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THE ASSOCIATED WEIGHT AND THE ESSENTIAL NORM OF WEIGHTED COMPOSITION OPERATORS

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Abstract. For an almost radial and typical weight v and any weight w, we characterize the continuity, compactness and we estimate the essential norm of weighted composition operators uC_φ, acting from the weighted Banach spaces of analytic functions H_v^∞ into H_w^∞, in terms of the quotients of the w-norm of the product of u with φ^n and the v-norm of the nth power of the identity function on $\mathbb{D}$, where $u : \mathbb{D} \to \mathbb{C}$ and $\varphi : \mathbb{D} \to \mathbb{D}$ are analytic. As a consequence, we estimate the essential norm of composition operators C_φ (in terms of φ^n) acting on μ-Bloch spaces, for very general weights μ. We also characterize continuity and compactness of weighted composition operators uC_φ acting on log-Bloch space.

1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$. We denote by $H(\mathbb{D})$ the space of analytic functions on $\mathbb{D}$. For an analytic selfmap φ of $\mathbb{D}$ and a function $u \in H(\mathbb{D})$ we define the linear operator uC_φ, by $uC_\varphi(f) := u(f \circ \varphi)$, for $f \in H(\mathbb{D})$. The expression uC_φ is known as *the weighted composition operator* induced by u and φ. Observe that when $u \equiv 1$, uC_φ is the composition operator C_φ. When φ is the identity function, we have the multiplication operator M_u defined by $M_u(f) := u \cdot f$. The properties of uC_φ that we study here are continuity, compactness and its essential norm. Applications of these operators have been found in semigroup theory and dynamical systems (see [11] and [16]). In this paper, we focus on the action of uC_φ on certain weighted growth spaces. More

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precisely, given a weight $v : \mathbb{D} \to \mathbb{R}^+$ which is a strictly positive, continuous and bounded function, the weighted Banach spaces of analytic functions, or growth spaces, are defined by

$$H_v^\infty := \{f \in H(\mathbb{D}) : \ \|f\|_{H_v^\infty} = \sup_{z\in\mathbb{D}} v(z)|f(z)| < \infty\}.$$

Initial interest in these Banach spaces focused on growth conditions of analytic functions and their duals, which are connected to various areas such as complex analysis, Fourier analysis, spectral theory and partial differential equations. See, for example, [3].

It is clear that for each $z \in \mathbb{D}$ fixed, the evaluation functional δ_z is bounded on H_v^∞ and satisfies $\|\delta_z\| \geq 1/\, \|v\|_\infty$, where $\|\cdot\|$ denotes the operator norm and $\|\cdot\|_\infty$ is the supremum norm. Thus, since the relation $z \mapsto \|\delta_z\|$ is a positive and continuous function, then the expression

$$\widetilde{v}(z) = \left(\sup_{\|f\|_{H_v^\infty}\leq 1} \{|f(z)|\} \right)^{-1}, \ f \in H_v^\infty, \ z \in \mathbb{D}$$

defines a positive continuous and bounded function $\widetilde{v}$ on $\mathbb{D}$. This function is called *the associated weight* of v. Associated weights were introduced by Anderson and Duncan in [1] and were studied further by Bierstedt, Bonet and Taskinen in [2]. The associated weight is a very important tool for studying the spaces H_v^∞. It is known that the space $H_{\widetilde{v}}^\infty$ is isometrically equal to H_v^∞; that is $\|f\|_{H_{\widetilde{v}}^\infty} = \|f\|_{H_v^\infty}$ for all $f \in H_v^\infty$. It is also known that $\widetilde{v} \geq v > 0$ on $\mathbb{D}$. We refer to the interested reader to [2] and [4] for more properties of the spaces H_v^∞ and their associated weights.

The μ-Bloch space, where μ is a weight on $\mathbb{D}$, consists of all holomorphic functions whose derivatives are in H_μ^∞. That is,

$$\mathcal{B}^\mu := \{f \in H(\mathbb{D}) : \ \|f\|_{\mathcal{B}^\mu} = \sup_{z\in\mathbb{D}} \mu(z)|f'(z)| < \infty\}.$$

When $\alpha > 0$ and $\mu(z) := (1 - |z|^2)^\alpha$, $\mathcal{B}^\mu$ is the so-called α-*Bloch space*, which is customarily denoted by $\mathcal{B}^\alpha$. Properties of $\mathcal{B}^\alpha$ can be found in [23]. When $\alpha = 1$, we have that $\mathcal{B}^\alpha = \mathcal{B}$, which is the classical *Bloch space*. Properties of $\mathcal{B}$ can be found in [22]. When $\mu(z) := (1 - |z|) \log \dfrac{2}{1-|z|}$, $\mathcal{B}^\mu$ is the so-called log-*Bloch space*, which is denoted by $\mathcal{B}^{\log}$.

The essential norm $\|T\|_e^{X\to Y}$ of a continuous linear operator $T : X \to Y$, where X and Y are Banach spaces, is the distance from T to the space of compact operators, that is, $\|T\|_e^{X\to Y} = \inf\{\|T - K\| \,/\, K : X \to Y \text{ is compact}\}$, where $\|\cdot\|$ denotes the operator norm. Notice that $\|T\|_e^{X\to Y} = 0$ if and only if T is compact. The essential norm of composition operators acting on weighted Banach spaces of analytic functions was estimated by Bonet *et al.* in [4]. Montes-Rodríguez in [14] and Contreras and Hernández-Díaz in [6] extended the results in [4] to weighted composition operators uC_φ mapping H_v^∞ into H_w^∞. More precisely, Montes-Rodríguez calculated the essential norm of $uC_\varphi : H_v^\infty \to H_w^\infty$ in terms of a quotient involving the associated weight of v, under the requirement that the weights v and w are radial, non-increasing and typical. Here, a weight v is called *typical* if $\lim_{|z|\to 1^-} v(z) = 0$. The above results have been extended in several directions. For instance, in [13] MacCluer and Zhao estimate the essential norm of C_φ acting on α-Bloch spaces. More recently, Manhas [12] was able to extend, to very general weighted spaces of holomorphic functions on Banach spaces, the results of Montes-Rodríguez in [14].

