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TOPOLOGICAL PROPERTIES OF OPERATIONS ON SPACES OF DIFFERENTIABLE FUNCTIONS

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Abstract. In this paper, we consider different notions of openness for the scalar multiplication on sequence spaces and spaces of continuous functions. We apply existing techniques to derive weak openness of multiplication on spaces of differentiable functions, endowed with a large collection of quasi-algebra norms.

1. Introduction

The classical open mapping theorem states that every continuous, linear surjective map between two Banach spaces is open. An example proposed by Rudin in [19] shows that this property does not extend to bilinear maps. In this paper, we study openness properties of several bilinear maps, defined on normed spaces. We start by reviewing some definitions.

Definition 1.1. Let E and F be topological spaces, and let $f : E \to F$ be a surjective map. Let $x \in E$.

(1) f is open at x if and only if f maps every neighborhood of x onto a neighborhood of $f(x)$, and x is said to be a point of local openness of f.
(2) f is densely open (d-open) at x if and only if f maps every neighborhood of x onto a dense subset of a neighborhood of $f(x)$, and x is a point of local d-openness of f.

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(3) f is weakly open (w-open) at x if and only if f maps every neighborhood of x onto a set with nonempty interior, and x is a point of local w-openness of f.

Then f is open (d-open or w-open) if f is open (respectively, d-open or w-open) at x, for every $x \in E$. Equivalently, the set of all points of local openness (respectively d- or w-openness) of f is the entire space E.

The notion of w-openness has been considered by several researchers (see, for example, the papers [6], [21], and also [11]). The notion of d-openness was proposed and studied in [18]. This concept was largely inspired by many different types of openness proposed by Ge, Gu, Lin, and Zhu in [14].

It is clear that "open" implies "d-open" and also "w-open". However, neither "d-open" implies "w-open" nor "w-open" implies "d-open". We give two examples to support the last claim. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x$ for $x \in \mathbb{Q}$, otherwise $f(x) = -x$. The function f is d-open but not w-open. Let $g : \mathbb{R} \to \mathbb{R}$ be given by $g(x) = |x|$ for $x \leq 2$ and $g(x) = 4 - x$ for $x > 2$. The function g is w-open but not d-open.

We now adapt Definition 1.1 to metric spaces. In a metric space, an open ball centered at point x and of radius $r > 0$ is denoted by $\mathcal{B}(x, r)$.

Definition 1.2. Let (E, d) and (F, D) be two metric spaces, and let $f : E \to F$ be a surjective map. Then

(1) f is open (d-open or w-open) if and only if, for every $x \in E$ and $\epsilon > 0$, there exists $\delta > 0$ such that

$$f(\mathcal{B}(x,\epsilon)) \supseteq \mathcal{B}(f(x),\delta) \quad \left(\overline{f(\mathcal{B}(x,\epsilon))} \supseteq \mathcal{B}(f(x),\delta) \text{ or } \operatorname{int}\left(f(\mathcal{B}(x,\epsilon))\right) \neq \emptyset, \text{ resp.}\right).$$

(2) f is uniformly open if, for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$f(\mathcal{B}(x,\epsilon)) \supseteq \mathcal{B}(f(x),\delta) \quad \text{for all } x \in E.$$

Uniform openness for the multiplication and for other multilinear maps was studied by Balcerzak, Majchrzycki, and Strobin; see [4] and [3].

In Section 2, we consider openness properties of scalar products on sequence spaces and spaces of vector-valued continuous functions. We characterize the points, where the product is (d-)open, and we establish the w-openness of the scalar product for arbitrary topological vector spaces over a field $\mathbb{F}$ ($\mathbb{R}$ or $\mathbb{C}$). We also observe that the scalar product is open if and only if the dimension of the vector space is 1.

In Section 3, we discuss results on the weak openness of the multiplication on spaces of functions, defined on the unit interval and with continuous n derivatives. We apply an approach, developed by A. Wachowicz in [21] to show that the multiplication on a collection of quasi-normed algebras is weakly open. Each space consists of all functions, defined on the unit interval with continuous n derivatives and endowed with a quasi-algebra norm. This norm is defined from a compact and connected subset of the $(n+1)$-dimensional hypercube. This extends an idea, presented by Kawamura, Koshimizu, and Miura in [17] for continuously differentiable functions on [0, 1].

In Section 4, we consider a problem, proposed by Behrends in [8] of characterizing those pairs of functions, where the multiplication is open. We derive necessary and sufficient conditions for a pair of functions to be a point of local openness for the multiplication on this new class of algebras.

2. Openness of the scalar product

We recall from [19, Problem 11 on p. 54] the following product: $\cdot : \mathbb{R}\times\mathbb{R}^2 \to \mathbb{R}^2$ is given by $\cdot(t,(x,y)) = t\cdot(x,y) = (tx,ty)$. It is easy to check that $\cdot$ is not open at $(0,(1,0))$. Set $\epsilon = 1/2$. For every $\delta > 0$, we have that $(0,\delta/2) \in \mathcal{B}((0,0),\delta)$, yet $(0,\delta/2) \notin \mathcal{B}(0,\epsilon)\cdot\mathcal{B}((1,0),\epsilon)$. Suppose the otherwise case. Then there exist $t \in \mathcal{B}(0,\epsilon)$ and $(x,y) \in \mathcal{B}((1,0),\epsilon)$ such that $(tx,ty) = (0,\frac{\delta}{2})$. Since $x \neq 0$, then $t = 0$. Therefore $ty = 0 \neq \frac{\delta}{2}$. A similar argument proves that $\cdot$ is not open at every point of the form $(0,(a,b)) \in \mathbb{R}\times(\mathbb{R}^2\setminus(0,0))$.

