{\mathcal{B}} \otimes s_{\mathcal{B}})^*(\Phi)$, then we have

$$
\begin{aligned}
\theta_{\mathcal{M}}(\Phi \cdot (a,b)) &= \big(\theta_{\mathcal{A}} \circ (\iota_{\mathcal{A}} \otimes \iota_{\mathcal{A}})^*(\Phi \cdot (a,b)), \theta_{\mathcal{B}} \circ (s_{\mathcal{B}} \otimes s_{\mathcal{B}})^*(\Phi \cdot (a,b)) \\
&\quad + \quad T^*(\theta_{\mathcal{A}} \circ (\iota_{\mathcal{A}} \otimes \iota_{\mathcal{A}})^*(\Phi \cdot (a,b)))\big) \\
&= \big(f \cdot [a + T(b)], g \cdot b + T^*(f \cdot [a + T(b)])\big) \\
&= \big(f \cdot [a + T(b)], [g + T^*(f)] \cdot b + T^*(f \cdot a)\big) \\
&= \big(f, g + T^*(f)\big) \cdot (a,b) \\
&= \theta_{\mathcal{M}}(\Phi) \cdot (a,b).
\end{aligned}
$$

Using the same proof as above, we can show that $\theta_{\mathcal{M}}((a,b) \cdot \Phi) = (a,b) \cdot \theta_{\mathcal{M}}(\Phi)$. Thus $\mathcal{M}$ is biflat. $\qquad\square$

3. SOME NOTIONS OF AMENABILITY ON $\mathcal{A} \times_T \mathcal{B}$

Let $\mathcal{A}$ be a Banach algebra and let X be a Banach $\mathcal{A}$-bimodule. A *derivation* from $\mathcal{A}$ into X is a linear map $D : \mathcal{A} \to X$ such that

$$D(ab) = a \cdot D(b) + D(a) \cdot b,$$

for all $a, b \in \mathcal{A}$. For example, the map $\mathrm{ad}_x : \mathcal{A} \to X$ defined by $\mathrm{ad}_x(a) = a \cdot x - x \cdot a$ is a derivation for all $a \in \mathcal{A}$ and $x \in X$. Such derivations are called *inner derivations*. Here, let us mention that X^*, the first dual space of X, becomes a Banach $\mathcal{A}$-bimodule in the natural way. The algebra $\mathcal{A}$ is called *amenable* if every continuous derivation $D : \mathcal{A} \to X^*$ is inner for each Banach $\mathcal{A}$-bimodule X, and is called *contractible* if every continuous derivation $D : \mathcal{A} \to X$ is inner for each Banach $\mathcal{A}$-bimodule X. A derivation $D : \mathcal{A} \to X$ is called *approximately inner* if there exists a net $(x_\alpha) \subseteq X$ such that $D(a) = \lim_\alpha \mathrm{ad}_{x_\alpha}(a)$ for all $a \in \mathcal{A}$. A Banach algebra $\mathcal{A}$ is *approximately amenable* if every continuous derivation $D : \mathcal{A} \to X^*$ is approximately inner. A Banach algebra $\mathcal{A}$ is *pseudo-amenable* if there is a net $(u_\alpha) \subseteq \mathcal{A} \widehat{\otimes} \mathcal{A}$, called an *approximate diagonal*, such that

$$a \cdot u_\alpha - u_\alpha \cdot a \longrightarrow 0, \quad \pi_{\mathcal{A}}(u_\alpha)a \longrightarrow a$$

for all $a \in \mathcal{A}$, and is *pseudo-contractible* if it has a central approximate diagonal, i.e. an approximate diagonal (u_α) satisfying $a \cdot u_\alpha = u_\alpha \cdot a$ for all $a \in \mathcal{A}$; see [9] for more details.

Remark 3.1. Let $\mathcal{A}$ be a Banach algebra with an idempotent element e and let $\theta \in \Delta(\mathcal{B})$, the character space of the Banach algebra $\mathcal{B}$. Then it is easy to check that the map $T_\theta : \mathcal{B} \to \mathcal{A}$ defined by $T_\theta(b) = \theta(b)e$ is a continuous homomorphism. In this case if e is an identity for $\mathcal{A}$, then $\mathcal{A} \times_{T_\theta} \mathcal{B}$ coincides with the θ-Lau product $\mathcal{A} \times_\theta \mathcal{B}$; see [17] for more details. In [15] the authors investigate biflatness and biprojectivity of $\mathcal{A} \times_\theta \mathcal{B}$ in the case where $\mathcal{A}$ is unital. Thus, in the previous section we generalized these results to that of T-Lau product of Banach algebras.