Also, in [4], Bonet *et al.* characterize the continuity and compactness of composition operators acting on H_v^∞ in terms of φ^n and the norm of the nth power of the identity function on $\mathbb{D}$, under the technical requirement that v is radial, typical and non-increasing. Recently, the above results have been extended by many authors to Bloch-type spaces. The first of these results was obtained by Wulan, Zheng, and Zhu ([19]) and is stated as follows:

Theorem 1.1. [19] *Let φ be an analytic self-map of $\mathbb{D}$. Then C_φ is compact on $\mathcal{B}$ if and only if*

$$\lim_{n\to\infty} \|\varphi^n\|_{\mathcal{B}} = 0.$$

Zhao in [21] gave a formula for the essential norm of $C_\varphi : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ in terms of an expression involving φ^n. More precisely, he showed that for $\alpha, \beta > 0$

$$\|C_\varphi\|_e^{\mathcal{B}^\alpha\to\mathcal{B}^\beta} = \left(\frac{e}{2\alpha}\right)^\alpha \limsup_{n\to\infty} n^{\alpha-1} \|\varphi^n\|_{\mathcal{B}^\beta} = \limsup_{n\to\infty} \frac{\|\varphi^n\|_{\mathcal{B}^\beta}}{\|g_n\|_{\mathcal{B}^\alpha}},$$

where $g_n(z) = z^n$ for $z \in \mathbb{D}$ and $n \in \mathbb{N}$. Hence, it is natural to ask if the above relation is also valid when C_φ maps $\mathcal{B}^{\mu_1}$ to $\mathcal{B}^{\mu_2}$, where μ_1 and μ_2 are weights on $\mathbb{D}$.

Recently Hyvärinen, Kemppainen, Lindström, Rautio, and Saukko [9] have extended Zhao's result above. Using new properties of the associated weight and Montes-Rodríguez's results, they calculated the essential norm of uC_φ on H_v^∞, in terms of φ^n, under the technical requirements that the weight v is radial, non-increasing and tends to zero toward the boundary of $\mathbb{D}$. We state their result here as follows

Theorem 1.2. [9] *Suppose that $u : \mathbb{D} \to \mathbb{C}$ and $\varphi : \mathbb{D} \to \mathbb{D}$ are holomorphic. Let v and w be radial, non-increasing and typical weights. Then*

$$\|uC_\varphi\|_e^{H_v^\infty\to H_w^\infty} = \limsup_{n\to\infty} \frac{\|u\varphi^n\|_{H_w^\infty}}{\|g_n\|_{H_v^\infty}}.$$

Evidently, the assumptions on the weights are imposed so that Montes-Rodríguez's theorem in [14] is applicable. Hyvärinen *et al.* do not consider the so-called *logarithmic Bloch spaces* $\mathcal{B}^\mu$ with μ given by

$$\mu(z) = \sup_{z\in\mathbb{D}} (1-|z|)^\alpha \left(\log \frac{2^{\beta/\alpha}}{1-|z|}\right)^\beta,$$

where the parameters $\alpha > 0$ and $\beta > 0$ are fixed, and we have replaced the number e, in the original definition given by Stević in [17], by the number 2. Clearly, the above weight is not decreasing on $(0, 1)$. Recent results concerning continuity and compactness of composition operators C_φ on logarithmic Bloch spaces were obtained by Castillo *et al.* in [5] and by García *et al.* in [8]. Furthermore, in [5], the authors give an estimation of $\|C_\varphi\|_e^{\mathcal{B}^{\log}\to\mathcal{B}^\mu}$ in terms of φ^n. Hyvärinen *et al.* in [9] furthermore did not consider the case when μ_1 is a non-radial weight given by

$$\mu_1(z) = (2 - \mathrm{Re}(z))(1 - |z|) \ \text{ or } \ \mu_1(z) = (2 - \mathrm{Im}(z))(1 - |z|).$$

The objective of this paper is to apply the arguments in [9] to more general, non-radial weights. In this paper, we estimate the essential norm of uC_φ between H_v^∞ and H_w^∞ when v is a typical and almost-radial weight, as defined in the next section. We apply this result to characterize the symbols φ for which C_φ is continuous, and the φ's for which C_φ is compact between $\mathcal{B}^{\mu_1}$ and $\mathcal{B}^{\mu_2}$, for more general weights μ_1 and μ_2 in terms of φ^n. We also are able to characterize the weights u and symbols φ for which uC_φ is continuous and compact on the log-Bloch space.

Throughout this paper, constants are denoted by C or C_v (if depending only on v). These constants are positive and may differ from one occurrence to the next.

2. Some additional properties of associated weight

A weight v is called *radial* if $v(|z|) = v(z)$ for all $z \in \mathbb{D}$. We say that two weights v and w are *equivalent* if there is a $C > 0$ such that

$$\frac{1}{C}v(z) \leq w(z) \leq Cv(z)$$

for all $z \in \mathbb{D}$. In this case, we write $v \sim w$. A weight v is called *essential* if $v \sim \widetilde{v}$. In [9], the authors showed that if a weight v is radial, typical and $v|_{(0,1)}$ is non-increasing, then v is essential. We call v *almost radial* if there is a radial $w \sim v$. For instance, the weight given by $v(z) = (2-\mathrm{Re}(z))(1-|z|)$ is typical and almost radial. We prove the following useful properties of associated weights of almost-radial weights. More properties of associated weights can be found in [2] and [4]. Here a real function h is called *almost decreasing* if there is a $C > 0$ such that $h(x) \geq Ch(y)$ for all $x < y$. We include a proof for the reader's convenience.

Lemma 2.1. *If v is an almost-radial weight, then the following statements hold:*

1. *$\widetilde{v}$ is almost radial,*
2. *$\widetilde{v}|_{(0,1)}$ is almost decreasing.*
3. *For each $r_0 \in (0,1)$ there are $C_v > 0$ and $c = c_{r_0,v} > 0$ such that*
$$r_0^c\widetilde{v}(r_0) \geq C_v r^c \widetilde{v}(r)$$
for all $r \in (0,1)$.
4. *If v is additionally typical, then $\widetilde{v}$ is typical.*

Proof. Assume that v is almost radial, so that in turn, there is a radial weight μ such that for all $z \in \mathbb{D}$

$$\frac{1}{C_v}\mu(z) \leq v(z) \leq C_v\mu(z). \tag{2.1}$$

We claim that for all $z \in \mathbb{D}$

$$\frac{1}{C_v}\widetilde{\mu}(z) \leq \widetilde{v}(z) \leq C_v\widetilde{\mu}(z). \tag{2.2}$$

Indeed, if $z_0 \in \mathbb{D}$ then (see [2], Property 1.2 (iv)) there is an $f_0 \in H_v^\infty$ such that $\|f_0\|_{H_v^\infty} \leq 1$ and $\widetilde{v}(z_0)|f_0(z_0)| = 1$. Therefore, since $\mu \leq C_v v$, the function $h_0 = \frac{1}{C_v} f_0$ satisfies $\|h_0\|_{H_\mu^\infty} \leq 1$. Hence, we have that

$$\frac{1}{\widetilde{\mu}(z_0)} \geq |h_0(z_0)| = \frac{1}{C_v\widetilde{v}(z_0)}.$$

This proves our claim above, and since $\widetilde{\mu}$ is a radial weight, (2.2) implies that $\widetilde{v}$ is almost radial. Thus the item (1) is proven.