We revisit the previous example for new settings. We consider a sequence of normed spaces A_k, over the same field $\mathbb{F}$, with corresponding norms $\|\cdot\|_k$ $(k \in \mathbb{N})$.

Let A denote one of the following sequence spaces: $c_0(\{A_k\}_k)$, $\ell_p(\{A_k\}_k)$ $(p \geq 1)$, and $\ell_\infty(\{A_k\}_k)$, endowed with the standard norms. We denote by $\overline{\mathbf{a}}$ the sequence $(a_1,a_2,\ldots) \in \Pi_n A^n$. We denote by $\overline{\mathbf{0}}$ the sequence of all zeros.

Proposition 2.1. *Let A be a sequence space with norm, denoted by $\|\cdot\|$. Let $T : \mathbb{F}\times A \to A$ be given by $T(t,\overline{\mathbf{a}}) = t\cdot\overline{\mathbf{a}} = (ta_i)_{i\in\mathbb{N}}$. Then the following statements are equivalent:*

(a) *$t \neq 0$ or $(t,\overline{\mathbf{a}}) = (0,\overline{\mathbf{0}})$.*
(b) *T is open at $(t,\overline{\mathbf{a}})$.*
(c) *T is d-open at $(t,\overline{\mathbf{a}})$.*

Proof. We show that (a) implies(b). We assume that $t \neq 0$. For every $\epsilon > 0$, we set $\delta = |t|\epsilon$. We claim that

$$\mathcal{B}(t\overline{\mathbf{a}},\delta) \subset \{\lambda\overline{\mathbf{y}} : \lambda \in \mathcal{B}(t,\epsilon) \text{ and } \overline{\mathbf{y}} \in \mathcal{B}(\overline{\mathbf{a}},\epsilon)\}.$$

Suppose $\overline{\mathbf{x}} \in \mathcal{B}(t\overline{\mathbf{a}},\delta)$; then $\|t\overline{\mathbf{a}}-\overline{\mathbf{x}}\| < \delta = |t|\epsilon$. We set $\overline{\mathbf{y}} = \dfrac{1}{t}\overline{\mathbf{x}}$ and $\lambda = t$. Clearly $\overline{\mathbf{y}} = (y_i)_{i\in\mathbb{N}} \in \mathcal{B}(\overline{\mathbf{a}},\epsilon)$.

If $t = 0$, $a_i = 0$ for all i, and $\epsilon > 0$, then we set $\delta = \dfrac{\epsilon}{k}$, with $\dfrac{1}{k} < \epsilon$. We show that

$$\mathcal{B}(\overline{\mathbf{0}},\delta) \subset \mathcal{B}(0,\epsilon)\cdot\mathcal{B}(\overline{\mathbf{0}},\epsilon) = \{(\lambda\overline{\mathbf{y}}) : \lambda \in \mathcal{B}(0,\epsilon) \text{ and } \overline{\mathbf{y}} \in \mathcal{B}(\overline{\mathbf{0}},\epsilon)\}.$$

Given $\overline{\mathbf{x}} \in \mathcal{B}(\overline{\mathbf{0}},\delta)$, we set $\lambda = \frac{1}{k} \in \mathcal{B}(0,\epsilon)$, and, hence, $\overline{\mathbf{y}} = k\overline{\mathbf{x}} \in \mathcal{B}(\overline{\mathbf{0}},\epsilon)$. This shows that T is open at $(0,\overline{\mathbf{0}})$.

We show that T is not open at $(0,\overline{\mathbf{a}})$, provided that some a_i is not equal to zero. We assume that $a_1 \neq 0$. We set $\epsilon = \dfrac{\|a_1\|}{2}$, and, for every $\delta > 0$, we consider $\overline{\mathbf{b}} \in \mathcal{B}(\overline{\mathbf{0}},\delta)$, given by $b_2 = \frac{\delta a_1}{2\|a_1\|}$ and $b_i = 0$ with $i \neq 2$. We claim that $\overline{\mathbf{b}}$ is not in $\mathcal{B}(0,\epsilon)\cdot\mathcal{B}(\overline{\mathbf{a}},\epsilon)$. Suppose the otherwise case. For some $t \in \mathcal{B}(0,\epsilon)$ and $\overline{\mathbf{x}} \in \mathcal{B}(\overline{\mathbf{a}},\epsilon)$, we have $tx_i = 0$ with $i \neq 2$. Since $x_1 \neq 0$ and $tx_1 = 0$, we obtain $t = 0$. Hence $0\cdot\overline{\mathbf{x}} = \overline{\mathbf{0}} \neq \overline{\mathbf{a}}$. This shows that (b) implies (a).

It is clear that “open” implies “d-open”. It remains to show that (c) implies (a). We show that T is not d-open at every point of the form $(0, \overline{\mathbf{a}})$, with $\overline{\mathbf{a}} \neq \overline{\mathbf{0}}$. Without loss of generality, we may assume that $\overline{\mathbf{a}} = (0, \ldots, 0, a_i, \ldots a_n, \ldots)$ with $a_i \neq 0$ and $i \neq 1$. We choose $\epsilon = \dfrac{\|a_i\|}{2}$. Then for every $\delta > 0$,

$$\overline{\mathbf{b}} = \left(\frac{\delta\, a_i}{2\|a_i\|}, 0, 0, \ldots \right) \notin \overline{\mathcal{B}(0, \epsilon) \cdot \mathcal{B}(\overline{\mathbf{a}}, \epsilon)}.$$

Suppose that there exist sequences $\{t_k\}$ and $\{\overline{\mathbf{c}}^k\}$, with $t_k \in \mathcal{B}(0, \epsilon)$ and $\overline{\mathbf{c}}^k \in \mathcal{B}(\overline{\mathbf{a}}, \epsilon)$, such that $t_k \overline{\mathbf{c}}^k \to \overline{\mathbf{b}}$. Since $\overline{\mathbf{c}}_i^k \neq 0$, then $t_k \to 0$. Therefore, the sequence $\{t_k \overline{\mathbf{c}}_1^k\}_k$ must converge to zero. This is impossible because $\overline{\mathbf{b}}_1 = \dfrac{\delta\, a_i}{2\|a_i\|} \neq 0$. This completes the proof. □

Remark 2.2. We observe that (a) implies (b) holds for an arbitrary normed space. Furthermore, a variation of the proof given for the aforementioned proposition and the Tietze extension theorem (see [13, 20]) imply the equivalence of the three statements in the Proposition 2.1 for spaces of vector-valued continuous functions, $C(\Omega, E)$, with Ω a compact metric space and E a normed space.