As a result of Theorems 2.2, 2.3 and Remark 2.4 we have the following corollary for the approximate biprojectivity of $\mathcal{A} \times_\theta \mathcal{B}$ and its relation with $\mathcal{A}$ and $\mathcal{B}$ in the sense of [21].

Corollary 3.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be two Banach algebras with $\theta \in \Delta(\mathcal{B})$. Then the following statements are equivalent.*

(i) $\mathcal{A}$ *is unital and* $\mathcal{A} \times_\theta \mathcal{B}$ *is approximately biprojective.*

(ii) $\mathcal{A}$ *is contractible and* $\mathcal{B}$ *is approximately biprojective.*

Proof. (i)$\Rightarrow$(ii). Suppose that e is the identity element of $\mathcal{A}$. Then $\mathcal{A} \times_{T_\theta} \mathcal{B}$ coincides with $\mathcal{A} \times_\theta \mathcal{B}$ as shown in above remark. Thus, approximate biprojectivity of $\mathcal{A} \times_\theta \mathcal{B}$ implies that both $\mathcal{A}$ and $\mathcal{B}$ are approximately biprojective by Theorem 2.2 and Remark 2.4. Since $\mathcal{A}$ is unital it follows that it is contractible by [9, Theorem 2.4 and Proposition 3.8]. The implication (ii)$\Rightarrow$(i) follows from Theorem 2.3 and Remark 2.4 with the fact that any contractible Banach algebra is unital and biprojective. $\qquad\square$

Proposition 3.3. $\mathcal{M}$ *is pseudo-amenable/pseudo-contractible if and only if* $\mathcal{A}$ *and* $\mathcal{B}$ *are pseudo-amenable/pseudo-contractible.*

Proof. We give the proof for the pseudo-amenable case. The proof for the other case is similar. Let $\mathcal{M}$ be pseudo-amenable. Then pseudo-amenability of $\mathcal{B}$ follows from [9, Proposition 2.2] together with the fact that the projection map $q_\mathcal{B}$ from $\mathcal{M}$ onto $\mathcal{B}$ is a continuous epimorphism. Now, we show that $\mathcal{A}$ is also pseudo-amenable. By assumption, $\mathcal{M}$ has an approximate diagonal, say $(u_\alpha) \subseteq \mathcal{M} \widehat{\otimes} \mathcal{M}$. Now, consider the net $r_\mathcal{A} \otimes r_\mathcal{A}(u_\alpha)$. Observe that

$$a \cdot (r_\mathcal{A} \otimes r_\mathcal{A})(u_\alpha) - (r_\mathcal{A} \otimes r_\mathcal{A})(u_\alpha) \cdot a = (r_\mathcal{A} \otimes r_\mathcal{A})\left((a,0) \cdot u_\alpha - u_\alpha \cdot (a,0)\right),$$

and

$$\pi_\mathcal{A} \circ (r_\mathcal{A} \otimes r_\mathcal{A})(u_\alpha) = r_\mathcal{A} \circ \pi_\mathcal{M}(u_\alpha)$$

for all $a \in \mathcal{A}$ and α, where the second equality follows from Lemma 2.1. These results imply that $\mathcal{A}$ is pseudo-amenable.

