FRAMES IN KREIN SPACES ARISING FROM A NON-REGULAR W-METRIC

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ABSTRACT. A definition of frames in Krein spaces is stated and a complete characterization is given by comparing them to frames in the associated Hilbert space. The basic tools of frame theory are described in the formalism of Krein spaces. It is shown how to transfer a frame for Hilbert spaces to Krein spaces given by a W-metric, where the Gram operator W is not necessarily regular and possibly unbounded.

1. INTRODUCTION

The theory of frames for Hilbert spaces originates from [7] and was further developed in [5, 6]. Frames can be thought of as "over-complete bases" and their over-completeness makes them more flexible than orthonormal bases. They proved to be a powerful tool, e.g., in signal processing and wavelet analysis [9]. It is only natural that one wants to have the same tools available for Krein spaces. This, of course, can be done without any changes by considering the associated Hilbert space. However we prefer a more direct approach by taking the structure of Krein spaces into account, avoiding thus the switching between Krein and Hilbert spaces. All results of the present paper are part of the first author's master thesis [10]. Similar ideas have been developed independently in [8] and [13].

Our approach to the theory of frames for Krein spaces is presented in Section 3. There we give a definition of frames for Krein spaces by replacing the positive definite inner product in the definition of a frame for a Hilbert space by a (possibly) indefinite inner product. As expected, we prove in Theorem 3.3 that the theory of frames for a Krein space and the theory of frames for the associated Hilbert space are equivalent. Then we reformulate the basic tools of frame theory in the language of Krein space. For instance, the pre-frame operator is allowed to have a Krein space as its domain. Our definitions are such that the frame operator (Definition 3.5) and the Frame Decomposition Theorem (Theorem 3.6) are exactly as in the Hilbert space case with the positive definite inner product replaced by the inner product of the Krein space. Furthermore, we discuss canonical dual frames (Proposition 3.8), tight frames and their relation to J-orthonormalized bases (Proposition 3.9), and the relation between frames and orthogonal projections commuting with the fundamental symmetry (Proposition 3.10).

In Section 4, we apply our theory to Krein spaces given by a non-regular or/and unbounded Gram operator W acting on a Hilbert space $\mathcal{H}$. The fundamental observation (Proposition 4.1) is that, whenever the Gram operator W is non-regular or unbounded, a frame for $\mathcal{H}$ can never be a frame for the Krein space $\mathcal{H}_W$ constructed from W. Nevertheless, we show how to transfer any frame for the Hilbert space $\mathcal{H}$ to a frame for the Krein space $\mathcal{H}_W$. The basic idea is to construct a unitary operator between $\mathcal{H}_W$ and $\mathcal{H}$ starting from the square root $\sqrt{|W|}$. For a better understanding, we distinguish between the cases where W is bounded (Theorem 4.2), and where W is unbounded but $0 \notin \mathrm{spec}(W)$ (Theorem 4.3). The general situation is treated in Theorem 4.4. Note that these results also apply to positive Gram operators W when $\mathcal{H}_W$ is actually a Hilbert space.

2. PRELIMINARIES

The purpose of this section is to fix notations and to recall the basic elements of frame theory. For more details on Krein spaces, we recommend [1] and [2]. A comprehensive introduction to frame theory can be found in [4].

2.1. **Krein spaces.** Throughout this paper, $(\mathfrak{K}, [\cdot, \cdot])$ denotes a Krein space with fundamental decomposition $\mathfrak{K}_+ \oplus \mathfrak{K}_-$ and fundamental symmetry J given by

$$J(k^+ + k^-) = k^+ - k^-, \qquad k^+ + k^- \in \mathfrak{K}_+ \oplus \mathfrak{K}_-, \tag{2.1}$$

such that the J-inner product

$$[h^+ + h^-, k^+ + k^-]_J := [h^+ + h^-, J(k^+ + k^-)] = [h^+, k^+] - [h^-, k^-], \quad h^\pm, k^\pm \in \mathfrak{K}_\pm, \tag{2.2}$$

turns $(\mathfrak{K}, [\cdot, \cdot]_J)$ into a Hilbert space. The positive definite inner product $[\cdot, \cdot]_J$ defines a topology on $\mathfrak{K}$ by the J-norm

$$\|k\|_J := \sqrt{[k, k]_J} = \sqrt{[k, Jk]}, \qquad k \in \mathfrak{K},$$

and $\mathfrak{K}_+ \oplus \mathfrak{K}_-$ becomes the orthogonal sum of Hilbert spaces. Note that $J^2 = 1$ by (2.1).

Topological concepts such as convergence and bounded linear operators refer to the topology induced by the Hilbert space norm $\| \cdot \|_J$. The angle brackets

$\langle \cdot, \cdot \rangle$ are reserved to denote the positive definite inner product of a given Hilbert space $\mathcal{H}$. When no confusion can arise, we write shortly $\mathfrak{K}$ for a Krein space $(\mathfrak{K}, [\cdot, \cdot])$, and $\mathcal{H}$ for a Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$. The symbol $\mathfrak{k}_2(\mathbb{N})$ will be used if $\mathfrak{K} = \ell_2(\mathbb{N})$ as a complex vector space. A J-orthonormal basis is a complete family $\{e_n\}_{n \in \mathfrak{N}} \subset \mathfrak{K}$ such that $[e_n, e_m] = 0$ for $n \neq m$ and $|[e_n, e_n]| = 1$. A subspace $V \subset \mathfrak{K}$ is said to be uniformly J-positive (resp. uniformly J-negative) if there exists an $\varepsilon > 0$ such that $[v, v] \geq \varepsilon \|v\|_J^2$ (resp. $-[v, v] \geq \varepsilon \|v\|_J^2$) for all $v \in V$.

The unique adjoint $T^* : (\mathfrak{K}_2, [\cdot, \cdot]_2) \to (\mathfrak{K}_1, [\cdot, \cdot]_1)$ of a bounded linear operator $T : (\mathfrak{K}_1, [\cdot, \cdot]_1) \to (\mathfrak{K}_2, [\cdot, \cdot]_2)$ is always taken with respect to the specified inner products, i.e.,

$$[T^*h, k]_1 = [h, Tk]_2 \quad \text{for all} \quad k \in \mathfrak{K}_1, \ h \in \mathfrak{K}_2.$$

For instance, it follows from (2.1) and (2.2) that $J : (\mathfrak{K}, [\cdot, \cdot]) \to (\mathfrak{K}, [\cdot, \cdot])$ and $J : (\mathfrak{K}, [\cdot, \cdot]_J) \to (\mathfrak{K}, [\cdot, \cdot]_J)$ are self-adjoint, i.e., $J^* = J$. Furthermore, the identity operator

$$I_J : (\mathfrak{K}, [\cdot, \cdot]) \longrightarrow (\mathfrak{K}, [\cdot, \cdot]_J), \qquad I_J k = k, \tag{2.3}$$

has the adjoint $J : (\mathfrak{K}, [\cdot, \cdot]_J) \to (\mathfrak{K}, [\cdot, \cdot])$ since $[I_J k, h]_J = [k, h]_J = [k, Jh]$ for all $h, k \in \mathfrak{K}$. A self-adjoint operator $A = A^*$ on $(\mathfrak{K}, [\cdot, \cdot])$ is called uniformly positive, if $[k, Ak] \geq \varepsilon[k, k]_J$ for a suitable constant $\varepsilon > 0$ and all $k \in \mathfrak{K}$. Equivalently, since $[k, Ak] = [k, JAk]_J$, we have $JA \geq \varepsilon$ on the Hilbert space $(\mathfrak{K}, [\cdot, \cdot]_J)$. As a consequence, A has a bounded inverse.