The next corollary follows easily from Proposition 2.1.

Corollary 2.3. *Let A and T be as in Proposition 2.1. Then T is weakly open.*

Previous observations can be extended to topological vector spaces.

Lemma 2.4 (see [18])**.** *Let V be a topological vector space over the field F. Then $T : \mathbb{F} \setminus \{0\} \times V \to V$, given by $T(\lambda, v) = \lambda \cdot v$, is an open map.*

Proof. We show that, for O_1, O_2 open, where $O_1 \subset \mathbb{F}$ and $O_2 \subset V$, $O_1 \cdot O_2 = \{\lambda \cdot y : \lambda \in O_1, y \in O_2\}$ is open. We notice that for a fixed $\lambda \in \mathbb{F} \setminus \{0\}$, the scalar multiplication, $M_\lambda : V \to V$, given by $M_\lambda(v) = \lambda \cdot v$, is a homeomorphism and is, thus, open. Therefore $O_1 \cdot O_2 = \cup_{\lambda \in O_1} \lambda \cdot O_2$ is open. This completes the proof. □

Remark 2.5. The map $T : \mathbb{F} \times V \to V$ is open at $(0, \overline{\mathbf{0}})$. Given W, a neighborhood of $\overline{\mathbf{0}}$, and $\mathcal{B}(0, \epsilon)$, an open ball in $\mathbb{F}$, we have

$$\frac{\epsilon}{2} \cdot W \subset \mathcal{B}(0, \epsilon) \cdot W.$$

Since $M_{\epsilon/2}$ is a homeomorphism, $\dfrac{\epsilon}{2} \cdot W$ is a neighborhood of $\overline{\mathbf{0}}$.

Proposition 2.6. *Let V be a topological vector space over the field $\mathbb{F}$. Then $T : \mathbb{F} \times V \to V$, given by $T(\lambda, \overline{\mathbf{x}}) = \lambda \cdot \overline{\mathbf{x}}$, is weakly open.*

Proof. From Lemma 2.4 and Remark 2.5, it remains to show that T is weakly open at every point of the form $(0, \overline{\mathbf{x}})$ with $\overline{\mathbf{x}} \neq \overline{\mathbf{0}}$. Given $\epsilon > 0$ and W, a neighborhood of $\overline{\mathbf{x}}$, we select $t_0 \in \mathcal{B}(0, \epsilon) \setminus \{0\}$. Then there exists $\delta > 0$ such that $\mathcal{B}(t_0, \delta) \subset \mathcal{B}(0, \epsilon)$. Since T is open at $(t_0, \overline{\mathbf{x}})$, there exists a neighborhood of $\overline{\mathbf{0}}$, U, such that

$$t_0 \overline{\mathbf{x}} + U \subset \mathcal{B}(t_0, \delta) \cdot W \subset \mathcal{B}(0, \epsilon) \cdot W.$$

This completes the proof. □

It is interesting to observe that $T\ :\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ and $T\ :\mathbb{C}\times\mathbb{C}\to\mathbb{C}$ are open maps. In fact, they are uniformly open. The proof for the real case is given in [4], and the complex case follows similarly. These two cases are examples of scalar multiplications that behave well relative to openness, contrarily to many previous examples (see Proposition 2.1).

We finish this section with a result on openness of addition.

Proposition 2.7. *The addition on a topological vector space is an open map.*

Proof. We show that $+ : V+V\to V$ is open, or, equivalently, that for every $O_1, O_2\subset V$ open, $O_1+O_2\subset V$ is open. For a fixed $y\in O_2$, the map $T:V\to V$, defined by $T(u)=u+y$, is a homeomorphism. Then $O_1+\{y\}$ is open. Therefore, O_1+O_2 is open, since $O_1+O_2=\cup_{y\in O_2}O_1+\{y\}$. □

Remark 2.8. Let X be a normed space. Then the addition on X is uniformly open. This follows from Proposition 1 in [6]: $B(x_1,r_1)+B(x_2,r_2)=B(x_1+x_2,r_1+r_2)$, for x_1 and $x_2\in X$, r_1 and r_2 positive numbers.

3. Weak-openness of multiplication on $C^{(n)}_{\langle D\rangle}[0,\ 1]$

In this section, we consider results that lead to the study of weak openness of multiplication on a class of quasi-normed algebras of continuously differentiable functions. We start by defining the spaces. Let $C^{(n)}[0,\ 1]$ denote the space of all functions on the unit interval with n-continuous derivatives. Let D be a compact and connected subset of the $(n+1)$-hypercube, $[0,\ 1]^{n+1}$. We set $\pi_j(D)=I_j$, with $j=0,\dots,n$, and π_j denotes the projection of D into the j^{th}-coordinate of $[0,\ 1]^{n+1}$. Then, for $f\in C^{(n)}[0,\ 1]$, we set $\|f^{(j)}\|_j=\max\{|f^{(j)}(x)| : x\in I_j\}$ and define

$$\|f\|_{\langle D\rangle}=\sum_{j=0}^{n}\|f^{(j)}\|_j.$$

Proposition 3.1. *Let D be a compact, connected subset of $[0,\ 1]^{n+1}$. Then $C^{(n)}[0,\ 1]$, endowed with $\|\cdot\|_{\langle D\rangle}$, is a normed space if and only if $\cup_{i=0}^{n}\pi_i(D)=[0,\ 1]$.*