Suppose that $0 < r_1 < r_2 < 1$. Then there is an $f_0 \in H_v^\infty$ such that $\|f_0\|_{H_v^\infty} \leq 1$ and $\sup_{\|f\|_{H_v^\infty} \leq 1}\{|f(r_1)|\} = |f_0(r_1)|$. Choose $\alpha \in \mathbb{R}$ such that $\sup_{|z|=r_2}|f_0(z)| = |f_0(r_2 e^{i\alpha})|$ and put $g(z) = \frac{1}{C_v^2} f_0(ze^{i\alpha})$ for all $z \in \mathbb{D}$. Then using the relations (2.1) we have $\|g\|_{H_v^\infty} \leq \|f_0\|_{H_v^\infty} \leq 1$. Hence, by definition of $\widetilde{v}$ and the Maximum Modulus Principle, we can write

$$\begin{aligned} \frac{1}{\widetilde{v}(r_2)} &\geq |g(r_2)| = \frac{1}{C_v^2}\left|f_0\left(r_2e^{i\alpha}\right)\right| \\ &= \frac{1}{C_v^2}\sup_{|z|=r_2}|f_0(z)| \geq \frac{1}{C_v^2}\sup_{|z|=r_1}|f_0(z)| \\ &\geq \frac{1}{C_v^2}|f_0(r_1)| = \frac{1}{C_v^2\widetilde{v}(r_1)}. \end{aligned}$$

Thus $\tilde{v}$ is almost decreasing on $(0,1)$ and the item (2) is showed.

Fix $r_0 \in (0,1)$ and put $t_0 = \log(r_0) < 0$. Since μ is radial, h given by $h(t) = -\log(\widetilde{\mu}(e^t))$ is convex on $(-\infty, 0)$. Thus by definition of support line of a convex function, there is a $c \in \mathbb{R}$, depending only on the lateral derivatives of h at t_0, such that $h(t) \geq h(t_0) + c(t - t_0)$ for all $t \in (-\infty, 0)$. Hence, setting $r = e^t$ yields

$$C_v r_0^c \widetilde{v}(r_0) \geq r_0^c \widetilde{\mu}(r_0) \geq r^c \widetilde{\mu}(r) \geq \frac{1}{C_v} r^c \widetilde{v}(r).$$

The proof of the item (3) is now completes.

Finally, since v is typical, then μ is typical and radial. It follows that $\widetilde{\mu}$ is typical. In turn, it also follows that $\widetilde{v}$ is typical, since $\widetilde{v} \sim \widetilde{\mu}$. □

Auxiliary weights $\bar{v}$ were considered by Bonet *et al.* in [4] (see also [9]).

Definition 2.2. Given a weight v on $\mathbb{D}$, the *auxiliary weight* of v, denoted by $\bar{v}$, is defined by

$$\bar{v}(z) = \left(\sup_{n \in \mathbb{W}} \frac{|g_n(z)|}{\|g_n\|_{H_v^\infty}}\right)^{-1},$$

where $\mathbb{W}$ is the whole numbers, $g_0(z) \equiv 1$ and $g_n(z) = z^n$ for $z \in \mathbb{D}$ and $n \in \mathbb{N}$.

From the definition of the auxiliary weight, it is clear that $\bar{v}$ satisfies the following useful properties:

(a) $\bar{v}$ is strictly positive.
(b) $\bar{v}$ is continuous and bounded.
(c) $\bar{v}$ is radial.

Furthermore, for each $z \in \mathbb{D}$, we have

$$\left\{ \frac{|g_n(z)|}{\|g_n\|_{H_v^\infty}} : n \in \mathbb{W} \right\} \subset \left\{ |f(z)| : \|f\|_{H_v^\infty} \leq 1 \right\}.$$

Therefore, for all $z \in \mathbb{D}$, we have that

$$\bar{v}(z) \geq \widetilde{v}(z).$$

The equivalence of $\widetilde{v}$ and $\bar{v}$ for an almost radial weight v, as well as the properties of both, will be crucial in the proofs of our results. Similar results were obtained by Hyvärinen *et al.* in [9] but under the condition that the weight v is radial, non-increasing and typical, in fact, their argument was taken from [4], however this last reference has a little mistake which was corrected in [9] imposing the non-increasing condition.

Lemma 2.3. *If v is an almost radial, then there is a $C_v > 0$ such that for all $z \in \mathbb{D}$*

$$\widetilde{v}(z) \geq C_v \bar{v}(z). \tag{2.3}$$

Proof. Since the function given by $\psi(z) = \frac{\bar{v}(z)}{\widetilde{v}(z)}$ is continuous on the compact set $\left\{ z \in \mathbb{D} : |z| \leq \frac{1}{2} \right\}$, there is a $C_1 > 0$ such that $\psi(z) \leq C_1$ for all such z. Hence it suffices to verify Inequality (2.3) for all $z \in \mathbb{D}$ such that $\frac{1}{2} < |z| < 1$.

Indeed, let z_0 be such a point, that is, let us suppose that $r_0 = |z_0| \in \left(\frac{1}{2}, 1\right)$. Lemma 2.1 implies that there are $c > 0$ and $K_v > 0$ such that

$$\widetilde{v}\left(r_0\right) \geq \frac{K_v}{r_0^c} \sup_{r \in (0,1)} \widetilde{v}(r) r^c.$$

Now, we choose $n_0 \in \mathbb{Z}$ such that $n_0 \leq c < n_0 + 1$. Then $r_0^c \leq r_0^{n_0}$ and $r^c \geq r^{n_0+1}$ for all $r \in (0,1)$. Hence we obtain that

$$\begin{aligned}
\widetilde{v}\left(r_0\right) &\geq \frac{K_v}{r_0^{n_0}} \sup_{r \in (0,1)} \widetilde{v}(r) r^{n_0+1} \\
&= K_v \frac{\|g_{n_0+1}\|_{H_{\widetilde{v}}^\infty}}{r_0^{n_0}} = K_v r_0 \frac{\|g_{n_0+1}\|_{H_{\widetilde{v}}^\infty}}{r_0^{n_0+1}} \\
&\geq \frac{K_v}{2} \left(\sup_{m \in \mathbb{W}} \frac{r_0^m}{\|g_m\|_{H_v^\infty}} \right)^{-1} = \frac{K_v}{2} \bar{v}\left(r_0\right) = \frac{K_v}{2} \bar{v}\left(z_0\right).
\end{aligned}$$

Here, we have used the assumption that $r_0 > \frac{1}{2}$ and the fact that $\bar{v}$ is a radial weight. The proof follows from the fact that there is a $C_v > 0$, depending only on v, such that $\widetilde{v}\left(|z_0|\right) \leq C_v \widetilde{v}\left(z_0\right)$, since $\widetilde{v}$ is an almost radial weight. ☐

Corollary 2.4. *If v is an almost radial weight, then $\widetilde{v} \sim \bar{v}$.*

3. Continuity of uC_φ between the spaces H_v^∞

Now that we have established that $\widetilde{v} \sim \overline{v}$ for an almost radial weight v, we can now extend the continuity results in [9]. Although the arguments below are standard, we provide them for the sake of completeness. Furthermore, we can apply our result to characterize continuous composition operators from the μ_1-Bloch space (being μ_1 an almost radial weight) into the μ_2-Bloch space, where μ_2 is any weight. This characterization generalizes recent results due to Castillo *et al.* in [5] and García *et al.* in [8]. Furthermore, as an application of our result, we obtain a new characterization of continuous weighted composition operators from $\mathcal{B}^{\log}$ into itself.