The fundamental projections

$$P_+ := \tfrac{1}{2}(1 + J), \qquad P_- := \tfrac{1}{2}(1 - J) \tag{2.4}$$

act on $\mathfrak{K} = \mathfrak{K}_+ \oplus \mathfrak{K}_-$ by $P_+(k^+ + k^-) = k^+$ and $P_-(k^+ + k^-) = k^-$. Equation (2.4) implies immediately that $P_\pm$ and J commute. Moreover, P_+ and P_- are orthogonal projections, i.e. $P_\pm^2 = P_\pm = P_\pm^*$, regardless of whether we consider $[\cdot, \cdot]$ or $[\cdot, \cdot]_J$ on $\mathfrak{K}$.

We close this subsection with a short review of Krein spaces given by a *regular* Gram operator W [1]. The case when W is unbounded or $0 \in \mathrm{spec}(W)$ will be discussed in Section 4. Let thus W be a bounded self-adjoint operator on a Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ such that $0 \notin \mathrm{spec}(W)$. Define a non-degenerate inner product on $\mathcal{H}$ by

$$[f, g] := \langle f, Wg \rangle, \quad f, g \in \mathcal{H}. \tag{2.5}$$

Since $W^* = W$ and $0 \notin \mathrm{spec}(W)$, the operator J given by the polar decomposition $W = J|W|$ is self-adjoint and unitary, i.e. $J^* = J$ and $J^2 = 1$. Hence the orthogonal projections P_+ and $P_- = 1 - P_+$ from (2.4) project onto the eigenspaces corresponding to the eigenvalues 1 and -1 of J, respectively. As a consequence, J acts on the orthogonal sum $\mathcal{H} = P_+\mathcal{H} \oplus P_-\mathcal{H}$ by $J(h^+ + h^-) = h^+ - h^-$, where $h^\pm \in P_\pm \mathcal{H}$. From $J^2 = 1$, we get $[f, g]_J := \langle f, JWg \rangle = \langle f, |W|g \rangle$ for all $f, g \in \mathcal{H}$. Moreover, the condition $0 \notin \mathrm{spec}(W)$ implies that $\varepsilon \leq |W| \leq \|W\|$ for some $\varepsilon > 0$, which means that $\varepsilon \langle h, h \rangle \leq \langle h, |W|h \rangle \leq \|W\| \langle h, h \rangle$ for all $h \in \mathcal{H}$. Therefore, the Hilbert space norm $\sqrt{\langle \cdot, \cdot \rangle}$ and the J-norm $\| \cdot \|_J$ are equivalent. In particular, $\mathcal{H} = P_+\mathcal{H} \oplus P_-\mathcal{H}$ remains an orthogonal sum of Hilbert spaces with respect to $[\cdot, \cdot]_J$. Summarizing, we have shown that $(\mathcal{H}, [\cdot, \cdot])$ is a Krein space

with fundamental decomposition $\mathcal{H} = P_+\mathcal{H} \oplus P_-\mathcal{H}$ and fundamental symmetry J such that $\langle \cdot, \cdot \rangle$ and $[\cdot, \cdot]_J$ define equivalent norms.

2.2. Frames in Hilbert spaces.
Let $\mathcal{H}$ be an (infinite-dimensional) Hilbert space with inner product $\langle \cdot, \cdot \rangle$. A frame for $\mathcal{H}$ is a countable sequence $\{f_n\}_{n\in\mathbb{N}}$ in $\mathcal{H}$ such that there exist constants $0 < A \leq B < \infty$ satisfying

$$A\|f\|^2 \leq \sum_{n\in\mathbb{N}} |\langle f_n, f \rangle|^2 \leq B\|f\|^2 \quad \text{for all } f \in \mathcal{H}. \tag{2.6}$$

Given a frame $\{f_n\}_{n\in\mathbb{N}}$ for $\mathcal{H}$, one defines the pre-frame operator by

$$T_0 : \ell_2(\mathbb{N}) \longrightarrow \mathcal{H}, \quad T_0(\alpha_n)_{n\in\mathbb{N}} = \sum_{n\in\mathbb{N}} \alpha_n f_n. \tag{2.7}$$

It follows from the Bessel condition $\sum_{n\in\mathbb{N}} |\langle f_n, f \rangle|^2 \leq B\|f\|^2$ that T_0 is well defined and bounded. A direct computation shows that

$$T_0^* : \mathcal{H} \longrightarrow \ell_2(\mathbb{N}), \quad T_0^*(f) = (\langle f_n, f \rangle)_{n\in\mathbb{N}} \tag{2.8}$$

yields its adjoint. The frame operator S is defined by $S := T_0 T_0^*$. By Equations (2.7) and (2.8),

$$Sf = \sum_{n\in\mathbb{N}} \langle f_n, f \rangle f_n, \quad f \in \mathcal{H}. \tag{2.9}$$

The inequalities in (2.6) imply that $A \leq S \leq B$, thus S has bounded inverse. Inserting (2.9) into $f = SS^{-1}f = S^{-1}Sf$ and using $(S^{-1})^* = S^{-1}$, we arrive at the so-called Frame Decomposition Theorem:

$$f = \sum_{n\in\mathbb{N}} \langle S^{-1}f_n, f \rangle f_n = \sum_{n\in\mathbb{N}} \langle f_n, f \rangle S^{-1}f_n \quad \text{for all } f \in \mathcal{H}. \tag{2.10}$$

From $S^{-1}f = \sum_{n\in\mathbb{N}} \langle S^{-1}f_n, f \rangle S^{-1}f_n$ and $B^{-1} \leq S^{-1} \leq A^{-1}$, it follows that $\{S^{-1}f_n\}_{n\in\mathbb{N}}$ is a frame for $\mathcal{H}$ admitting the frame bounds $B^{-1} \leq A^{-1}$. A frame $\{g_n\}_{n\in\mathbb{N}} \subset \mathcal{H}$ satisfying

$$f = \sum_{n\in\mathbb{N}} \langle g_n, f \rangle f_n = \sum_{n\in\mathbb{N}} \langle f_n, f \rangle g_n \quad \text{for all } f \in \mathcal{H}. \tag{2.11}$$

is called a dual frame of $\{f_n\}_{n\in\mathbb{N}}$. By (2.10), $\{S^{-1}f_n\}_{n\in\mathbb{N}}$ yields an example of a dual frame of $\{f_n\}_{n\in\mathbb{N}}$, and one usually refers to it as the canonical dual frame.

3. Frames in Krein spaces

In this section we state a definition of frames in Krein spaces and prove that they are in one-to-one correspondence to frames in the associated Hilbert space. Some basic tools of frame theory such as pre-frame operator, frame operator and dual frame are then described in the language of Krein spaces.

Definition 3.1. Let $\mathfrak{K}$ be a Krein space. A countable sequence $\{k_n\}_{n\in\mathfrak{N}} \subset \mathfrak{K}$ is called a *frame for* $\mathfrak{K}$, if there exist constants $0 < A \leq B < \infty$ such that

$$A\|k\|_J^2 \leq \sum_{n\in\mathfrak{N}} |[k_n, k]|^2 \leq B\|k\|_J^2 \quad \text{for all } k \in \mathfrak{K}. \tag{3.1}$$

Remark 3.2. Since we are mostly interested in infinite-dimensional spaces, and since one can always fill up a finite frame with zero elements, we assume that $\mathfrak{N} = \mathbb{N}$ whenever the finiteness of $\mathfrak{N}$ is of no importance.

As in the Hilbert space case, we refer to A and B as frame bounds. The greatest constant A and the smallest constant B satisfying (3.1) are called optimal lower frame bound and optimal upper frame bound, respectively. A frame is tight, if one can choose $A = B$. If a frame ceases to be a frame when an arbitrary element is removed, the frame is said to be exact.

The next theorem shows that frames for a Krein space are essentially the same objects as frames for the associated Hilbert space.