Proof. We prove the sufficiency by contrapositive. Suppose that $\cup_{i=0}^{n}\pi_i(D)\neq[0,\ 1]$. Since $\cup_{i=0}^{n}\pi_i(D)\neq[0,\ 1]$, there exists $x\in[0,\ 1]$ such that $x\notin\cup_{i=1}^{n}\pi_i(D)$. Let O be an open set containing $\cup_{i=0}^{n}\pi_i(D)$ such that $x\in\operatorname{int}([0,\ 1]\setminus O)$. We consider $f\in C^{(n)}[0,\ 1]$ such that $\|f\|_\infty\neq 0$ and $f|_O\equiv 0$. Thus, $\|f\|_{\langle D\rangle}=0$, but $f\neq 0$, and $\|f\|_{\langle D\rangle}$ is not a norm.

Now we prove the reverse implication. Suppose $\cup_{i=0}^{n}\pi_i(D)=[0,\ 1]$. First, we show that $\|f\|_{\langle D\rangle}=0$ implies $f\equiv 0$. Suppose that $\|f\|_{\langle D\rangle}=0$ (i.e., $\sup_{r\in D}\{|f(r_0)|+\Sigma_{i=1}^{n}|f^{(i)}(r_i)|\}=0$). This implies that $f^{(i)}|_{\pi_i(D)}=0$, with $i\in\{0,1,\dots,n\}$. Since D is compact and connected, $\pi_i(D)$ is a closed subinterval of $[0,\ 1]$, possibly a singleton.

We set $\pi_j(D)=I_j$, where I_j is a closed interval. We first show that, given two nontrivial intervals I_p and I_q with nonempty intersection and $p<q\leq n$, then f restricted to the union $I_p\cup I_q$ is a polynomial of degree $p-1$. This is clear if the intersection of the two intervals is a non-degenerate interval.

We assume that $x_0 \in I_p \cap I_q$. This assumption contains both possible cases of whether the intersection is a non-degenerate interval or a singleton. Without loss of generality, we may assume that $x_0 \neq 0$ (since both intervals are non-degenerate and zero is the leftmost point in [0, 1]). Since $\|f\|_{\langle D\rangle} = 0$, $f^{(p)}|_{I_p} = 0$ implies $f|_{I_p} = \sum_{i=0}^{p-1} a_i\, x^i$; also, $f^{(q)}|_{I_q} = 0$ implies $f|_{I_q} = \sum_{i=0}^{q-1} b_i x^i$. Since $f \in C^{(n)}[0,\, 1]$, the values of these two polynomials and all their derivatives, up to order n, computed at x_0, must coincide. Therefore, we have

$$\begin{gathered}
(a_0 - b_0) + (a_1 - b_1)x_0 + \cdots + (a_{p-1} - b_{p-1})x_0^{p-1} - b_p x_0^p - \cdots - b_{q-1}x_0^{q-1} = 0 \\
(a_1 - b_1) + \cdots + (p-1)(a_{p-1} - b_{p-1})x_0^{p-2} - b_p p x_0^{p-1} - \cdots - (q-1)b_{q-1}x_0^{q-2} = 0 \\
\vdots \\
(p-1)!(a_{p-1} - b_{p-1}) - p! b_p x_0 - \cdots - (q-1)(q-2)\cdots(q-p+1)\, b_{q-1}x_0^{q-p} = 0 \\
-p! b_p - (p+1)! b_{p+1} x_0 - \cdots - (q-1)(q-2)\cdots(q-p)\, b_{q-1}\, x_0^{q-p-1} = 0 \\
\vdots \\
-(q-1)!\, b_{q-1} = 0.
\end{gathered}$$

The solution set of this homogeneous system consists of sequences (a_i) and (b_i) such that $a_i = b_i$, for $i \in \{0, 1, \ldots, p-1\}$, and $b_i = 0$, for $i \in \{p, \ldots, q-1\}$. This proves our original claim that f, restricted to the union of the two intervals with nonempty intersection, is a polynomial of degree $p - 1$ with p being smaller than the two indices.

Therefore, we conclude that f must be a polynomial on [0, 1] of degree less than n. We choose points $x_i \in [0,\, 1]$, such that $f^{(i)}(x_i) = 0$ with $i \in \{0, 1, \ldots, n-1\}$. The system that, translates this, is as follows:

$$\begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^{n-1} \\ 0 & 1 & 2x_1 & \cdots & (n-1)x_1^{n-2} \\ 0 & 0 & 2 & \cdots & (n-1)(n-2)x_2^{n-3} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & (n-1)! \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{n-1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

This system has the zero solution, implying $f = 0$. The remaining properties of a norm are straightforward. □

Remark 3.2. The space $C^{(n)}[0,\, 1]$ with a norm $\|\cdot\|_{\langle D\rangle}$ is denoted by $C^{(n)}_{\langle D\rangle}[0,\, 1]$. If D has surjective projections along each component (i.e. $\pi_j(D) = [0,\, 1]$ for every $j = 0, \ldots n$), then $\|\cdot\|_{\langle D\rangle}$ is denoted by $\|\cdot\|_1$ and the corresponding space $C^{(n)}_1[0,\, 1]$. We denote by $C^{(n)}_m[0,\, 1]$ the space $C^{(n)}[0,\, 1]$, equipped with $\|f\|_m = \max\{\|f^{(i)}\|_\infty \,:\, i = 0, \ldots, n\}$. We observe that, in general, $C^{(n)}_{\langle D\rangle}[0,\, 1]$ is not complete. We just consider $n = 1$ and $D = [0, 1] \times \{0\}$, and let $f_n(x) = \frac{x}{1+nx^2}$.