Theorem 3.1. *Let φ be a holomorphic self-map of $\mathbb{D}$, $u \in H(\mathbb{D})$. Let v, w be weights on $\mathbb{D}$, where v is almost radial. Then uC_φ maps H_v^∞ into H_w^∞ if and only if*

$$K = \sup_{n\in\mathbb{W}} \frac{\|u\varphi^n\|_{H_w^\infty}}{\|g_n\|_{H_v^\infty}} < \infty. \tag{3.1}$$

Proof. Let us suppose first that $uC_\varphi : H_v^\infty \to H_w^\infty$ is continuous. For each $n \in \mathbb{W}$ we set $f_n(z) = \frac{g_n(z)}{\|g_n\|_{H_v^\infty}}$, where g_n are the functions in Definition 2.2. Then $f_n \in H_v^\infty$, because $\|f_n\|_{H_v^\infty} = 1$. Hence, we have

$$\sup_{n\in\mathbb{W}} \frac{\|u\varphi^n\|_{H_w^\infty}}{\|g_n\|_{H_v^\infty}} = \sup_{n\in\mathbb{W}} \|uC_\varphi(f_n)\|_{H_w^\infty} \leq \|uC_\varphi\| < \infty.$$

On the other hand, suppose that Relation (3.1) holds. Since v is an almost radial weight, then by Lemma 2.3, there is a constant $C_v > 0$, depending only on v, such that

$$\frac{1}{\widetilde{v}\,(\varphi(z))} \leq \frac{C_v}{\overline{v}\,(\varphi(z))} \quad (z \in \mathbb{D}).$$

Hence

$$\begin{aligned} \sup_{z\in\mathbb{D}} \frac{w(z)|u(z)|}{\widetilde{v}(\varphi(z))} &\leq C_v \sup_{z\in\mathbb{D}} \frac{w(z)|u(z)|}{\overline{v}\,(\varphi(z))} \\ &\leq C_v \sup_{n\in\mathbb{W}} \sup_{z\in\mathbb{D}} \frac{w(z)|u(z)|\,|g_n\,(\varphi(z))|}{\|g_n\|_{H_v^\infty}} \\ &= C_v \sup_{n\in\mathbb{W}} \frac{\|u\varphi^n\|_{H_w^\infty}}{\|g_n\|_{H_v^\infty}} \leq C_v K < \infty. \end{aligned} \tag{3.2}$$

Therefore, for any $f \in H_v^\infty$ we have

$$\begin{aligned} \|uC_\varphi(f)\|_{H_w^\infty} &= \sup_{z\in\mathbb{D}} w(z)\,|u(z)|\,|f\,(\varphi(z))|\,\frac{\widetilde{v}(\varphi(z))}{\widetilde{v}(\varphi(z))} \\ &\leq \|f\|_{H_{\widetilde{v}}^\infty} \sup_{z\in\mathbb{D}} \frac{w(z)|u(z)|}{\widetilde{v}(\varphi(z))} \\ &\leq C_v K \|f\|_{H_v^\infty}, \end{aligned}$$

and the operator $uC_\varphi : H_v^\infty \to H_w^\infty$ is continuous. □

We now point out two applications of Theorem 3.1.

Corollary 3.2. *Let φ be an analytic self-map of $\mathbb{D}$. Let μ_1, μ_2 be weights on $\mathbb{D}$, where μ_1 is almost radial. Then C_φ is continuous from $\mathcal{B}^{\mu_1}$ to $\mathcal{B}^{\mu_2}$ if and only if*

$$\sup_{n\in\mathbb{W}} \frac{\|\varphi^n\|_{\mathcal{B}^{\mu_2}}}{\|g_n\|_{\mathcal{B}^{\mu_1}}} < \infty.$$

Proof. Indeed, C_φ is continuous from $\mathcal{B}^{\mu_1}$ to $\mathcal{B}^{\mu_2}$ if and only if there exists $M \geq 0$ such that $\|C_\varphi(f)\|_{\mathcal{B}^{\mu_2}} \leq M \|f\|_{\mathcal{B}^{\mu_1}}$ for all $f \in \mathcal{B}^{\mu_1}$. But $f \in \mathcal{B}^{\mu_1}$ if and only if $f' \in H^\infty_{\mu_1}$, $\|f\|_{\mathcal{B}^{\mu_1}} = \|f'\|_{H^\infty_{\mu_1}}$ and $\|C_\varphi(f)\|_{\mathcal{B}^{\mu_2}} = \|\varphi' C_\varphi(f')\|_{H^\infty_{\mu_2}}$. Hence $C_\varphi : \mathcal{B}^{\mu_1} \to \mathcal{B}^{\mu_2}$ is continuous if and only if $\varphi' C_\varphi$ is continuous from $H^\infty_{\mu_1}$ to $H^\infty_{\mu_2}$. By Theorem 3.1, this continuity occurs if and only if

$$\sup_{n\in\mathbb{W}} \frac{\frac{1}{n+1}\|\varphi^{n+1}\|_{\mathcal{B}^{\mu_2}}}{\frac{1}{n+1}\|g_{n+1}\|_{\mathcal{B}^{\mu_1}}} = \sup_{n\in\mathbb{W}} \frac{\|\varphi'\varphi^n\|_{H^\infty_{\mu_2}}}{\|g_n\|_{H^\infty_{\mu_1}}} < \infty.$$

The statement of the corollary follows. □

Example 3.3. The statement of Corollary 3.2 in the special case that $\mu_1(z) = (1-|z|)\log\left(\frac{3}{1-|z|}\right)$ appeared recently in [5]. A result appearing in [8] gives the statement of the corollary in the case that $\mu_1(z) = (1-|z|)^\alpha \log^\beta\left(\frac{3}{1-|z|}\right)$ with $\alpha > 0$ and $\beta \geq 0$.