Conversely, suppose that both $\mathcal{A}$ and $\mathcal{B}$ are pseudo-amenable. Let $(u_\alpha)_{\alpha \in I} \subseteq \mathcal{A} \widehat{\otimes} \mathcal{A}$ and $(u_\beta)_{\beta \in J} \subseteq \mathcal{B} \widehat{\otimes} \mathcal{B}$ be approximate diagonal for $\mathcal{A}$ and $\mathcal{B}$, respectively. Given the directed set

$$\Gamma := \{\gamma = (\alpha, \beta) : \alpha \in I, \beta \in J\},$$

with the order

$$(\alpha, \beta) \preceq (\alpha', \beta') \iff \alpha \leq \alpha', \beta \leq \beta'.$$

Then the net $(u_\gamma) \subseteq \mathcal{M} \widehat{\otimes} \mathcal{M}$ defined by

$$u_\gamma := (\iota_\mathcal{A} \otimes \iota_\mathcal{A})(u_\alpha) + (s_\mathcal{B} \otimes s_\mathcal{B})(u_\beta)$$

is an approximate diagonal for $\mathcal{M}$. In fact, by Lemma 2.1 we have

$$
\begin{aligned}
(a,b) \cdot u_\gamma - u_\gamma \cdot (a,b) &= (\iota_\mathcal{A} \otimes \iota_\mathcal{A})\left([a + T(b)] \cdot u_\alpha - u_\alpha \cdot [a + T(b)]\right) \\
&\quad + (s_\mathcal{B} \otimes s_\mathcal{B})(b \cdot u_\beta - u_\beta \cdot b),
\end{aligned}
$$

and again Lemma 2.1 implies that

$$
\begin{aligned}
\pi_{\mathcal{M}}(u_\gamma)(a,b) &= \pi_{\mathcal{M}}\left((\iota_{\mathcal{A}} \otimes \iota_{\mathcal{A}})(u_\alpha) + (s_{\mathcal{B}} \otimes s_{\mathcal{B}})(u_\beta)\right)(a,b) \\
&= [\iota_{\mathcal{A}} \circ \pi_{\mathcal{A}}(u_\alpha) + s_{\mathcal{B}} \circ \pi_{\mathcal{B}}(u_\beta)](a,b) \\
&= [(\pi_{\mathcal{A}}(u_\alpha),0) + (-T(\pi_{\mathcal{B}}(u_\beta)), \pi_{\mathcal{B}}(u_\beta))](a,b) \\
&= (\pi_{\mathcal{A}}(u_\alpha)a + \pi_{\mathcal{A}}(u_\alpha)T(b),0) \\
&+ (-T(\pi_{\mathcal{B}}(u_\beta))T(b), \pi_{\mathcal{B}}(u_\beta)b) \\
&\longrightarrow (a+T(b),0) + (-T(b),b) = (a,b),
\end{aligned}
$$

for all $(a,b) \in \mathcal{M}$. Hence, $\mathcal{M}$ is pseudo-amenable. $\qquad\square$

Proposition 3.4. *Let $\mathcal{M}$ be approximately amenable. Then $\mathcal{A}$ and $\mathcal{B}$ are approximately amenable.*

Proof. Since $\mathcal{M}/\mathcal{A}$ is isometrically isomorphic to $\mathcal{B}$, approximate amenability of $\mathcal{M}$ implies that $\mathcal{B}$ is also approximately amenable by [8, Corollary 2.1]. Now, we show that $\mathcal{A}$ is also approximately amenable. Let X be a Banach $\mathcal{A}$-bimodule and $D : \mathcal{A} \longrightarrow X^*$ be a continuous derivation. Then it is easy to show that X is an $(\mathcal{A} \times_T \mathcal{B})$-bimodule with the module actions

$$
(a,b) \cdot x = [a+T(b)] \cdot x, \qquad x \cdot (a,b) = x \cdot [a+T(b)],
$$

for all $a \in \mathcal{A}$, $b \in \mathcal{B}$ and $x \in X$. We prove that the map

$$
\widetilde{D} : \mathcal{M} \longrightarrow X^*
$$

defined by $\widetilde{D} = D \circ r_{\mathcal{A}}$ is a derivation. In fact, for every (a,b) and (a',b') in $\mathcal{M}$ we have

$$
\begin{aligned}
\widetilde{D}((a,b)(a',b')) &= \widetilde{D}\big((aa' + T(b)a' + aT(b'), bb')\big) &\text{(3.1)} \\
&= D(aa') + D(T(b)a') + D(aT(b')) + D(T(b)T(b')) &\text{(3.2)} \\
&= a \cdot D(a') + D(a) \cdot a' + T(b)D(a') + D(T(b))a' + D(a)T(b') &\text{(3.3)} \\
&+ a \cdot D(T(b')) + T(b) \cdot D(T(b')) + D(T(b)) \cdot T(b'). &\text{(3.4)}
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
(a,b) \cdot \widetilde{D}((a',b')) &= (a,b) \cdot D(a'+T(b')) &\text{(3.5)} \\
&= [a+T(b)] \cdot [D(a') + D(T(b'))] &\text{(3.6)}
\end{aligned}
$$