This sequence is Cauchy in this norm. We assume that $m > n$; then

$$\begin{aligned}\|f_n - f_m\|_{\langle D\rangle} &= \|f_n - f_m\|_\infty + |f_n'(0) - f_m'(0)| = \|f_n - f_m\|_\infty \\ &= \frac{x^3\,(m-n)}{(1+nx^2)(1+mx^2)} \\ &\leq \frac{x}{(1+nx^2)} \cdot \frac{x^2 m}{(1+mx^2)} \leq \frac{x}{(1+nx^2)} \leq \frac{1}{2\sqrt{n}}.\end{aligned}$$

This sequence converges to the constant function being equal to zero and $f_n'(0) = 1$ for every n. Therefore $\{f_n\}$ does not converge to the zero function in $C^{(n)}_{\langle D\rangle}[0,\,1]$.

We now give some examples.

Example 3.3. Let $n = 1$, $D_1 = [0,\,1] \times \{0\}$, and $D_2 = [0,\,1] \times \{1\}$. The corresponding norms on $C^{(n)}[0,\,1]$ are $\|\cdot\|_1$ and $\|\cdot\|_2$. Then $f_k(x) = x^k$ and $g_k(x) = (x-1)^k$ (with k an integer greater than 1) show that these norms are not equivalent. We have $\|f_k\|_{\langle D_1\rangle} = 1$ and $\|f_k\|_{\langle D_2\rangle} = 1+k$. Also $\|g_k\|_{\langle D_1\rangle} = 1+k$ and $\|g_k\|_{\langle D_2\rangle} = 1$.

Let $D = \{0\} \times [0,\,1]$ and $f(x) = x$. Then $\|f\|_{\langle D\rangle} = 1$, and $\|f^2\|_{\langle D\rangle} = 2$. This shows that $\|\cdot\|_{\langle D\rangle}$ is not an algebra norm.

Definition 3.4 (see [1] and [22]). If $(X, \|\cdot\|)$ is a normed space and X is an algebra, then we say that X is a quasi-normed algebra if there exists a constant $C > 0$ such that $\|xy\| \leq C\|x\|\|y\|$ for all $x, y \in X$.

We identify in the next lemma a collection of quasi-normed algebras.

Lemma 3.5. *Let D be a connected and compact subset of $[0,\,1]^{n+1}$ such that $[0,\,1] = \pi_0(D) \supseteq \pi_1(D) \supseteq \cdots \supseteq \pi_n(D)$. Then, for every $f, g \in C^{(n)}[0,\,1]$, we have $\|f\cdot g\|_{\langle D\rangle} \leq (n+1)2^n\|f\|_{\langle D\rangle}\|g\|_{\langle D\rangle}$.*

Proof. Let $f \in C^{(n)}[0,\,1]$ be given. For $n = 0$, we have $\|f\cdot g\|_\infty \leq \|f\|_\infty\|g\|_\infty$, and for $n = 1$, we have $\|f\cdot g\|_{\langle D\rangle} = \|f\cdot g\|_\infty + \|f\cdot g' + g\cdot f'\|_1 \leq \|f\|_{\langle D\rangle}\,\|g\|_{\langle D\rangle}$. We justify this inequality as follows. Since $\pi_0(D) \supseteq \pi_1(D)$, we have $\|f\cdot g\|_\infty + \|f\cdot g' + g\cdot f'\|_1 \leq \|f\|_\infty\|g\|_\infty + \|f\|_\infty\|g'\|_1 + \|g\|_\infty\|f'\|_1$. Hence

$$\|f\|_\infty\|g\|_\infty + \|f\|_\infty\|g'\|_1 + \|g\|_\infty\|f'\|_1 \leq \|f\|_\infty\|g\|_{\langle D\rangle} + \|g\|_\infty\|f'\|_1 \leq \|f\|_{\langle D\rangle}\,\|g\|_{\langle D\rangle}.$$

This implies $\|f \cdot g\|_{\langle D\rangle} \leq \|f\|_{\langle D\rangle} \|g\|_{\langle D\rangle}$. For $D \subset [0,\,1]^{n+1}$, we set $\|f^{(j)}\|_k = \max\{|f^{(j)}(x)|, \text{ with } x \in \pi_k(D)\}$. Then

$$\begin{aligned}
&\|f \cdot g\|_{\langle D\rangle}\\
&= \|f \cdot g\|_0 + \|(f \cdot g)'\|_1 + \cdots + \|(f \cdot g)^{(n)}\|_n\\
&\leq \|f\|_0 \|g\|_0 + (\|f'\|_1 \|g\|_1 + \|f\|_1 \, \|g'\|_1) + \cdots + \sum_{k=0}^{n} \binom{n}{k} \|f^{(n-k)}\|_n \, \|g^{(k)}\|_n\\
&= \sum_{j=0}^{n} \sum_{k=j}^{n} \binom{k}{j} \|f^{(k-j)}\|_k \, \|g^{(j)}\|_k\\
&\leq \sum_{j=0}^{n} \left(\sum_{k=j}^{n} \binom{k}{j} \|f^{(k-j)}\|_k \right) \|g^{(j)}\|_j\\
&\text{(since } k \geq j, \text{ we have } ||g^{(j)}||_k \leq ||g^{(j)}||_j).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\|f \cdot g\|_{\langle D\rangle} &\leq \sum_{j=0}^{n} \left(\sum_{k=j}^{n} \binom{k}{j} \|f^{(k-j)}\|_k \right) \|g^{(j)}\|_j\\
&\leq \sum_{j=0}^{n} \|g^{(j)}\|_j \sum_{i=0}^{n-j} \binom{i+j}{j} \|f^{(i)}\|_i\\
&\quad \text{(by setting } i = k-j \text{ and using that } ||f^{(i)}||_{i+j} \leq ||f^{(i)}||_i)\\
&\leq (n+1) \sum_{j=0}^{n} \|g^{(j)}\|_j \sum_{j=0}^{n} \binom{n}{j} \sum_{i=0}^{n} \|f^{(i)}\|_i = (n+1)2^n \|f\|_{\langle D\rangle} \|g\|_{\langle D\rangle},
\end{aligned}$$

since $\binom{i+j}{j} \leq \binom{n}{j}$ and $\sum_{i=0}^{n-j} \binom{i+j}{j} \leq (n+1)\binom{n}{j}$. This completes the proof. □