A second application of Theorem 3.1 is the following new characterization of the holomorphic functions u and self-maps φ on $\mathbb{D}$ such that uC_φ is continuous on $\mathcal{B}^{\log}$. The characterization involves the weight $w_{\log}$, which is given by

$$w_{\log}(z) = \left[\log\log\left(\frac{4}{1-|z|}\right)\right]^{-1}.$$

This weight is radial and typical. It is also non-increasing, that is, $w_{\log}(z) \geq w_{\log}(s)$ for all $z, s \in \mathbb{D}$ such that $|z| \leq |s|$. Therefore, $w_{\log}$ is essential (see [9]); that is, we have that $w_{\log} \sim \widetilde{w}_{\log}$.

Corollary 3.4. *Suppose that $u : \mathbb{D} \to \mathbb{C}$ and $\varphi : \mathbb{D} \to \mathbb{D}$ are holomorphic. Then uC_φ is continuous on $\mathcal{B}^{\log}$ if and only if the following are satisfied:*

(1) $\displaystyle\sup_{n\in\mathbb{W}} \frac{(n+1)\|J_u(\varphi^n)\|_{\mathcal{B}^{\log}}}{\|g_{n+1}\|_{\mathcal{B}^{w_{\log}}}} < \infty$, *and*

(2) $\displaystyle\sup_{n\in\mathbb{N}} \frac{\|I_u(\varphi^n)\|_{\mathcal{B}^{\log}}}{\|g_n\|_{\mathcal{B}^{\log}}} < \infty$,

where $I_u, J_u : H(\mathbb{D}) \to \mathbb{C}$ are defined by

$$I_u(f(z)) = \int_0^z f'(s)u(s)ds, \ \text{ and } \ J_u(f(z)) = \int_0^z f(s)u'(s)ds.$$

Proof. Recently, Ye showed in [20] that under the present assumptions on u and φ, continuity of uC_φ on $\mathcal{B}^{\log}$ holds if and only if both of the following conditions are satisfied

(1) $\sup_{z\in\mathbb{D}} \frac{v_{\log}(z)}{w_{\log}(\varphi(z))} |u'(z)| < \infty$, and

(2) $\sup_{z\in\mathbb{D}} \frac{v_{\log}(z)}{v_{\log}(\varphi(z))} |\varphi'(z)u(z)| < \infty$,

where $v_{\log}$ is the weight on $\mathbb{D}$ given by

$$v_{\log}(z) = (1-|z|)\log\left(\frac{2}{1-|z|}\right).$$

This weight is essential, because it is equivalent to the weight $\mu_{\log}$ on $\mathbb{D}$ given by

$$\mu_{\log}(z) = (1-|z|)\log\left(\frac{e}{1-|z|}\right),$$

which is radial, typical and non-increasing. Thus since the weight $w_{\log}$ is also essential (because is radial, typical and non-increasing), the Proposition 3.1 in [6] implies that $u'C_\varphi$ is continuous from $H^\infty_{w_{\log}}$ to $H^\infty_{v_{\log}}$ and that ψC_φ is continuous from $H^\infty_{v_{\log}}$ to $H^\infty_{v_{\log}}$, where $\psi = \varphi' u$. By Theorem 3.1, the continuity of these two operators is equivalent to the compound statement

$$\sup_{n\in\mathbb{W}} \frac{\|u'\varphi^n\|_{H^\infty_{v_{\log}}}}{\|g_n\|_{H^\infty_{w_{\log}}}} < \infty \text{ and } \sup_{n\in\mathbb{W}} \frac{\|\psi\varphi^n\|_{H^\infty_{v_{\log}}}}{\|g_n\|_{H^\infty_{v_{\log}}}} < \infty.$$

However, we have that

$$\begin{aligned} \|g_n\|_{H^\infty_{w_{\log}}} &= \frac{1}{n+1}\|g_{n+1}\|_{\mathcal{B}^{w_{\log}}}, \\ \|u'\varphi^n\|_{H^\infty_{v_{\log}}} &= \|J_u(\varphi^n)\|_{\mathcal{B}^{\log}}, \text{ and} \\ \|\psi\varphi^n\|_{H^\infty_{v_{\log}}} &= \frac{1}{n+1}\left\|I_u\left(\varphi^{n+1}\right)\right\|_{\mathcal{B}^{\log}}. \end{aligned}$$

The statement of the corollary follows. □

We note here that the technique used in the proof of Corollary 3.4 was used in [10].

4. The essential norm of uC_φ

In this section, we extend Theorem 1.2 to the case where v is almost radial and typical and w is any weight on $\mathbb{D}$. To do so, we will use the following well-known fact (see [7, 15] or [18]). The proof of a similar result for Bloch-type space is actually explicitly stated in the reference [5].

Lemma 4.1 ([18]). *Let v, w be weights on $\mathbb{D}$. Let $T : H^\infty_v \to H^\infty_w$ be a continuous operator when H^∞_v and H^∞_w are given the topology of uniform convergence on compact subsets of $\mathbb{D}$. $T : H^\infty_v \to H^\infty_w$ is compact if and only if*

$$\lim_{n\to\infty} \|Tf_n\|_{H^\infty_w} = 0$$

for each bounded sequence $\{f_n\} \subset H^\infty_v$ such that $f_n \to 0$ uniformly on compact subsets in $\mathbb{D}$.

Lemma 4.1 will now be used to prove the following essential norm estimate for uC_φ

Lemma 4.2. *Suppose that $u : \mathbb{D} \to \mathbb{C}$ and $\varphi : \mathbb{D} \to \mathbb{D}$ are holomorphic. Let v, w be weights on $\mathbb{D}$, where v is almost radial, and suppose that uC_φ is bounded from H_v^∞ to H_w^∞. Then there is an $M_v > 0$ such that*

$$\|uC_\varphi\|_e^{H_v^\infty \to H_w^\infty} \leq M_v \lim_{r \to 1^-} \sup_{|\varphi(z)|>r} \frac{w(z)}{\widetilde{v}(\varphi(z))} |u(z)|.$$

Proof. For each $r \in (0,1)$, define $K_r : H(\mathbb{D}) \to H(\mathbb{D})$ by $K_r(f) = f_r$, where f_r is the dilation of f given by $f_r(z) = f(rz)$. Since f_r for these values of r is bounded and v is bounded by definition, we have that $f_r \in H_v^\infty$ for such r's. Furthermore, if $r \in (0,1)$ is fixed and $\{f_n\}$ is a bounded sequence in H_v^∞ such that $f_n \to 0$ uniformly on compact subsets in $\mathbb{D}$, then

$$\|K_r(f_n)\|_{H_v^\infty} \leq \|v\|_\infty \sup_{|z|\leq r} |f_n(z)| \to 0,$$

as $n \to \infty$. Hence, for these values of r, K_r is compact on H_v^∞ by Lemma 4.1. By the boundedness assumption on uC_φ, we have that $uC_\varphi K_r$ is compact between these spaces as well, for all $r \in (0,1)$.