and

$$
\begin{aligned}
\widetilde{D}((a,b)) \cdot (a',b') &= D(a+T(b)) \cdot (a',b') &\text{(3.7)} \\
&= [D(a) + D(T(b))] \cdot [a'+T(b')] &\text{(3.8)}
\end{aligned}
$$

for each $a,a' \in \mathcal{A}$ and $b,b' \in \mathcal{B}$. Adding (3.5) to (3.7) and comparing with (3.1), we conclude that $\widetilde{D}$ is a derivation. From the approximate amenability of $A \times_T \mathcal{B}$, it follows that there is a net $(\xi_\alpha) \subseteq X^*$ such that $\widetilde{D}((a,b)) = \lim_\alpha \mathrm{ad}_{\xi_\alpha}((a,b))$ for all $(a,b) \in \mathcal{M}$. We claim that $D(a) = \lim_\alpha \mathrm{ad}_{\xi_\alpha}(a)$ for all $a \in \mathcal{A}$; indeed,

$$
\begin{aligned}
D(a) = \widetilde{D}((a,0)) &= \lim_\alpha [(a,0) \cdot \xi_\alpha - \xi_\alpha \cdot (a,0)] \\
&= \lim_\alpha [a \cdot \xi_\alpha - \xi_\alpha \cdot a]
\end{aligned}
$$

for all $a \in \mathcal{A}$, as required.   □

We do not know if the converse of the Proposition 3.4 is valid, however following results are a partial converse for the Proposition 3.4. Before we give the next result, recall that a Banach $\mathcal{A}$-bimodule X is called *neo-unital* if

$$X = \mathcal{A} \cdot X \cdot \mathcal{A} = \{a \cdot x \cdot a : a, b \in \mathcal{A}, x \in X\}.$$

Proposition 3.5. *Let $\mathcal{A}$ and $\mathcal{B}$ be approximately amenable Banach algebras. Then, for any neo-unital Banach $\mathcal{M}$-bimodule X, continuous derivations from $\mathcal{M}$ into X^* are weak* approximately inner.*

Proof. Let X be a neo-unital Banach $\mathcal{M}$-bimodule and let $D : \mathcal{M} \to X^*$ be a continuous derivation. Then it is easy to check that X^* is both $\mathcal{A}$-bimodule and $\mathcal{B}$-bimodule with the module actions

$$a \cdot \xi = (a, 0) \cdot \xi, \quad \xi \cdot a = \xi \cdot (a, 0), \quad b \cdot \xi = (-T(b), b) \cdot \xi, \quad \xi \cdot b = \xi \cdot (-T(b), b)$$

for all $a \in \mathcal{A}$, $b \in \mathcal{B}$ and $\xi \in X^*$. Now, consider the linear maps $D_{\mathcal{A}} : \mathcal{A} \to X^*$ and $D_{\mathcal{B}} : \mathcal{B} \to X^*$ defined by $D_{\mathcal{A}}(a) = D((a, 0))$ and $D_{\mathcal{B}}(b) = D((-T(b), b))$. Trivially $D_{\mathcal{A}}$ and $D_{\mathcal{B}}$ are continuous derivations. Thus by assumption there are nets (ξ_γ) and (η_γ) in X^* such that

$$
\begin{aligned}
D_{\mathcal{A}}(a) &= D((a, 0)) = \lim_\gamma [(a, 0) \cdot \xi_\gamma - \xi_\gamma \cdot (a, 0)], \\
D_{\mathcal{B}}(b) &= D((-T(b), b)) = \lim_\gamma [(-T(b), b) \cdot \eta_\gamma - \eta_\gamma \cdot (-T(b), b)],
\end{aligned}
$$