Theorem 3.3. *Let $\mathfrak{K}$ be a Krein space and $\{k_n\}_{n\in\mathbb{N}}$ a sequence in $\mathfrak{K}$. The following statements are equivalent:*

   i) $\{k_n\}_{n\in\mathbb{N}}$ is a frame for the Krein space $\mathfrak{K}$ with frame bounds $A \leq B$.

   ii) $\{Jk_n\}_{n\in\mathbb{N}}$ is a frame for the Krein space $\mathfrak{K}$ with frame bounds $A \leq B$.

   iii) $\{k_n\}_{n\in\mathbb{N}}$ is a frame for the Hilbert space $(\mathfrak{K}, [\cdot, \cdot]_J)$ with frame bounds $A \leq B$.

   iv) $\{Jk_n\}_{n\in\mathbb{N}}$ is a frame for the Hilbert space $(\mathfrak{K}, [\cdot, \cdot]_J)$ with frame bounds $A \leq B$.

Proof. The equivalence of i) and iv) follows from

$$|[k, k_n]|^2 = |[Jk, Jk_n]|^2 = |[k, Jk_n]_J|^2.$$

The same argument applied to $\{Jk_n\}_{n\in\mathbb{N}}$ together with $J^2 = 1$ proves the equivalence of ii) and iii). Since J is a unitary operator on $(\mathfrak{K}, [\cdot, \cdot]_J)$, the equivalence iii) and iv) is obvious. This finishes the proof. $\qquad\square$

Recall that, given a frame $\{f_n\}_{n\in\mathbb{N}}$ for a Hilbert space $\mathcal{H}$, the pre-frame operator $T_0 : \ell_2(\mathbb{N}) \to \mathcal{H}$ is defined by (2.7). In the special case when $\mathcal{H} = \ell_2(\mathbb{N})$ and $\{f_n\}_{n\in\mathbb{N}}$ coincides with the standard basis of $\ell_2(\mathbb{N})$, T_0 is just the identity. By Theorem 3.3, we could define a pre-frame operator T for frames in a Krein space in the same way. But then, if the Hilbert space $\mathcal{H}$ is replaced by a non-trivial Krein space of sequences $\mathfrak{k}_2(\mathbb{N})$, the pre-frame operator $T : \ell_2(\mathbb{N}) \to \mathfrak{k}_2(\mathbb{N})$ can never be the identity operator in the strict sense. Therefore we also allow non-trivial Krein spaces for the domain of the pre-frame operator.

Definition 3.4. Let $(\mathfrak{K}, [\cdot, \cdot])$ be a Krein space with fundamental symmetry J and let $(\mathfrak{k}_2(\mathbb{N}), [\cdot, \cdot])$ be a Krein space with fundamental symmetry $\tilde{J}$ such that $[\cdot, \cdot]_{\tilde{J}}$ agrees with the standard inner product $\langle \cdot, \cdot \rangle$ on $\ell_2(\mathbb{N})$. Given a frame $\{k_n\}_{n\in\mathbb{N}}$ for $\mathfrak{K}$, the linear map

$$T : \mathfrak{k}_2(\mathbb{N}) \longrightarrow \mathfrak{K}, \quad T(\alpha_n)_{n\in\mathbb{N}} = \sum_{n\in\mathbb{N}} \alpha_n k_n \tag{3.2}$$

is called *pre-frame operator*.

Note that T is well defined and bounded since it factorizes as

$$(\mathfrak{k}_2(\mathbb{N}), [\cdot, \cdot]) \xrightarrow{I_{\tilde{J}}} (\ell_2(\mathbb{N}), \langle \cdot, \cdot \rangle) \xrightarrow{T_0} (\mathfrak{K}, [\cdot, \cdot]_J) \xrightarrow{I_J^{-1}} (\mathfrak{K}, [\cdot, \cdot]) \tag{3.3}$$

and all operators in (3.3) are bounded. Here we use the fact that T_0 is a pre-frame operator by Theorem 3.3.iii), apply [4, Theorem 5.5.1] to T_0, and observe that $I_{\tilde{j}}$ and I_J defined by (2.3) are bijections. Since $T = I_J^{-1} T_0 I_{\tilde{j}}$, it follows from the Hilbert space frame theory applied to T_0 that a sequence $\{k_n\}_{n\in\mathbb{N}}$ is a frame for $\mathfrak{K}$, if and only if T is well defined (i.e. bounded) and surjective.

The adjoint of T is given by

$$T^*k = \tilde{J}([k_n, k])_{n\in\mathbb{N}}, \quad k \in \mathfrak{K}. \tag{3.4}$$

Indeed, for all $(\alpha_n)_{n\in\mathbb{N}} \in \mathfrak{k}_2(\mathbb{N})$ and $k \in \mathfrak{K}$, one has

$$[T(\alpha_n)_{n\in\mathbb{N}}, k] = \sum_{n\in\mathbb{N}} \bar{\alpha}_n [k_n, k] = \langle(\alpha_n)_{n\in\mathbb{N}}, ([k_n, k])_{n\in\mathbb{N}}\rangle = [(\alpha_n)_{n\in\mathbb{N}}, \tilde{J}([k_n, k])_{n\in\mathbb{N}}].$$

To obtain a frame operator given by a formula analogous to (2.9) with $\langle\cdot,\cdot\rangle$ replaced by $[\cdot,\cdot]$, we make the following definition:

Definition 3.5. Let $(\mathfrak{K}, [\cdot,\cdot])$ be a Krein space with fundamental symmetry J, $(\mathfrak{k}_2(\mathbb{N}), [\cdot,\cdot])$ a Krein space with fundamental symmetry $\tilde{J}$ such that $[\cdot,\cdot]_{\tilde{j}}$ agrees with the standard inner product $\langle\cdot,\cdot\rangle$ on $\ell_2(\mathbb{N})$, and $\{k_n\}_{n\in\mathbb{N}} \subset \mathfrak{K}$ a frame for $\mathfrak{K}$. The operator

$$S := T \tilde{J} T^*$$

is called *frame operator*.

It follows immediately from (3.2), (3.4) and $\tilde{J}^2 = \text{id}$ that

$$Sk = \sum_{n\in\mathbb{N}} [k_n, k] k_n, \quad k \in \mathfrak{K}, \tag{3.5}$$

as desired. Moreover, S is clearly self-adjoint. If $(\mathfrak{k}_2(\mathbb{N}), [\cdot,\cdot]) = (\ell_2(\mathbb{N}), \langle\cdot,\cdot\rangle)$, then $S = TT^*$, exactly as in the Hilbert space case.

Equation (3.5) yields $[k, Sk] = \sum_{n\in\mathbb{N}} |[k_n, k]|^2$ for all $k \in \mathfrak{K}$. Replacing k by Jk and writing $[k_n, Jk] = [k_n, k]_J$, we get from Equation (3.1) and Theorem 3.3.iii) that

$$A\|k\|_J^2 \leq [k, SJk]_J \leq B\|k\|_J^2, \quad k \in \mathfrak{K}.$$

Hence SJ is a strictly positive operator on the Hilbert space $(\mathfrak{K}, [\cdot,\cdot]_J)$ satisfying

$$A \leq SJ \leq B. \tag{3.6}$$

Therefore S is invertible with bounded inverse, and

$$B^{-1} \leq JS^{-1} \leq A^{-1},$$

where we used $J^{-1} = J$. Equivalently, when viewed as positive operators on the Krein space $(\mathfrak{K}, [\cdot,\cdot])$, we have $AJ \leq S \leq BJ$ and

$$B^{-1}J \leq S^{-1} \leq A^{-1}J.$$

The invertibility of the operator S allows us to state the Frame Decomposition Theorem – the most important theorem of frame theory – exactly as in the Hilbert space case, cf. [4, Theorem 5.1.6].

Theorem 3.6 (Frame Decomposition Theorem)**.** *Let $\{k_n\}_{n\in\mathbb{N}}$ be a frame for the Krein space $\mathfrak{K}$, and let S be defined as in Definition 3.5. Then*

$$k = \sum_{n\in\mathbb{N}} [k_n, k] S^{-1} k_n, \tag{3.7}$$

$$k = \sum_{n\in\mathbb{N}} [S^{-1}k_n, k] k_n, \tag{3.8}$$

and both series converge unconditionally for all $k \in \mathfrak{K}$.