Now let $r \in (0,1)$ be fixed, and let $\{r_n\}$ be a sequence in $(0,1)$ such that $r_n \nearrow 1^-$. Then there is an $N \in \mathbb{W}$ such that

$$\sup_{|w|\leq r} |f_{r_N}(w) - f(w)| < \frac{L}{\|u\|_{H_w^\infty}},$$

where

$$L = \lim_{s \to 1^-} \sup_{|\varphi(z)|>s} \frac{w(z)}{\widetilde{v}(\varphi(z))} |u(z)|.$$

Since $uC_\varphi K_{r_N} : H_v^\infty \to H_w^\infty$ is a compact operator, then

$$\|uC_\varphi\|_e^{H_v^\infty \to H_w^\infty} \leq \|uC_\varphi - uC_\varphi K_{r_N}\|.$$

Furthermore, if $f \in H_v^\infty$ satisfies $\|f\|_{H_v^\infty} = 1$, then

$$\|(uC_\varphi - uC_\varphi K_{r_N})(f)\|_{H_w^\infty} = \sup_{z\in\mathbb{D}} w(z)\,|u(z)|\,|f_{r_N}(\varphi(z)) - f(\varphi(z))|.$$

Now, if $|\varphi(z)| \leq r$, then

$$w(z)\,|u(z)|\,|f_{r_N}(\varphi(z)) - f(\varphi(z))| \leq L.$$

On the other hand, if $|\varphi(z)| > r$ we have that

$$\begin{aligned} w(z)\,|u(z)|\,|f(\varphi(z))| &\leq \|f\|_{H_{\widetilde{v}}^\infty} \frac{w(z)}{\widetilde{v}(\varphi(z))} |u(z)| \\ &= \frac{w(z)}{\widetilde{v}(\varphi(z))} |u(z)|. \end{aligned}$$

Moreover, since v is almost radial, there is a $C_v > 0$ such that for all $z \in \mathbb{D}$,

$$w(z)\,|u(z)|\,|f_{r_N}(\varphi(z))| \leq \|f\|_{H_{\widetilde{v}}^\infty} \frac{w(z)}{\widetilde{v}(r_N\varphi(z))} |u(z)| \leq C_v \frac{w(z)}{\widetilde{v}(\varphi(z))} |u(z)|.$$

It follows that

$$\|(uC_\varphi - uC_\varphi K_{r_N})(f)\|_{H_w^\infty} \le \left(L + (1+C_v) \sup_{|\varphi(z)|>r} \frac{w(z)}{\widetilde{v}(\varphi(z))} |u(z)| \right) \|f\|_{H_v^\infty}.$$

Hence, we obtain the estimate

$$\|uC_\varphi\|_e^{H_v^\infty \to H_w^\infty} \le \|uC_\varphi - uC_\varphi K_{r_N}\| \le L + (1+C_v) \sup_{|\varphi(z)|>r} \frac{w(z)}{\widetilde{v}(\varphi(z))} |u(z)|.$$

Letting $r \to 1^-$, we can deduce that there is an $M_v > 0$ such that

$$\|uC_\varphi\|_e^{H_v^\infty \to H_w^\infty} \le M_v \lim_{r\to 1^-} \sup_{|\varphi(z)|>r} \frac{w(z)}{\widetilde{v}(\varphi(z))}|u(z)|.$$

□

Remark 4.3. If φ is an analytic self-map of $\mathbb{D}$, μ_1, μ_2 are weights on $\mathbb{D}$ with μ_1 almost radial and C_φ is continuous from $\mathcal{B}^{\mu_1}$ to $\mathcal{B}^{\mu_2}$, then since $\|f\|_{\mathcal{B}^{\mu_1}} = \|f'\|_{H_{\mu_1}^\infty}$ and $\|C_\varphi(f)\|_{\mathcal{B}^{\mu_2}} = \|\varphi' C_\varphi(f')\|_{H_{\mu_2}^\infty}$ for all $f \in \mathcal{B}^{\mu_1}$, we have that $\varphi' C_\varphi$ is continuous from $H_{\mu_1}^\infty$ to $H_{\mu_2}^\infty$. Hence the argument in the proof of the above result implies that

$$\|C_\varphi\|_e^{\mathcal{B}^{\mu_1} \to \mathcal{B}^{\mu_2}} \le M_{\mu_1} \lim_{r\to 1^-} \sup_{|\varphi(z)|>r} \frac{\mu_2(z)}{\widetilde{\mu_1}(\varphi(z))}|\varphi'(z)|. \tag{4.1}$$

Now we are prepared to state and prove the main result of this section.

Theorem 4.4. *Suppose that $u : \mathbb{D} \to \mathbb{C}$ and $\varphi : \mathbb{D} \to \mathbb{D}$ are holomorphic. If v, w are weights on $\mathbb{D}$ and v is typical and almost radial and uC_φ is bounded from H_v^∞ to H_w^∞, then*

$$\|uC_\varphi\|_e^{H_v^\infty \to H_w^\infty} \simeq \limsup_{n\to\infty} \frac{\|u\varphi^n\|_{H_w^\infty}}{\|g_n\|_{H_v^\infty}}. \tag{4.2}$$

Relation (4.2) denotes that there is an M_v such that

$$\frac{1}{M_v} \limsup_{n\to\infty} \frac{\|u\varphi^n\|_{H_w^\infty}}{\|g_n\|_{H_v^\infty}} \le \|uC_\varphi\|_e^{H_v^\infty \to H_w^\infty} \le M_v \limsup_{n\to\infty} \frac{\|u\varphi^n\|_{H_w^\infty}}{\|g_n\|_{H_v^\infty}}. \tag{4.3}$$

Proof. Let $n \in \mathbb{N}$. Then $f_n = g_n / \|g_n\|_{H_v^\infty}$ satisfies $\|f_n\|_{H_v^\infty} = 1$. Now, for any compact operator K from H_v^∞ to H_w^∞, we have that

$$\|uC_\varphi - K\| \ \ge \ \|(uC_\varphi - K)(f_n)\|_{H_w^\infty} \ge \|uC_\varphi(f_n)\|_{H_w^\infty} - \|K(f_n)\|_{H_w^\infty}.$$

The reader can verify that $f_n \to 0$ uniformly on compacta. Using this fact and Lemma 4.1, we can take the lim sup of all three quantities in the above chain of inequalities as $n \to \infty$ to deduce that

$$\|uC_\varphi\|_e^{H_v^\infty \to H_w^\infty} \ge \limsup_{n\to\infty} \|uC_\varphi(f_n)\|_{H_w^\infty} = \limsup_{n\to\infty} \frac{\|u\varphi^n\|_{H_w^\infty}}{\|g_n\|_{H_v^\infty}}.$$

Therefore, the leftmost inequality in the Relations (4.3) is proven. We now prove the rightmost inequality there.