for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$. Since $\mathcal{A}$ is approximately amenable by [8, Lemma 2.2] there are right and left approximate identities say $(r_\beta^{\mathcal{A}})$ and $(\ell_\beta^{\mathcal{A}})$ respectively for $\mathcal{A}$. Similarly, let $(r_\beta^{\mathcal{B}})$ and $(\ell_\beta^{\mathcal{B}})$ respectively be left and right approximate identities for $\mathcal{B}$. Therefore,

$$
\begin{aligned}
(a + T(b), 0) &= \lim_\beta (a, b)(r_\beta^{\mathcal{A}}, 0) = \lim_\beta (\ell_\beta^{\mathcal{A}}, 0)(a, b), \\
(-T(b), b) &= \lim_\beta (a, b)(0, r_\beta^{\mathcal{B}}) = \lim_\beta (0, \ell_\beta^{\mathcal{B}})(a, b),
\end{aligned}
$$

for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$. Above equalities imply that there are nets (ζ_α) and (ς_α) in X^* such that

$$D((a, b)) = D_{\mathcal{A}}(a + T(b)) + D_{\mathcal{B}}(b) = \lim_\alpha [(a, b) \cdot \zeta_\alpha - \varsigma_\alpha \cdot (a, b)]$$

for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$. Since D is a derivation we conclude that

$$\lim_\alpha (a, b) \cdot (\zeta_\alpha - \varsigma_\alpha) \cdot (a', b') = 0$$

for all $a, a' \in \mathcal{A}$ and $b, b' \in \mathcal{B}$. Hence,

$$D((a, b)) \cdot (a', b') = \lim_\alpha [(a, b) \cdot \varsigma_\alpha - \varsigma_\alpha \cdot (a, b)] \cdot (a', b')$$

for all $a, a' \in \mathcal{A}$ and $b, b' \in \mathcal{B}$. Since X is neo-unital it follows that

$$D((a, b)) = \text{weak}^* - \lim_\alpha \text{ad}_{\varsigma_\alpha}((a, b))$$

for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$ as desired.   □

Proposition 3.6. *Let $\mathcal{A}$ and $\mathcal{B}$ be approximately amenable Banach algebras. Suppose that one of the following statements holds. Then $\mathcal{M}$ is approximately amenable.*

(i) $\mathcal{A}$ *is amenable.*

(ii) $\mathcal{A}$ *and* $\mathcal{B}$ *have a central approximate identity.*

(iii) $\mathcal{B}$ *has a bounded approximate identity.*

Proof. (i). Since $\mathcal{A}$ is amenable and $\mathcal{M}/\mathcal{A} \cong \mathcal{B}$ is approximately amenable, the result follows from [8, Corollary 2.1] together with the fact that $\mathcal{A}$ is a closed two-sided ideal in $\mathcal{M}$.

(ii). It follows from [20, Proposition 3.4] and Theorem 2.3 that $\mathcal{M}$ is approximately biprojective in the sense of [20]. Moreover, in this case $\mathcal{M}$ has a central approximate identity. This is because of, if nets (a_α) and (b_β) are central approximate identities for $\mathcal{A}$ and $\mathcal{B}$, respectively. Then the net $((a_\alpha - T(b_\beta), b_\beta))$ is a central approximate identity for $\mathcal{M}$. Indeed, for each $(a, b) \in \mathcal{M}$ we have

$$(a_\alpha - T(b_\beta), b_\beta)(a, b) = (a_\alpha[a + T(b)] - T(b_\beta b), b_\beta b).$$

Hence [20, Corollary 3.7] implies that $\mathcal{M}$ is approximately amenable.

(iii). Let X be a Banach $\mathcal{M}$-bimodule, and let $D : \mathcal{M} \to X^*$ be a continuous derivation. Suppose that (b_α) is a bounded approximate identity for $\mathcal{B}$. Without loss of generality we assume that there are $E \in \mathcal{B}^{**}$, $F \in \mathcal{A}^{**}$ and $\xi, \eta \in X^{***}$ such that

$$b_\alpha \xrightarrow{w^*} E, \quad T(b_\alpha) \xrightarrow{w^*} F, \quad D(b_\alpha) \xrightarrow{w^*} \xi, \quad D(T(b_\alpha)) \xrightarrow{w^*} \eta.$$