Proof. Equation (3.7) is proved by applying (3.5) to $k = S^{-1}Sk$. Since the self-adjointness of S implies the self-adjointness of S^{-1}, Equation (3.8) follows from (3.5) applied to $k = SS^{-1}k$. For the proof of the unconditional convergence, we replace again k by Jk, write $[k_n, Jk] = [k_n, k]_J$, invoke Theorem 3.3.iii), and refer to the Hilbert space case [4, Theorem 5.1.6]. $\square$

By Theorem 3.3, each frame $\{k_n\}_{n\in\mathbb{N}}$ for $(\mathfrak{K}, [\cdot,\cdot])$ gives rise to three other frames with slightly different frame operators. In the following, we will relate these frame operators to S from (3.5). First, consider the frame $\{Jk_n\}_{n\in\mathbb{N}}$ for $(\mathfrak{K}, [\cdot,\cdot])$. Denoting the corresponding frame operator by S_0, we get from (3.5)

$$S_0 k = \sum_{n\in\mathbb{N}} [Jk_n, k] Jk_n = \sum_{n\in\mathbb{N}} [k_n, Jk] Jk_n, \quad k \in \mathfrak{K}.$$

Comparing this equation with (3.5) shows that the two frame operators are related by

$$S_0 = JSJ.$$

Next, let S_1 be the frame operator of the frame $\{k_n\}_{n\in\mathbb{N}}$ for $(\mathfrak{K}, [\cdot,\cdot]_J)$. Then, by Equation (2.9), we have

$$S_1 k = \sum_{n\in\mathbb{N}} [k_n, k]_J k_n = \sum_{n\in\mathbb{N}} [k_n, Jk] k_n, \quad k \in \mathfrak{K},$$

and thus

$$S_1 = SJ.$$

Finally, with S_2 denoting the frame operator of the frame $\{Jk_n\}_{n\in\mathbb{N}}$ for $(\mathfrak{K}, [\cdot,\cdot]_J)$, we get from (2.9)

$$S_2 k = \sum_{n\in\mathbb{N}} [Jk_n, k]_J Jk_n = \sum_{n\in\mathbb{N}} [k_n, k] Jk_n, \quad k \in \mathfrak{K},$$

so that

$$S_2 = JS.$$

Recall that, by Equation (2.11), a dual frame of the frame $\{k_n\}_{n\in\mathbb{N}}$ for the Hilbert space $(\mathfrak{K}, [\cdot,\cdot]_J)$ is a frame $\{g_n\}_{n\in\mathbb{N}}$ for $(\mathfrak{K}, [\cdot,\cdot]_J)$ satisfying

$$k = \sum_{n\in\mathbb{N}} [g_n, k]_J k_n, \quad \text{for all } k \in \mathfrak{K}.$$

We will now state an analogous definition for Krein spaces and then describe dual frames in terms of the fundamental symmetry J and the frame operator S. These dual frames are called *canonical* dual frames.

Definition 3.7. Let $\{k_n\}_{n\in\mathbb{N}}$ be a frame for the Krein space $\mathfrak{K}$. A frame $\{h_n\}_{n\in\mathbb{N}}$ for $\mathfrak{K}$ is called a *dual frame* of $\{k_n\}_{n\in\mathbb{N}}$ if

$$k = \sum_{n\in\mathbb{N}}[h_n, k]\, k_n,$$

for all $k \in \mathfrak{K}$.

Proposition 3.8. *Let S be the frame operator of the frame $\{k_n\}_{n\in\mathbb{N}}$ for the Krein space $\mathfrak{K}$. Then*

i) $\{S^{-1}k_n\}$ is a dual frame of $\{k_n\}_{n\in\mathbb{N}}$ for the Krein space $\mathfrak{K}$.

ii) $\{JS^{-1}k_n\}$ is a dual frame of $\{Jk_n\}_{n\in\mathbb{N}}$ for the Krein space $\mathfrak{K}$.

iii) $\{JS^{-1}k_n\}$ is a dual frame of $\{k_n\}_{n\in\mathbb{N}}$ for the Hilbert space $(\mathfrak{K}, [\cdot, \cdot]_J)$.

iv) $\{S^{-1}k_n\}$ is a dual frame of $\{Jk_n\}_{n\in\mathbb{N}}$ for the Hilbert space $(\mathfrak{K}, [\cdot, \cdot]_J)$.

If $0 < A \leq B < \infty$ are frame constants for $\{k_n\}_{n\in\mathbb{N}}$, then all these dual frames admit the frame constants $0 < B^{-1} \leq A^{-1} < \infty$.

Proof. First, since $S_1 = SJ$ is the frame operator of $\{k_n\}_{n\in\mathbb{N}}$ for $(\mathfrak{K}, [\cdot, \cdot]_J)$ with inverse $S_1^{-1} = JS^{-1}$, it follows from the Hilbert space theory [4, Lemma 5.1.5] that $\{S_1^{-1}k_n\}_{n\in\mathbb{N}} = \{JS^{-1}k_n\}_{n\in\mathbb{N}}$ is a dual frame of $\{k_n\}_{n\in\mathbb{N}}$ for $(\mathfrak{K}, [\cdot, \cdot]_J)$ with frame constants $0 < B^{-1} \leq A^{-1} < \infty$. This proves iii). Moreover, by Theorem 3.3, all the other dual frames given in the proposition are indeed frames for the corresponding spaces and admit the same frame constants. Now i) is a direct consequence of (3.8), and ii) follows by inserting the frame $\{Jk_n\}_{n\in\mathbb{N}}$ and the inverse frame operator $S_0^{-1} = JS^{-1}J$ into (3.8). Finally iii) implies iv) by applying the unitary transformation J to the Hilbert space $(\mathfrak{K}, [\cdot, \cdot]_J)$. $\qquad\square$

If $\{k_n\}_{n\in\mathbb{N}}$ is a tight frame for $(\mathfrak{K}, [\cdot, \cdot])$, i.e. $A = B$ in (3.1), then $S = AJ$ by (3.6), and Equations (3.7) and (3.8) become

$$k = AJ\Big(\sum_{n\in\mathbb{N}}[k_n, k]\, k_n\Big) = A\sum_{n\in\mathbb{N}}[k_n, k]_J\, k_n.$$

Note that a frame is tight in $(\mathfrak{K}, [\cdot, \cdot])$ if and only if it is tight in $(\mathfrak{K}, [\cdot, \cdot]_J)$. To see this, it suffices to replace k by Jk in (3.1). Therefore, structure results for tight frames on a Hilbert space can be transferred to the Krein space setting.

Let us also discuss the situation of tight frames with normalized elements. The next proposition shows that these frames are actually J-orthonormalized bases.

Proposition 3.9. *Let $\{k_n\}_{n\in\mathfrak{N}}$ be a tight frame for the Krein space $\mathfrak{K}$ with frame bounds $A = B = 1$. Assume that $|[k_n, k_n]| = 1$ for all $n \in \mathfrak{N}$. Then $\{k_n\}_{n\in\mathfrak{N}}$ is a J-orthonormalized basis for $\mathfrak{K}$.*

Proof. For $n \in \mathfrak{N}$, write $k_n = P_+k_n + P_-k_n =: k_n^+ + k_n^- \in \mathfrak{K}_+ \oplus \mathfrak{K}_-$, where P_+ and P_- denote the fundamental projections of (2.4). Set $\mathfrak{N}_+ := \{n \in \mathfrak{N} : [k_n, k_n] = 1\}$ and $\mathfrak{N}_- := \{n \in \mathfrak{N} : [k_n, k_n] = -1\}$. Let $m \in \mathfrak{N}_+$. Then

$$1 = [k_m^+, k_m^+] + [k_m^-, k_m^-] \leq [k_m^+, k_m^+].$$

Inserting $k := k_m^+$ and $A = B = 1$ into (3.1) gives

$$[k_m^+, k_m^+] = \left|[k_m^+, k_m^+]\right|^2 + \sum_{n\in\mathfrak{N}\setminus\{m\}}\left|[k_m^+, k_n]\right|^2. \tag{3.9}$$

If we had $[k_m^+, k_m^+] > 1$, we would get a contradiction. Hence $[k_m^+, k_m^+] = 1$ and therefore $k_m^- = 0$. Now (3.9) implies that $[k_m, k_n] = [k_m^+, k_n] = 0$ for all $n \in \mathfrak{N}$, $n \neq m$. Replacing $(\mathfrak{K}, [\cdot, \cdot])$ by $(\mathfrak{K}, -[\cdot, \cdot])$ shows that the same holds for all $m \in \mathfrak{N}_-$. The rest of the proof is routine. $\qquad\square$

The following proposition establishes a link to J-frames as defined in [8]. We will explain this after the proof.