Remark 3.6. A challenging problem seems to be a characterization of the compact and connected subsets of $[0,\,1]^{n+1}$ that define a quasi-algebra norm on $C^{(n)}[0,\,1]$.

We now review some definitions. We consider V a normed space with a binary operation $P : V \times V \to V$, which is associative, commutative, and satisfies the distributive properties relatively to the addition on V. We also assume that V contains a neutral element $\mathbf{1}$ for P (i.e., $P(\mathbf{x}, \mathbf{1}) = P(\mathbf{1}, \mathbf{x}) = \mathbf{x}$ for every $\mathbf{x} \in V$). We call such a map P, a multiplication on V.

If P can only be defined on a dense subset of V, containing $\mathbf{1}$, then P is called a densely-defined multiplication. The domain of P is denoted by Dom(P).

We first consider the standard pointwise multiplication of the functions $P : C_m^{(n)}[0,\,1] \times C_m^{(n)}[0,\,1] \to C_m^{(n)}[0,\,1]$, where $P(f,g) = f \cdot g$.

The well-known Fremlin function, $f(x) = x - \dfrac{1}{2}$, shows that P is not open. This follows because $P(f,f) = f^2 \notin int\,(\mathcal{B}(f,1/2) \cdot \mathcal{B}(f,1/2))$, and P is not open at (f, f) (see [6]). It is interesting to observe that $f^2 \in \overline{\mathcal{B}(f, 1/2) \cdot \mathcal{B}(f, 1/2)}$, since $f^2 = \lim_n \left(f + \frac{1}{n}\right)^2$.

Lemma 3.7. *P on $C_m^{(n)}[0,\, 1]$ is not densely open.*

Proof. Let $\delta > 0$, and let f be the Fremlin function. We show that

$$\mathcal{B}(f^2, \delta) \not\subset \overline{\mathcal{B}(f, 1/4) \cdot \mathcal{B}(f, 1/4)}.$$

We consider the function $f^2 + \delta/2$. We show that, for every sequence $\{f_n\}$ and $\{g_n\}$ in $\mathcal{B}(f, 1/4)$, the sequence $\{f_n \cdot g_n\}$ does not converge to $f^2 + \delta/2$. Since $\delta/2 \leq \min_{x\in[0,\,1]} (f^2 + \delta/2)$ and, for every n, there exists x_n such that $f_n(x_n) \cdot g_n(x_n) = 0$, we have $\|f^2 + \frac{\delta}{2} - f_n \cdot g_n\|_m \geq \frac{\delta}{2}$. This proves the statement. □

If D satisfy $\pi_k(D) = I_k$, an interval in $[0,\, 1]$ such that $\cup_{i=1,\dots,n} \pi_k(D) = [0,\, 1]$, and f is a function with n-continuous derivatives, then we have

$$\|f\|_{\langle D\rangle} = \sum_{i=0}^{n} \|f^{(i)}\|_i \leq (n+1)\|f\|_m,$$

with $\|f^{(i)}\|_i = \max_{x\in I_i} |f^{(i)}(x)|$.

Lemma 3.8. *$P_* : C_{\langle D\rangle}^{(n)}[0,\, 1] \times C_{\langle D\rangle}^{(n)}[0,\, 1] \to C_m^{(n)}[0,\, 1]$, given by $P_*(f, g) = f \cdot g$, is weakly open.*

Proof. Given $\epsilon > 0$, we have

$$P_* \left(\mathcal{B}_{\langle D\rangle}(f, \epsilon) \times \mathcal{B}_{\langle D\rangle}(g, \epsilon)\right) \supset P_* \left(\mathcal{B}_m \left(f, \frac{\epsilon}{n+1}\right) \times \mathcal{B}_m \left(g, \frac{\epsilon}{n+1}\right)\right).$$

Since the multiplication in $(C^{(n)}[0,\, 1], \|\cdot\|_m)$ is weakly open, there exist $h \in C^{(n)}[0,\, 1]$ and $\delta > 0$ such that

$$\mathcal{B}_m(h, \delta) \subset P_* \left(\mathcal{B}_m \left(f, \frac{\epsilon}{n+1}\right) \times \mathcal{B}_m \left(g, \frac{\epsilon}{n+1}\right)\right) \subset P_* \left(\mathcal{B}_{\langle D\rangle}(f, \epsilon) \times \mathcal{B}_{\langle D\rangle}(g, \epsilon)\right).$$

This completes the proof. □

Example 3.9. We now give an example of a compact and connected set D such that the identity map $\mathrm{id} : (C^{(n)}[0,\, 1], \|\cdot\|_{\langle D\rangle}) \to (C^{(n)}[0,\, 1], \|\cdot\|_m)$ is not continuous. As expected, these two norms are not equivalent. Let $0 < \delta < 1$ and $D = [0,\, 1] \times \{0\} \times \cdots \times \{0\}$. Then $\mathcal{B}_{\langle D\rangle}(h, \eta)$ is not contained in $\mathcal{B}_m(h, \delta)$ for every $\eta > 0$ and $h \in C^{(n)}[0,\, 1]$. Let k be a positive integer, and let $f(x) = \frac{1}{k+1} x^{n+k}$. Then $\|f\|_{\langle D\rangle} = \frac{1}{k+1}$, and $f \in \mathcal{B}_{\langle D\rangle}\left(\mathbf{0}, \frac{1}{k}\right)$. The m-norm of f is equal to $\max\left\{\frac{1}{k+1}, \frac{n+k}{k+1}, \dots, \frac{(n+k)(n+k-1)\cdots(k+1)}{k+1}\right\} > 1$. Therefore, $f \notin \mathcal{B}_m(\mathbf{0}, \delta)$.