Let $r \in (0,1)$ be fixed again, and let $N \in \mathbb{W}$. Then by Lemma 2.3 there is a $C_v > 0$ such that

$$\begin{aligned}\sup_{|\varphi(z)|>r} \frac{w(z)|u(z)|}{\widetilde{v}(\varphi(z))} &\leq C_v \sup_{|\varphi(z)|>r} \frac{w(z)|u(z)|}{\bar{v}(\varphi(z))} \\ &\leq C_v \sup_{n>N} \frac{\|u\varphi^n\|_{H_w^\infty}}{\|g_n\|_{H_v^\infty}} + C_v \sup_{|\varphi(z)|>r} \sup_{0\leq n\leq N} \frac{|u(z)|w(z)}{\|g_n\|_{H_v^\infty}}.\end{aligned}$$

The second inequality above uses the fact that $|\varphi(z)| < 1$ for all $z \in \mathbb{D}$. On the other hand, uC_φ is bounded from H_v^∞ to H_w^∞ by assumption, so by Inequality (3.2), there is a $C_v > 0$ such that

$$w(z)|u(z)| \leq C_v \widetilde{v}\left(\varphi(z)\right).$$

Hence, we have that

$$\sup_{|\varphi(z)|>r} \sup_{0\leq n\leq N} \frac{|u(z)|w(z)}{\|g_n\|_{H_v^\infty}} \leq C_v \sup_{|\varphi(z)|>r} \frac{\widetilde{v}(\varphi(z))}{\|g_{n_0}\|_{H_v^\infty}},$$

where $\|g_{n_0}\|_{H_v^\infty} = \max\left\{\|g_n\|_{H_v^\infty} : 0 \leq n \leq N\right\}$. Therefore, since $\widetilde{v}$ is a typical weight, we conclude that

$$\sup_{|\varphi(z)|>r} \frac{\widetilde{v}(\varphi(z))}{\|g_{n_0}\|_{H_v^\infty}} \to 0 \text{ as } r \to 1^-.$$

This fact and Lemma 4.2 together imply that there is a $C_v > 0$ such that for all $N \in \mathbb{W}$

$$\|uC_\varphi\|_e^{H_v^\infty \to H_w^\infty} \leq C_v \lim_{r\to 1} \sup_{|\varphi(z)|>r} \frac{w(z)|u(z)|}{\widetilde{v}(\varphi(z))} \leq C_v \sup_{n>N} \frac{\|u\varphi^n\|_{H_w^\infty}}{\|g_n\|_{H_v^\infty}}. \tag{4.4}$$

The statement of the theorem now follows by letting $N \to \infty$ in the above chain of inequalities. □

Since a linear operator on a normed vector space is compact if and only if its essential norm is 0, the following statement is an immediate consequence of Theorem 4.4:

Corollary 4.5. *Suppose that $u : \mathbb{D} \to \mathbb{C}$ and $\varphi : \mathbb{D} \to \mathbb{D}$ are holomorphic. If v, w are weights on $\mathbb{D}$ and v is typical and almost radial and uC_φ is bounded from H_v^∞ to H_w^∞, then uC_φ is compact from H_v^∞ to H_w^∞ if and only if*

$$\lim_{n\to\infty} \frac{\|u\varphi^n\|_{H_w^\infty}}{\|g_n\|_{H_v^\infty}} = 0.$$

Next, we apply Theorem 4.4 to obtain the following statement, which estimates the essential norm and characterizes the compactness of C_φ between μ-Bloch spaces:

Corollary 4.6. *Suppose that $\varphi : \mathbb{D} \to \mathbb{D}$ is holomorphic. Let μ_1, μ_2 be weights on $\mathbb{D}$, and suppose that μ_1 is typical and almost radial and C_φ is bounded from $\mathcal{B}^{\mu_1}$ to $\mathcal{B}^{\mu_2}$. Then the following statements hold:*

(a) *We have that*

$$\|C_\varphi\|_e^{\mathcal{B}^{\mu_1}\to\mathcal{B}^{\mu_2}} \simeq \limsup_{n\to\infty} \frac{\|\varphi'\varphi^n\|_{H^\infty_{\mu_2}}}{\|g_n\|_{H^\infty_{\mu_1}}} = \limsup_{n\to\infty} \frac{\|\varphi^n\|_{\mathcal{B}^{\mu_2}}}{\|g_n\|_{\mathcal{B}^{\mu_1}}}.$$

(b) $C_\varphi : \mathcal{B}^{\mu_1} \to \mathcal{B}^{\mu_2}$ *is compact if and only if*

$$\lim_{n\to\infty} \frac{\|\varphi^n\|_{\mathcal{B}^{\mu_2}}}{\|g_n\|_{\mathcal{B}^{\mu_1}}} = 0.$$

Proof. Statement (a) above immediately implies (b), so we prove (a). We proceed in a way that is similar to the start of the proof of Theorem 4.4. That is, we have that

$$\|C_\varphi\|_e^{\mathcal{B}^{\mu_1}\to\mathcal{B}^{\mu_2}} \geq \limsup_{n\to\infty} \frac{\|\varphi^n\|_{\mathcal{B}^{\mu_2}}}{\|g_n\|_{\mathcal{B}^{\mu_1}}}.$$

On the other hand, Remark 4.3 implies that exists a constant $C_{\mu_1} > 0$ such that

$$\|C_\varphi\|_e^{\mathcal{B}^{\mu_1}\to\mathcal{B}^{\mu_2}} \leq C_{\mu_1} \lim_{r\to1^-} \sup_{|\varphi(z)|>r} \frac{\mu_2(z)}{\widetilde{\mu_1}(\varphi(z))} |\varphi'(z)|.$$

Thus, the inequalities (4.4) in the second part of the proof of Theorem 4.4, implies that

$$\|C_\varphi\|_e^{\mathcal{B}^{\mu_1}\to\mathcal{B}^{\mu_2}} \leq C_{\mu_1} \limsup_{n\to\infty} \frac{\|\varphi'\varphi^n\|_{H^\infty_{\mu_2}}}{\|g_n\|_{H^\infty_{\mu_1}}}.$$

The statement in the corollary follows. □

Example 4.7. When $\mu_1(z) := (1-|z|)\log\left(\frac{3}{1-|z|}\right)$ and μ_2 is any weight, Corollary 4.6 reduces to a result appearing in [5]. When $\mu_1(z) := (1-|z|)^\alpha \log^\beta\left(\frac{3}{1-|z|}\right)$ with $\alpha > 0$ and $\beta \geq 0$, and μ_2 is any weight, Corollary 4.6 reduces to a recent result in [8] about compactness of C_φ.