Proposition 3.10. *Let $\mathfrak{K}$ be a Krein space with fundamental symmetry J, and let P be an orthogonal projection commuting with J.*

If $\{k_n\}_{n \in \mathbb{N}}$ is a frame for $\mathfrak{K}$ with frame bounds $A \leq B$, then $\{Pk_n\}_{n \in \mathbb{N}}$ is a frame for $P\mathfrak{K}$ and $\{(1-P)k_n\}_{n \in \mathbb{N}}$ is a frame for $(1-P)\mathfrak{K}$, both admitting the same frame bounds.

Conversely, if $\{k_n^+\}_{n \in \mathfrak{N}_+}$ is a frame for $P\mathfrak{K}$ and $\{k_n^-\}_{n \in \mathfrak{N}_-}$ is one for $(1-P)\mathfrak{K}$, both with frame bounds $A \leq B$, then $\{k_n^+\}_{n \in \mathfrak{N}_+} \cup \{k_n^-\}_{n \in \mathfrak{N}_-}$ is a frame for $\mathfrak{K}$ admitting the same frame bounds.

Proof. Since P commutes with J, the subspaces $P\mathfrak{K}$ and $(1-P)\mathfrak{K}$ of $\mathfrak{K}$ are Krein spaces with fundamental symmetry PJ and $(1-P)J$, respectively. Given a frame $\{k_n\}_{n \in \mathfrak{N}}$ for $\mathfrak{K}$ with frame bounds $A \leq B$, we have for all $k \in P\mathfrak{K}$

$$A\|k\|_J^2 = A\|Pk\|_J^2 \leq \sum_{n \in \mathbb{N}} |[Pk, k_n]|^2 = \sum_{n \in \mathbb{N}} |[k, Pk_n]|^2 \leq B\|Pk\|_J^2 = B\|k\|_J^2,$$

hence $\{Pk_n\}_{n \in \mathbb{N}}$ is a frame for $P\mathfrak{K}$ with frame bounds $A \leq B$. The same remains true for P replaced by $1 - P$. This completes the prove of i).

Now let $\{k_n^+\}_{n \in \mathfrak{N}_+}$ and $\{k_n^-\}_{n \in \mathfrak{N}_-}$ be two frames satisfying the assumptions stated in the proposition. For $k \in \mathfrak{K}$, set $k^+ := Pk$ and $k^- := (1-P)k$. Note that $[k, k_n^+] = [k, Pk_n^+] = [Pk, k_n^+] = [k^+, k_n^+]$ and, similarly, $[k, k_n^-] = [k^-, k_n^-]$. From $PJ = JP$, it follows that $\|k\|_J^2 = \|k^+\|_J^2 + \|k^-\|_J^2$. Therefore

$$\begin{aligned}
A\|k\|_J^2 &= A\|k^+\|_J^2 + A\|k^-\|_J^2 \\
&\leq \sum_{n \in \mathfrak{N}_+} \left|[k^+, k_n^+]\right|^2 + \sum_{n \in \mathfrak{N}_-} \left|[k^-, k_n^-]\right|^2 = \sum_{n \in \mathfrak{N}_+} \left|[k, k_n^+]\right|^2 + \sum_{n \in \mathfrak{N}_-} \left|[k, k_n^-]\right|^2 \\
&\leq B\|k^+\|_J^2 + B\|k^-\|_J^2 = B\|k\|_J^2,
\end{aligned}$$

which finishes the proof. $\qquad\square$

The definition of a J-frame given in [8] can be rephrased as follows: Given a Bessel sequence $\{f_n\}_{n \in \mathbb{N}}$ in $(\mathfrak{K}, [\cdot, \cdot]_J)$ (i.e., $\{f_n\}_{n \in \mathbb{N}}$ satisfies the upper bound condition in (2.6)), consider the inner product on $\mathfrak{k}_2(\mathbb{N})$ given by

$$[e_n, e_n] := \begin{cases} 1 & \text{if } [f_n, f_n] \geq 0, \\ -1 & \text{if } [f_n, f_n] < 0, \end{cases}$$

where $\{e_n\}_{n \in \mathbb{N}}$ denotes the standard basis of the sequence space $\mathfrak{k}_2(\mathbb{N})$. Let $\tilde{J}$ denote the fundamental symmetry of $(\mathfrak{k}_2(\mathbb{N}), [\cdot, \cdot])$ such that $[\cdot, \cdot]_{\tilde{J}}$ becomes the standard inner product $\langle \cdot, \cdot \rangle$ on $\ell_2(\mathbb{N})$. Then the fundamental projection $P_+ = \frac{1}{2}(1 + \tilde{J})$ projects onto the closed subspace generated by those e_n satisfying $[e_n, e_n] = 1$. Now, $\{f_n\}_{n \in \mathbb{N}}$ is a J-frame if the ranges $\operatorname{ran}(TP_+)$ and $\operatorname{ran}(T(1-P_+))$ are maximal

uniformly J-positive and J-negative subspaces of $\mathfrak{K}$, respectively, where T denotes the pre-frame operator (3.2). In [8, Example 3.3], it was shown that not every frame in $\mathfrak{K}$ is a J-frame. As a consequence, Definition 3.1 and the definition of a J-frame are not equivalent. However, by Proposition 3.10, each frame $\{f_n\}_{n\in\mathbb{N}}$ in $\mathfrak{K}$ gives rise to a J-frame by considering $\{\frac{1}{2}(1+J)f_n\}_{n\in\mathbb{N}} \cup \{\frac{1}{2}(1-J)f_n\}_{n\in\mathbb{N}}$ and omitting zero elements if required.

We close this section with a few remarks concerning examples. Let $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ be a Hilbert space and J a bounded linear operator on $\mathcal{H}$ satisfying $J^2 = 1$ and $J^* = J$. Define $[h, k] := \langle h, Jk\rangle$ for all $h, k \in \mathcal{H}$. Then $(\mathcal{H}, [\cdot,\cdot])$ is a Krein space with fundamental symmetry J. Note that, by considering the associated Hilbert space with inner product $[\cdot,\cdot]_J$, each Krein space is of this type. By Theorem 3.3, all frames for the Krein space $(\mathcal{H}, [\cdot,\cdot])$ are obtained from frames for the Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$.

Our interest in frames for Krein spaces originates from L_2-spaces $\mathcal{L}_2(\Omega, \mu)$. If μ is a positive measure on a σ-algebra over Ω and φ is a measurable real function such that $0 < \operatorname{ess\,inf}|\varphi| \leq \operatorname{ess\,sup}|\varphi| < \infty$, then

$$[f, g] := \int \bar{f} g \varphi \,\mathrm{d}\mu, \quad f, g \in \mathcal{L}_2(\Omega, \mu), \tag{3.10}$$

defines an (indefinite) inner product such that $(\mathcal{L}_2(\Omega, \mu), [\cdot,\cdot])$ becomes a Krein space with fundamental symmetry J given by the multiplication by the sign function of φ.