Fremlin's example shows that multiplication in $C[0,\, 1]$ is not open, but Wachowicz, in [21], showed that the multiplication on $C_m^{(n)}[0,\, 1]$ is weakly open. In this section, we study this property for the multiplication on $C^{(n)}_{\langle D\rangle}[0,\,1]$. We recall the statement of Wachowicz's theorem. We also refer the reader to [2].

Theorem 3.10. (*see*[21, Theorem 1]). *The multiplication on* $C^{(n)}_m[0,\,1]$ *is weakly open.*

It is an easy observation that openness properties are invariant under homeomorphisms. We conclude that the same results hold for $C^{(n)}_1[0,\,1]$, endowed with the norm $\|f\|_1 = \sum_{0\le i\le n} \max_{x\in[0,\,1]} |f^{(i)}(x)|$, since $\|f\|_m \le \|f\|_1 \le (n+1)\|f\|_m$ for every $f \in C^{(n)}[0,\,1]$.

A minor adjustment of the proof, presented in [21], shows the following corollary.

Corollary 3.11 (see [21]). *Let D be a connected and compact subset of* $[0,\,1]^{n+1}$ *such that* $[0,\,1] = \pi_0(D) \supseteq \pi_1(D) \supseteq \cdots \supseteq \pi_n(D)$, *and let* $C^{(n)}_{\langle D\rangle}[0,\,1]$ *denote the space of all n-continuously differentiable functions endowed with the* $\|\cdot\|_{\langle D\rangle}$. *Then the multiplication in* $C^{(n)}_{\langle D\rangle}[0,\,1]$ *is weakly open.*

We use the steps of the proof in [21], which we outline next.

Step 1. Let f and g be functions in $C^{(n)}[0,\,1]$. Bernstein polynomials uniformly approximate the functions and their derivatives, up to order n (see [12] and references therein). Each polynomial has a decomposition into a product of irreducible factors, either quadratic polynomials or $x-\alpha$. Then a polynomial $p(x)$ can be written as $p(x) = \Pi_{i=1}^n (x^2 + a_i x + b_i)^{k_i} \Pi_{j=1}^m (x-\alpha_j)^{l_j}$. If p is $\epsilon/2$-close to f, and p has multiple roots, say $(x-\alpha)^l$, then a perturbation of the roots defines an arbitrarily close polynomial with simple roots: $p_1(x) = (x-\alpha)(x-(\alpha+\epsilon_0))\cdots(x-(\alpha+(l-1)\epsilon_0))$, for a conveniently small ϵ_0, so that $\|f-p_1\|_1 < \epsilon$, a given positive number. For simplicity of notation, we also denote by f and g the polynomials with simple zeros and disjoint zero sets ϵ-close to the original functions. We denote the zero sets of f and g by $Z(f)$ and $Z(g)$, respectively.

Step 2. Define a partition of $[0,\,1]$, $0 = x_0 < x_1 < x_2 < \cdots < x_m = 1$, such that

$$Z(f) \subset \cup_{k\in\{0,2,\dots\}} [x_k,\, x_{k+1}] \quad\text{and}\quad Z(g) \subset \cup_{k\in\{1,3,\dots\}} [x_k,\, x_{k+1}].$$

Step 3. Extension Lemma. Let φ and h be functions in $C^{(n)}[0,\,1]$. Let $\eta > 0$ and $x_0 \in [0,\,1]$ such that $|\varphi^{(j)}(x_0) - h^{(j)}(x_0)| < \eta$, with $j = 0,\dots,n$. For every $x \in [0,\,1]$, we set

$$k(x) = h(x) + \sum_{j=0}^{n} \left(\varphi^{(j)}(x_0) - h^{(j)}(x_0)\right) \frac{(x-x_0)^j}{j!}.$$

Then for every $j \in \{0,1,\dots,n\}$ we have $k^{(j)}(x_0) = \varphi^{(j)}(x_0)$ and $k \in \mathcal{B}(h, e\eta)$. The function k is an extension of φ to the interval $[0,\,1]$.

Step 4. Given φ, δ-close to the product $f\cdot g$, the construction of $\tilde{f}$ and $\tilde{g}$, ϵ-close to f and g, respectively, is done as follows: Since $f|_{[0,x_1]} \neq 0$, we set $f_1 = f$ and $g_1 = \frac{\varphi}{f}$ over the interval $[x_0, x_1]$. Then, applying the Extension Lemma, there exists g_2 that extends g_1 to the interval $[x_1, x_2]$

so that $Z(g_2) \cap [x_1, x_2] = \emptyset$ and $g_2 \in \mathcal{B}(g, e\epsilon)$. Now extend f_1 to the interval $[x_1, x_2]$ by setting $f_2 = \frac{\varphi}{g_2}$ on $[x_1, x_2]$. This procedure repeats until we reach $[x_{m-1}, x_m]$. At each step of the construction, we decrease the value of δ in order to get the extended function and the ratio in the corresponding ϵ-ball (i.e. $\mathcal{B}(f, \epsilon)$ and $\mathcal{B}(g, \epsilon)$).