It is known by [6] that X^{***} is a Banach $\mathcal{A}^{**} \times_{T^{**}} \mathcal{B}^{**}$-bimodule. We can extend T to a homomorphism from $\mathcal{B}$ into $\mathcal{A}^\sharp$, the unitization of $\mathcal{A}$, denoted still by T. Moreover, it is easy to show that X^{***} is an $\mathcal{A}^\sharp \times_T \mathcal{B}$-bimodule with the module actions

$$e_\mathcal{A} \cdot G = G + F \cdot G - E \cdot G, \quad G \cdot e_\mathcal{A} = G + G \cdot F - G \cdot E$$

for all $G \in X^{***}$, where $e_\mathcal{A}$ is the identity element of $\mathcal{A}^\sharp$. We can consider D as a derivation from $\mathcal{M}$ into X^{***} and extend it to a derivation $\widetilde{D}$ from $\mathcal{A}^\sharp \times_T \mathcal{B}$ into X^{***} defined by $\widetilde{D}(e_\mathcal{A}) = \eta - \xi$. For example, for each $a \in \mathcal{A}$ we have

$$
\begin{aligned}
a \cdot \widetilde{D}(e_\mathcal{A}) + \widetilde{D}(a) \cdot e_\mathcal{A} &= a \cdot \eta - a \cdot \xi + D(a) + D(a) \cdot F - D(a) \cdot E \\
&= D(a) + \text{weak}^* - \lim_\alpha D((aT(b_\alpha), 0) - (a, 0)(0, b_\alpha)) \\
&= D(a) = \widetilde{D}(ae_\mathcal{A}).
\end{aligned}
$$

Since both $\mathcal{A}^\sharp$ and $\mathcal{B}$ are pseudo-amenable by [9, Theorem 3.1 and Proposition 3.2], $\mathcal{A}^\sharp \times_T \mathcal{B}$ is pseudo-amenable by Proposition 3.3 and consequently again by [9, Proposition 3.2] it is approximately amenable. Thus, there is a net (G_γ) in X^{***} such that

$$D((a, b)) = \lim_\gamma \text{ad}_{G_\gamma}((a, b))$$

for all $(a, b) \in \mathcal{M}$. Applying Goldstine's theorem we can obtain (G_γ) in X^*, and replace norm convergence in above equation by weak convergence. This implies, via Mazur's theorem, that $\mathcal{M}$ is approximately amenable. $\qquad\square$

It is known that every approximately amenable Banach algebra have right and left approximate identities. However, it has been an open question whether approximately amenable Banach algebras must also have two-sided approximate identities. In the following result we show that if the T-Lau product $\mathcal{A} \times_T \mathcal{A}$ of Banach algebra $\mathcal{A}$ is approximately amenable, then $\mathcal{A}$ has a two-sided approximate identity.

Proposition 3.7. *Let $\mathcal{A}$ be a Banach algebra and let $T : \mathcal{A} \to \mathcal{A}$ be a continuous homomorphism with $\|T\| \leq 1$. Then $\mathcal{A}$ has a two-sided approximate identity if $\mathcal{A} \times_T \mathcal{A}$ is approximately amenable.*

Proof. First note that $X = \mathcal{A}$ is an $\mathcal{A} \times_T \mathcal{A}$-bimodule by the following module actions.

$$(a,b) \cdot x = [a + T(b)]x \quad \text{and} \quad x \cdot (a,b) = xb$$

for all $a, b \in \mathcal{A}$ and $x \in X$. Now, consider the linear map $D : \mathcal{A} \times_T \mathcal{A} \to X$ defined by $D((a,b)) = a + T(b) - b$. It is not hard to check that D is a continuous derivation. Applying [9, Theorem 3.1(iii)], we obtain that there is a net (x_α) in X such that $D((a,b)) = \lim_\alpha \operatorname{ad}_{x_\alpha}((a,b))$. It follows that

$$a - b = D((a - T(b), b)) = \lim_\alpha [(a - T(b), b) \cdot x_\alpha - x_\alpha \cdot (a - T(b), b)] = \lim_\alpha [a x_\alpha - x_\alpha b]$$

for all $a, b \in \mathcal{A}$. Trivially (x_α) is a two-sided approximate identity for $\mathcal{A}$. $\qquad \square$