Now, if we allow $\operatorname{ess\,inf}|\varphi| = 0$ (but require without loss of generality that $\mu\{\varphi = 0\} = 0$), then $(\mathcal{L}_2(\Omega, \mu), [\cdot,\cdot]_J)$ is not complete. In order to obtain a Krein space with the inner product determined by (3.10), one has to take a completion. On the dense subspace $\mathcal{L}_2(\Omega, \mu)$, the positive inner product is then given by

$$[f, g]_J = \int \bar{f} g |\varphi| \,\mathrm{d}\mu.$$

By Theorem 3.3, we may apply the frame theory either to the Krein space or to the associated Hilbert space. However, by passing from φ to $|\varphi|$, one might lose some desired properties of φ. For instance, if φ is differentiable, $|\varphi|$ does not necessarily have to be so. Therefore we prefer to work in the Krein space.

On an abstract level, we may consider the multiplication by φ as an operator on $\mathcal{L}_2(\Omega, \mu)$,

$$(W_\varphi f)(x) := \varphi(x) f(x). \tag{3.11}$$

The operator W_φ is unbounded, if $\operatorname{ess\,sup}|\varphi| = \infty$, and self-adjoint, if we set $\operatorname{dom}(W_\varphi) := \{f \in \mathcal{L}_2(\Omega, \mu) : \int |f|^2 |\varphi|^2 \,\mathrm{d}\mu < \infty\}$. With $\langle\cdot,\cdot\rangle$ denoting the standard inner product on $\mathcal{L}_2(\Omega, \mu)$, Equation (3.10) reads

$$[f, g] = \langle f, W_\varphi g\rangle, \quad f, g \in \operatorname{dom}(W_\varphi).$$

Furthermore, $\operatorname{ess\,inf}|\varphi| = 0$ if and only if 0 belongs to the spectrum of W_φ. This is the general setting which will concern us in the next section.

4. Frames in Hilbert spaces with W-metric

The objective of this section is to show how to transfer a frame for a Hilbert space $\mathcal{H}$ to a non-regular Krein space $\mathcal{H}_W$. Although, on a technical level, we could treat the bounded and the unbounded case at once, we start by distinguishing between the following situations: First we suppose that the Gram operator W is bounded to illustrate the completion process enforced by $0 \in \mathrm{spec}(W)$. Second we allow W to be an unbounded but assume that $0 \notin \mathrm{spec}(W)$. The difference between these two cases is that in the latter, we can identify $\mathcal{H}_W$ with a subspace of $\mathcal{H}$, and in the former, we have to extend $\mathcal{H}$ by taking a completion. The general case will be obtained by combining both situations.

Throughout this section, $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ stands for a separable Hilbert space, and W denotes a self-adjoint operator with domain $\mathrm{dom}(W) \subset \mathcal{H}$ and polar decomposition $W = J|W|$. We assume that $\ker(W) = \{0\}$. Then J is a unitary self-adjoint operator. The letter E will be used for the projection-valued measure on the Borel σ-algebra $\Sigma(\mathbb{R})$ such that

$$W = \int \lambda \, dE(\lambda). \tag{4.1}$$

Analogous to (2.5), we define

$$[f, g] := \langle f, W g \rangle, \quad f, g \in \mathrm{dom}(W). \tag{4.2}$$

In this way, $\mathrm{dom}(W)$ becomes a decomposable non-degenerate inner product space with fundamental decomposition $\mathrm{dom}(W) = \mathcal{D}_+ \oplus \mathcal{D}_-$ and fundamental symmetry J, where

$$\mathcal{D}_+ := E(0, \infty)\mathrm{dom}(W), \quad \mathcal{D}_- := E(-\infty, 0)\mathrm{dom}(W), \quad J = E(0, \infty) - E(-\infty, 0). \tag{4.3}$$

Here, $\ker(W) = \{0\}$ is necessary since otherwise $\mathrm{dom}(W)$ would be degenerate. From $J^2 = 1$, the polar decomposition $W = J|W|$ and Equation (4.2), it follows that

$$[f, g]_J := \langle f, |W| g \rangle, \quad f, g \in \mathrm{dom}(W). \tag{4.4}$$

Taking the closure under the norm defined by $[\cdot, \cdot]_J$ and extending J to the closure (without changing the notation), we obtain a Krein space $(\mathcal{H}_W, [\cdot, \cdot])$ with fundamental symmetry J and fundamental decomposition $\mathcal{H}_W = \mathcal{H}_+ \oplus \mathcal{H}_-$ such that $\mathcal{D}_+$ and $\mathcal{D}_-$ are dense in $\mathcal{H}_+$ and $\mathcal{H}_-$, respectively.

Recall from the end of Section 2.1 that, for a regular Gram operator, $\langle \cdot, \cdot \rangle$ and $[\cdot, \cdot]_J$ define equivalent norms on $\mathcal{H}$. It follows from (2.6) (for instance, by applying the restriction $\||W|^{-1}\|^{-1} \leq \lambda_1 \leq \lambda_2 \leq \|W\|$ in the next proof) that equivalent norms admit the same set of frames. Hence, by Theorem 3.3, any frame for $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ yields one for $(\mathcal{H}_W, [\cdot, \cdot])$ and vice versa. The next proposition tells us that the same cannot hold for Gram operators which are not regular or unbounded.

Proposition 4.1. *Let $(\mathcal{H}_W, [\cdot, \cdot])$ denote the Krein space described above.*

    i) If W is bounded and $0 \in \mathrm{spec}(W)$, then any frame $\{f_n\}_{n \in \mathbb{N}}$ for $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ is not a frame for $(\mathcal{H}_W, [\cdot, \cdot])$.

ii) If W is unbounded, then any frame $\{f_n\}_{n\in\mathbb{N}} \subset \operatorname{dom}(W)$ for $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ is not a frame for $(\mathcal{H}_W, [\cdot,\cdot])$.

Proof. Using the spectral theorem, we write $|W| = \int_{[0,\infty)} \lambda \, dF(\lambda)$, where F denotes the corresponding projection valued measure. Given $\lambda_2 > \lambda_1 > 0$ such that $F([\lambda_1,\lambda_2]) \neq 0$, choose $h \in F([\lambda_1,\lambda_2])\mathcal{H}$ with $\|h\| = 1$. Then h belongs to the domain of $\sqrt{|W|}^{-1}$ so that $g := \sqrt{|W|}^{-1} h$ is well defined. From (4.4), we get

$$\|g\|_J^2 = \langle g, |W|g\rangle = \langle h, h\rangle = 1. \tag{4.5}$$

Furthermore, by the spectral theorem,

$$\lambda_1 \leq \int_{[\lambda_1,\lambda_2]} \lambda \, d\langle h, F(\lambda)h\rangle = \|\sqrt{|W|}h\|^2 = \||W|g\|^2 \leq \lambda_2. \tag{4.6}$$

Now let $\{f_n\}_{n\in\mathbb{N}}$ be a frame for $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ with frame bounds $0 < A \leq B < \infty$. Suppose that $0 \in \operatorname{spec}(W)$. Since 0 is not an eigenvalue of W, there exists for each $\lambda_2 > 0$ a $\lambda_1 \in (0,\lambda_2)$ such that $F([\lambda_1,\lambda_2]) \neq 0$. Using Equations (2.6), (4.5) and (4.6), we get for g as above

$$\sum_{n\in\mathbb{N}} |[f_n, g]|^2 = \sum_{n\in\mathbb{N}} |\langle f_n, Wg\rangle|^2 \leq B\|Wg\|^2 = B\||W|g\|^2 \leq B\lambda_2\|g\|_J^2, \tag{4.7}$$

and taking the limit $\lambda_2 \to 0$ shows that there cannot exist a lower frame bound satisfying Definition 3.1.

If the Gram operator W is unbounded, then for each $\lambda_1 > 0$ we find a $\lambda_2 > \lambda_1$ such that $F([\lambda_1,\lambda_2]) \neq 0$. Similarly to (4.7), we compute

$$\sum_{n\in\mathbb{N}} |[f_n, g]|^2 = \sum_{n\in\mathbb{N}} |\langle f_n, Wg\rangle|^2 \geq A\|Wg\|^2 = A\||W|g\|^2 \geq A\lambda_1\|g\|_J^2,$$

and since $\lambda_1 > 0$ was arbitrary, an upper frame bound does not exist. $\square$

Although $\{f_n\}_{n\in\mathbb{N}}$ loses the property of being a frame for the Hilbert space $(\mathcal{H}_W, [\cdot,\cdot]_J)$ if Proposition 4.1.i) or ii) applies, one can still work with this sequence by considering for instance the dual pair $\{(JS^{-1}f_n, f_n))\}_{n\in\mathbb{N}}$ as a *framing* in $(\mathcal{H}_W, [\cdot,\cdot]_J)$, see [3, 11] for more details. In view of Theorem 3.3, one would expect that the structural results on framings in Hilbert spaces can be generalized to Krein spaces.