As an additional application of Theorem 4.4, we give the following characterization of the holomorphic self-maps φ of $\mathbb{D}$ and the holomorphic functions $u : \mathbb{D} \to \mathbb{C}$ that induce a compact uC_φ on $\mathcal{B}^{\log}$.

Corollary 4.8. *Suppose that* $u : \mathbb{D} \to \mathbb{C}$ *and* $\varphi : \mathbb{D} \to \mathbb{D}$ *are holomorphic. Then* uC_φ *is compact on* $\mathcal{B}^{\log}$ *if and only if the following conditions hold*

(1) $\displaystyle\lim_{n\to\infty} \frac{(n+1)\|J_u(\varphi^n)\|_{\mathcal{B}^{\log}}}{\|g_{n+1}\|_{\mathcal{B}^{w_{\log}}}} = 0,$ *and*

(2) $\displaystyle\lim_{n\to\infty} \frac{\|I_u(\varphi^n)\|_{\mathcal{B}^{\log}}}{\|g_n\|_{\mathcal{B}^{\log}}} = 0,$

where I_u *and* J_u *are defined as in Corollary 3.4.*

Proof. Ye in [20] showed that uC_φ is compact on $\mathcal{B}^{\log}$ if and only if the following are satisfied:

(1) $\displaystyle\lim_{|\varphi(z)|\to1^-} \frac{v_{\log}(z)}{w_{\log}(\varphi(z))} |u'(z)| = 0,$ and

(2) $\displaystyle\lim_{|\varphi(z)|\to 1^-} \frac{v_{\log}(z)}{v_{\log}(\varphi(z))} |\varphi'(z)u(z)| = 0.$

The above conditions together Corollary 4.3 in [6] imply that $u'C_\varphi : H^\infty_{w_{\log}} \to H^\infty_{v_{\log}}$ and $\psi C_\varphi : H^\infty_{v_{\log}} \to H^\infty_{v_{\log}}$ are compact, where $\psi = u\varphi'$. By Corollary 4.6 (b), the compactness of these two operators is equivalent to the statement that both of the following conditions hold:

$$\lim_{n\to\infty} \frac{\|u'\varphi^n\|_{H^\infty_{v_{\log}}}}{\|g_n\|_{H^\infty_{w_{\log}}}} = 0, \quad \text{and} \quad \lim_{n\to\infty} \frac{\|\psi\varphi^n\|_{H^\infty_{v_{\log}}}}{\|g_n\|_{H^\infty_{v_{\log}}}} = 0.$$

The statement in the corollary now follows. □

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References

1. M. Anderson and J. Duncan, *Duals of Banach spaces of entire functions*, Glasgow Math. J. **32** (1990), 215–220.
2. K.D. Bierstedt, J. Bonet and J. Taskinen, *Associated weights and spaces of holomorphic functions*, Studia Math. **127** (1998), 137–168.
3. K.D. Bierstedt, R. Meise and W.H. Summers, *A projective description of weighted inductive limits*, Trans. Amer. Math. Soc. **272** (1982), 107–160.
4. J. Bonet, P. Domański and M. Lindström, *Essential norm and weak compactness of composition operators on weighted Banach spaces of analytic function*, Canad. Math. Bull. **42** (1999), 139–148.
5. R. Castillo, D. Clahane, J. Farías López and J.C. Ramos-Fernández, *Composition operators from logarithmic Bloch spaces to weighted Bloch spaces*, Appl. Math. Comput. **219** (2013), 6692–6706.
6. M. Contreras and A. Hernández-Díaz, *Weighted composition operators in weighted Banach spaces of analytic functions*, J. Austral. Math. Soc. Ser. A, **69** (2000), 41–60.
7. C.C. Cowen and B.D. MacCluer, *Composition operators on spaces of analytic functions*, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1995.
8. A. García Ortiz and J.C. Ramos-Fernández, *Composition operators from logarithmic Bloch spaces to Bloch-type spaces*, Georgian Math. J. **20** (2013), 671–686.
9. O. Hyvärinen, M. Kemppainen, M. Lindström, A. Rautio and E. Saukko, *The essential norm of weighted composition operators on weighted Banach spaces of analytic functions*, Integral Equations Operator Theory **72** (2012), 151–157.
10. O. Hyvärinen and M. Lindström, *Estimates of essential norms of weighted composition operators between Bloch-type spaces*, J. Math. Anal. Appl. **393** (2012), 38-44.
11. F. Jafari, T. Tonev, E. Toneva and K. Yale, *Holomorphic flows, cocycles, and coboundaries*, Michigan Math. J. **44** (1997), no. 2, 239–253.
12. J.S. Manhas, *Weighted composition operators and dynamical systems on weighted spaces of holomorphic functions on Banach spaces*, Ann. Funct. Anal. **4** (2013), no. 2, 58–71.
13. B. MacCluer and R. Zhao, *Essential norms of weighted composition operators between Bloch-type spaces*, Rocky Mountain J. Math. **33** (2003), 1437–1458.
14. A. Montes-Rodríguez, *Weighted composition operators on weighted Banach spaces of analytic functions*, J. London Math. Soc. (2) **61** (2000), 872-884.

15. J.H. Shapiro, *Composition Operators and Classical Function Theory*, Springer-Verlag, New York, 1993.
16. A. Siskakis, *Weighted composition semigroups on Hardy spaces*, Proceedings of the symposium on operator theory (Athens, 1985). Linear Algebra Appl. **84** (1986), 359–371.
17. S. Stević, *On new Bloch-type spaces*, Appl. Math. Comput. **215** (2009), 841–849.
18. M. Tjani, *Compact composition operators on Besov spaces*, Trans. Amer. Math. Soc. **355** (2003), 4683–4698.
19. H. Wulan, D. Zheng and K. Zhu, *Compact composition operators on BMOA and the Bloch space*, Proc. Amer. Math. Soc. **137** (2009), 3861–3868.
20. S. Ye, *A weighted composition operator on the logarithmic Bloch space*, Bull. Korean Math. Soc. **47** (2010), no. 3, pp. 527–540.
21. R. Zhao, *Essential norms of composition operators between Bloch type spaces*, Proc. Amer. Math. Soc. **138** (2010), 2537–2546.
22. K. Zhu, *Operator theory in function spaces*, Marcel Dekker, New York, 1990.
23. K. Zhu, *Bloch type spaces of analytic functions*, Rocky Mountain J. Math. **23** (1993), 1143–1177.

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