References

1. W.G. Bade and P.C. Curtis, *Homomorphisms of commutative Banach algebras,* Amer. J. Math. **82** (1960), 589–608.
2. W.G. Bade and P.C. Curtis, *The Wedderburn decomposition of commutative Banach algebras,* Amer. J. Math. **82** (1960), 851–866.
3. W.G. Bade, P.C. Curtis and A.M. Sinclair, *Raising bounded groups and splitting of radical extensions of commutative Banach algebras,* Studia Math. **141** (2000), 85–98.
4. W.G. Bade and H.G. Dales, *The Wedderburn decomposition of some commutative Banach algebras,* J. Funct. Anal. **107** (1992), 105–121.
5. S.J. Bhatt and P.A. Dabhi, *Arens regularity and amenability of Lau product of Banach algebras defined by a Banach algebra morphism,* Bull. Austral. Math. Soc. **87** (2013), 195–206.
6. H.G. Dales, F. Ghahramani and N. Grønbæk, *Derivations into iterated duals of Banach algebras,* Studia Math. **128** (1998), 19–53.
7. C. Feldman, *The Wedderburn principal theorem in Banach algebras,* Proc. Amer. Math. Soc. **2** (1951), 771–777.
8. F. Ghahramani and R.J. Loy, *Generalized notions of amenability,* J. Funct. Anal. **208** (2004), 229–260.
9. F. Ghahramani and Y. Zhang, *Pseudo-amenable and pseudo-contractible Banach algebras,* Math. Proc. Cambridge Philos. Soc. **142** (2007), 111–123.
10. F. Gourdeau, *Amenability and the second dual of a Banach algebras,* Studia Math. **125** (1997), 75–81.
11. A.Ya. Helemskii, *The Homology of Banach and Topological Algebras,* Kluwer Academic Publishers, Dordrecht, 1989.

12. H. Javanshiri and M. Nemati, *On a certain product of Banach algebras and some of its properties,* Proc. Rom. Acad. Ser. A Math. Phys. Tech. Sci. Inf. Sci. **15** (2014), 219–227.
13. B.E. Johnson, *The Wedderburn decomposition of Banach algebras with finite dimensional radical,* Amer. J. Math. **90** (1968), 866–876.
14. B.E. Johnson, *Cohomology in Banach Algebras*, Memoirs of the American Mathematical Society, No. 127. American Mathematical Society, Providence, R.I., 1972.
15. A.R. Khoddami and H.R. Ebrahimi Vishki, *Biflatness and biprojectivity of Lau product of Banach algebras,* Bull. Iranian Math. Soc. **39** (2013), 559–568.
16. A.T.-M. Lau, *Analysis on a class of Banach algebras with applications to harmonic analysis on locally compact groups and semigroups,* Fund. Math. **118** (1983), 161–175.
17. M.S. Monfared, *On certain products of Banach algebras with applications to harmonic analysis*, Studia Math. **178** (2007), 277–294.
18. M.S. Monfared, *Extensions and isomorphisms for the generalized Fourier algebras of a locally compact group*, J. Funct. Anal. **198** (2003), 413–444.
19. A.Yu. Pirkovskii, *Approximate characterizations of projectivity and injectivity for Banach modules,* Math. Proc. Cambridge Philos. Soc. **143** (2007), 375–385.
20. H. Pourmahmood-Aghababa, *Approximately biprojective Banach algebras and nilpotent ideals,* Bull. Austral. Math. Soc. **87** (2013), 158–173.
21. Y. Zhang, *Nilpotent ideals in a class of Banach algebras,* Proc. Amer. Math. Soc. **127** (1999), 2337–3242.
22. Y. Zhang, *Weak amenability of module extensions of Banach algebras,* Trans. Amer. Math. Soc. **354** (2002), 4131–4151.

[1]Department of Mathematical Sciences, Isfahan Uinversity of Technology, Isfahan 84156-83111, Iran;
School of Mathematics, Institute for Research in Fundamental Sciences (IPM), P.O. Box: 19395–5746, Tehran, Iran.
   E-mail address: m.nemati@cc.iut.ac.ir

[2]Department of Mathematics, Yazd University, P.O. Box: 89195-741, Yazd, Iran.
   E-mail address: h.javanshiri@yazd.ac.ir