We will now show how to transfer frames for the Hilbert space $\mathcal{H}$ to frames for the Krein space $\mathcal{H}_W$. As outlined in the beginning of this section, we start by considering a bounded Gram operator.

Theorem 4.2. *Let W be a bounded self-adjoint operator on the Hilbert space $\mathcal{H}$ such that $\ker(W) = \{0\}$.*

i) The inclusion is $\mathcal{H} \subset \mathcal{H}_W$ is an equality if and only if $0 \notin \operatorname{spec}(W)$.

ii) $\sqrt{|W|} : \mathcal{H} \subset \mathcal{H}_W \longrightarrow \mathcal{H}, \ h \longmapsto \sqrt{|W|}h$ defines an isometric operator and its extension

$$U := \overline{\sqrt{|W|}} : \mathcal{H}_W \longrightarrow \mathcal{H}$$

is a unitary operator.

iii) $\{k_n\}_{n\in\mathbb{N}} \subset \mathcal{H}$ *is a frame for the Hilbert space* $\mathcal{H}$ *with frame bounds* $A \leq B$
if and only if $\{U^{-1}k_n\}_{n\in\mathbb{N}} \subset \mathcal{H}_W$ *is a frame for the Krein space* $\mathcal{H}_W$ *with*
frame bounds $A \leq B$.

Proof. Item iii) is an immediate consequence of ii) since unitary operators transfer frames into frames with the same frame bounds.

We next show ii). By (4.4), we have for all $h \in \mathcal{H}$

$$\left\langle \sqrt{|W|}\,h \,, \sqrt{|W|}\,h \right\rangle = \langle h \,, |W|\,h \rangle = [h, h]_J.$$

Hence $\sqrt{|W|} : \mathcal{H} \subset \mathcal{H}_W \longrightarrow \mathcal{H}$ is an isometry and can be extended to the closure $\overline{\mathcal{H}_W}$ of $\mathcal{H}$. It only remains to verify that its extension $\overline{\sqrt{|W|}}$ is surjective. For this, it suffices to show that $\sqrt{|W|}\mathcal{H}$ is dense in $\mathcal{H}$. Since $\ker(W) = \{0\}$, we have $(\sqrt{|W|}\mathcal{H})^\perp = \ker(\sqrt{|W|}) = \{0\}$, from which the result follows.

The equality $\mathcal{H} = \mathcal{H}_W$ for regular Gram operators W has already been discussed before Proposition 4.1. Suppose now that $0 \in \operatorname{spec}(W)$. As well known, the Hilbert space norm $\|\cdot\| = \sqrt{\langle\cdot,\cdot\rangle}$ is stronger than $\|\cdot\|_J = \sqrt{[\cdot,\cdot]_J}$ since $\|h\|_J^2 = \langle h, |W|\,h\rangle \leq \|W\|\,\|h\|^2$ for all $h \in \mathcal{H}$. From Proposition 4.1.i), it follows that the norms are not equivalent because equivalent norms give rise to the same set of frames. Therefore $\mathcal{H} \neq \mathcal{H}_W$, since otherwise id $: \mathcal{H} \to \mathcal{H}_W$ would be a continuous bijection and would thus have a bounded inverse, contradicting the fact that the norms are not equivalent. $\qquad\square$

The following theorem treats the case when W is unbounded and $0 \notin \operatorname{spec}(W)$. In contrary to the above situation, the Krein space $\mathcal{H}_W$ can then be identified with a subspace of $\mathcal{H}$.

Theorem 4.3. *Let* $W : \operatorname{dom}(W) \longrightarrow \mathcal{H}$ *be a self-adjoint operator on the Hilbert space* $\mathcal{H}$ *such that* $0 \notin \operatorname{spec}(W)$. *Then*

i) $\mathcal{H}_W$ *can be identified with* $\operatorname{dom}(\sqrt{|W|}) \subset \mathcal{H}$.

ii) $\sqrt{|W|} : \operatorname{dom}(\sqrt{|W|}) = \mathcal{H}_W \longrightarrow \mathcal{H}$ *is a unitary operator.*

iii) $\{k_n\}_{n\in\mathbb{N}} \subset \mathcal{H}$ *is a frame for the Hilbert space* $\mathcal{H}$ *with frame bounds* $A \leq B$
if and only if $\{\sqrt{|W|}^{-1}k_n\}_{n\in\mathbb{N}} \subset \mathcal{H}_W$ *is a frame for the Krein space* $\mathcal{H}_W$
with frame bounds $A \leq B$.

Proof. For the proof of i), we will use the fact that a densely defined linear operator T is closed if and only if its domain is complete with respect to the graph norm $(\|\cdot\|^2 + \|T(\,\cdot\,)\|^2)^{1/2}$ [14, Theorem 5.1]. As by assumption $0 \notin \operatorname{spec}(W)$, there exists an $\epsilon > 0$ such that $|W| \geq \epsilon$. Therefore, for all $h \in \operatorname{dom}(W)$, we have $\langle h, |W|\,h\rangle \geq \epsilon\langle h, h\rangle$. Recall that $\|h\|_J^2 = \langle h, |W|\,h\rangle = \|\sqrt{|W|}\,h\|^2$ for all $h \in \operatorname{dom}(W)$. Thus

$$\|h\|_J^2 = \|\sqrt{|W|}\,h\|^2 \leq \|h\|^2 + \|\sqrt{|W|}\,h\|^2 \leq (\epsilon^{-1}+1)\|\sqrt{|W|}\,h\|^2 = (\epsilon^{-1}+1)\|h\|_J^2,$$

hence the J-norm $\|\cdot\|_J$ and the graph norm of $\sqrt{|W|}$ are equivalent on its common domain of definition. As remarked above, $\operatorname{dom}(\sqrt{|W|})$ is complete with respect to the graph norm since a self-adjoint operator is always closed. From

the equivalence of norms on the dense subspace $\mathrm{dom}(W) \subset \mathrm{dom}(\sqrt{|W|})$, it follows that $\mathrm{dom}(\sqrt{|W|})$ is complete with respect to (the extension of) the norm $\|\cdot\|_J$. Taking the closure of $\mathrm{dom}(W)$ allows us therefore to identify $\mathcal{H}_W$ with $\mathrm{dom}(\sqrt{|W|})$.

To show ii), note $\sqrt{|W|}$ has a bounded inverse $\sqrt{|W|}^{-1} : \mathcal{H} \longrightarrow \mathrm{dom}(\sqrt{|W|})$ since $0 \notin \mathrm{spec}(W)$. From

$$\left[\sqrt{|W|}^{-1} h, \sqrt{|W|}^{-1} h\right]_J = \left\langle \sqrt{|W|}^{-1} h, |W| \sqrt{|W|}^{-1} h \right\rangle = \langle h, h \rangle$$

for all $h \in \mathrm{dom}(\sqrt{|W|})$, we conclude that $\sqrt{|W|}^{-1}$ and thus $\sqrt{|W|}$ are unitary. Now iii) follows from the unitarity of $\sqrt{|W|}^{-1}$ as in the previous theorem. $\square$

Our last theorem deals with the general situation, where W may be unbounded and 0 may belong to $\mathrm{spec}(W)$.

Theorem 4.4. *Let $W : \mathrm{dom}(W) \longrightarrow \mathcal{H}$ be a self-adjoint operator on the Hilbert space $\mathcal{H}$ such that $\ker(W) = \{0\}$. Then*

i) $\mathrm{dom}(\sqrt{|W|})$ *is complete in the norm $\|\cdot\|_J$ if and only if $0 \notin \mathrm{spec}(W)$.*

ii) $\sqrt{|W|} : \mathrm{dom}(\sqrt{|W|}) \longrightarrow \mathcal{H}$ *can be extended to a unitary operator*

$$U := \overline{\sqrt{|W|}} : \mathcal{H}_W \longrightarrow \mathcal{H}.$$

iii) $\{k_n\}_{n \in \mathbb{N}} \subset \mathcal{H}$ *is a frame for the Hilbert space $\mathcal{H}$ with frame bounds $A \leq B$ if and only if $\{U^{-1} k_n\}_{n \in \mathbb{N}} \subset \mathcal{H}_W$ is a frame for the Krein space $\mathcal{H}_W$ with frame bounds $A \leq B$.*

iv) *If $\{k_n\}_{n \in \mathbb{N}} \subset \mathrm{dom}(\sqrt{|W|}^{-1})$ is a frame for the Hilbert space $\mathcal{H}$, then the frame $\{U^{-1} k_n\}_{n \in \mathbb{N}}$ for the Krein space $\mathcal{H}_W$ is given by $\{\sqrt{|W|}^{-1} k_n\}_{n \in \mathbb{N}}$.*

Proof. The first part will be proven by reducing to the previous theorems. For this, we decompose $\mathcal{H}$ into the orthogonal sum

$$\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2, \qquad \mathcal{H}_1 := E([-1, 1])\mathcal{H}, \qquad \mathcal{H}_2 := E(\mathbb{R} \setminus [-1, 1])\mathcal{H},$$

where E denotes the projection valued measure from (4.1). Since the spectral projections commute with W and $J = E(0, \infty) - E(-\infty, 0)$, the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ are invariant under the action of these operators:

$$W_1 := W{\restriction}_{\mathcal{H}_1} : \mathcal{H}_1 \longrightarrow \mathcal{H}_1, \qquad W_2 := W{\restriction}_{\mathcal{H}_2} : \mathcal{H}_2 \cap \mathrm{dom}(W) \longrightarrow \mathcal{H}_2,$$
$$J_1 := J{\restriction}_{\mathcal{H}_1} : \mathcal{H}_1 \longrightarrow \mathcal{H}_1, \qquad J_2 := J{\restriction}_{\mathcal{H}_2} : \mathcal{H}_2 \longrightarrow \mathcal{H}_2.$$

Also, the inner products respect this decomposition, that is,

$$[h_1, h_2] = \langle h_1, W h_2 \rangle = 0, \quad [h_1, h_2]_J = \langle h_1, |W| h_2 \rangle = 0,$$

for all $h_1 \in \mathcal{H}_1$, and $h_2 \in \mathcal{H}_2 \cap \mathrm{dom}(W)$. Therefore we can consider the closures of $\mathcal{H}_1$ and $\mathcal{H}_2 \cap \mathrm{dom}(W)$ under the norm $\|\cdot\|_J$ separately. Note that, by the spectral theorem, $\mathrm{dom}(W) = \mathcal{H}_1 \oplus (\mathcal{H}_2 \cap \mathrm{dom}(W))$. Taking the norm closure gives

$$\mathcal{H}_W = \mathcal{H}_{W_1} \oplus \mathcal{H}_{W_2},$$

where $(\mathcal{H}_{W_i}, [\cdot, \cdot])$ denotes the Krein space constructed from the Gram operator W_i on $\mathcal{H}_i$, $i = 1, 2$. Applying Theorem 4.2.ii) and Theorem 4.3.ii), we obtain a unitary operator

$$U := \overline{\sqrt{|W_1|}} \oplus \sqrt{|W_2|} \; : \; \mathcal{H}_W = \mathcal{H}_{W_1} \oplus \mathcal{H}_{W_2} \longrightarrow \mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \qquad (4.8)$$

such that its restriction to $\mathcal{H}_1 \oplus \big(\mathcal{H}_2 \cap \mathrm{dom}(\sqrt{|W|})\big) = \mathrm{dom}(\sqrt{|W|})$ is given by $\sqrt{|W_1|} \oplus \sqrt{|W_2|} = \sqrt{|W|}$. By the density of $\mathrm{dom}(\sqrt{|W|})$ in $\mathcal{H}_W$, the unitary operator U in (4.8) is the unique extension of $\sqrt{|W|}$. This proves ii).

Now, i) follows from Theorem 4.2.i) applied to $\mathcal{H}_1 \subset \mathrm{dom}(\sqrt{|W|})$ and the bounded Gram operator $W_1 = W \!\restriction_{\mathcal{H}_1}$. Item iii) is a direct consequence of the unitarity of U in ii), and iv) follows from $U \!\restriction_{\mathrm{dom}(\sqrt{|W|})} = \sqrt{|W|}$, as observed in the previous paragraph. $\qquad \square$

By Theorem 4.4.i), if $0 \in \mathrm{spec}(W)$, then the completion process will always require to add "abstract" elements to $\mathrm{dom}(\sqrt{|W|})$. One of the standard procedures is to view these elements as equivalence classes of Cauchy sequences. However, for $\mathcal{L}_2$-spaces and multiplication operators W_φ as defined in (3.11), one can give an explicit description of $\mathcal{H}_{W_\varphi}$ in terms of measurable functions. In our final example, we illustrate this for $\mathcal{H} = \mathcal{L}_2(\mathbb{R}, \mu)$. In a certain sense, this is the generic case since any self-adjoint operator on a separable Hilbert space is unitarily equivalent to a direct sum of multiplication operators [12, Theorem VII.3].

Example 4.5. Let μ be a locally finite Borel measure on $\Sigma(\mathbb{R})$ and let $\mathcal{F}(\mathbb{R})$ denote the space of all Borel measurable complex functions on $\mathbb{R}$. For $\varphi \in \mathcal{F}(\mathbb{R})$, we define as in the end of the last section

$$\mathrm{dom}(W_\varphi) := \{f \in \mathcal{L}_2(\mathbb{R}, \mu) : \int |f|^2 |\varphi|^2 \mathrm{d}\mu < \infty\},$$
$$W_\varphi : \mathrm{dom}(W_\varphi) \longrightarrow \mathcal{L}_2(\mathbb{R}, \mu), \quad (W_\varphi f)(t) := \varphi(t)\, f(t).$$

Assume that φ is real-valued and that $\mu(\{x \in \mathbb{R} : \varphi(x) = 0\}) = 0$. Then $W_\varphi^* = W_\varphi$ and $\ker(W_\varphi) = \{0\}$ so that the assumptions of Theorem 4.4 are satisfied. Up to unitary equivalence, we may write

$$\mathcal{H}_{W_\varphi} = \left\{ f \in \mathcal{F}(\mathbb{R}) : \int |f|^2 |\varphi| \mathrm{d}\mu < \infty \right\},$$
$$U = W_{|\varphi|^{1/2}} \; : \; \mathcal{H}_{W_\varphi} \longrightarrow \mathcal{L}_2(\mathbb{R}, \mu), \quad (W_{|\varphi|^{1/2}} f)(t) := \sqrt{|\varphi(t)|}\, f(t),$$
$$U^{-1} = W_{|\varphi|^{-1/2}} \; : \; \mathcal{L}_2(\mathbb{R}, \mu) \longrightarrow \mathcal{H}_{W_\varphi}, \quad (W_{|\varphi|^{-1/2}} f)(t) := \frac{1}{\sqrt{|\varphi(t)|}}\, f(t).$$

By Theorem 4.4, any frame $\{f_n\}_{n \in \mathbb{N}}$ for $\mathcal{L}_2(\mathbb{R}, \mu)$ determines a frame for the Krein space $\mathcal{H}_{W_\varphi}$ with the same frame bounds. It can be given by $\{\frac{1}{\sqrt{|\varphi|}} f_n\}_{n \in \mathbb{N}} \subset \mathcal{H}_{W_\varphi}$.

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