The distances $\|f - f_i\|_m$ and $\|g - g_i\|_m$ are bounded by a constant c times δ. The constant c depends only on f and g. Hence, the value of δ can be adjusted so that $c\delta < \epsilon$. In this proof, it is crucial that $\|\cdot\|_m$ is a quasi-algebra norm, more precisely, $\|f \cdot g\|_m \leq 2^n \|f\|_m \|g\|_m$.

Proof of Corollary 3.11. Lemma 3.5 asserts that $\|\cdot\|_{\langle D\rangle}$ is a quasi-algebra norm. Given $\epsilon > 0$ and $f, g \in C^{(n)}[0, 1]$, we apply Step 1 to approximate f and g with polynomials with simple zeros and disjoint zero sets. We apply Step 2 to define a partition that also includes all the end points of the interval $\pi_i(D) = I_i = [a_i, b_i]$. Given $\varphi \in \mathcal{B}(f \cdot g, \delta)$, we define $\tilde{f}_n$ and $\tilde{g}_n$ so that $\|\tilde{f}_n - f|_{I_n}\|_{\langle D\rangle} < \epsilon$ and $\|\tilde{g}_n - g|_{I_n}\|_{\langle D\rangle} < \epsilon$. We then pursue with the construction to extend to $[a_{n-1}, a_n]$ and $[b_n, b_{n+1}]$, by applying Steps 3 and 4. Hence, we extend $\tilde{f}_n$ and $\tilde{g}_n$ to these two intervals. We continue with the procedure until we cover the entire interval $[0, 1]$. By finitely many adjustments of δ, we have that $\varphi = \tilde{f} \cdot \tilde{g}$, with these two functions within ϵ from f and g, respectively. □

4. Openness of multiplication on $C^{(n)}_{\langle D\rangle}[0, 1]$

In [8], Behrends considered pairs of functions $(f, g) \in C[0, 1] \times C[0, 1]$ with the property that, for every $\epsilon > 0$, the product $f \cdot g$ is in the interior of the product of the balls $\mathcal{B}(f, \epsilon) \cdot \mathcal{B}(f, \epsilon)$. Pairs of continuous functions satisfying this condition are said to have the property $(*)$. Behrends, following an interesting approach, characterized those pairs of functions $(f, g) \in C[0, 1] \times C[0, 1]$ with the property $(*)$. As pointed out by a referee, this property can be rephrased in terms of local openness for the multiplication. Indeed, a pair (f, g) satisfying $(*)$ is a point of local openness for the multiplication. Characterizations of points of local openness of various nonopen bilinear and multilinear maps have been obtained by Behrends in [9] and [8]; see also [7] and [10].

Given f and g in $C[0, 1]$, $\gamma(t) = (f(t), g(t))$ with $t \in [0, 1]$ describes a path in $\mathbb{R}^2$. The question can now be formulated as under what conditions on f and g, a continuous perturbation of the product $f \cdot g$, say $f \cdot g + p$, with p continuous such that $\|p\|_\infty < \delta$, does there exist a small perturbation of γ, $(\gamma + \tau)(t) = (f_1(t), g_1(t))$, such that $f \cdot g + p = f_1 \cdot g_1$. Equivalently, we ask the following question: Under what conditions on f and g is (f, g) a point of local openness? In [8], it was shown that the pairs (f, g) such that the corresponding path does not cross the origin, have this property. This is a consequence of the implicit function theorem (see [8, Lemma 2.1]). The difficulty relies on those paths that pass through the origin. In that case, there are two essentially distinct possibilities. One can be labeled as an acceptable crossing (AC) and the other an unacceptable crossing (UC). A UC is a crossing where the path crosses the origin, using the first and third quadrants or using the second and fourth. All the other crossings are acceptable (i.e. AC). More precisely, $\gamma(t) = (f(t), g(t))$ has a UC if for some $t_0 \in (0,1)$, $\gamma(t_0) = (0,0)$ and there exists $\epsilon > 0$ such that $f(t) \cdot g(t) \geq 0$, for every $t \in (t_0 - \epsilon, t_0 + \epsilon)$, or $f(t) \cdot g(t) \leq 0$ for every $t \in (t_0 - \epsilon, t_0 + \epsilon)$. In [8], Behrends proved the following amazing result.

Theorem 4.1. *Consider f and g in $C[0,\,1]$. Then, for every $\epsilon > 0$, the product $f \cdot g$ is in the interior of $\mathcal{B}(f \cdot g, \epsilon)$ if and only if γ has only acceptable crossings.*

Our goal is to extend some of these ideas to the new class of spaces $C^{(n)}_{\langle D\rangle}([0,\,1])$. We start by recalling property $(*)$, or local openness for the multiplication.

Definition 4.2. Let X be a quasi-normed algebra. Let x and y be two elements in X. We say that (x, y) has the property $(*)$ if, for every $\epsilon > 0$, $x \cdot y \in \text{int}\, (\mathcal{B}(x, \epsilon) \cdot \mathcal{B}(y, \epsilon))$, or, equivalently, (x, y) is a point of local openness for the multiplication.

We consider the space $C^{(n)}_{\langle D\rangle}[0,\,1]$. Then, we denote by $f \cdot g$ the product of f and g. We consider the function $H : C^{(n)}_{\langle D\rangle}[0,\,1] \times C^{(n)}_{\langle D\rangle}[0,\,1] \to C^{(n)}_{\langle D\rangle}[0,\,1]$, given by $H(f, g) = f \cdot g$. For simplicity of notation, the symbols E and F denote, respectively, the spaces $C^{(n)}_{\langle D\rangle}[0,\,1] \times C^{(n)}_{\langle D\rangle}[0,\,1]$ and $C^{(n)}_{\langle D\rangle}[0,\,1]$.

Given $(f_0, g_0) \in E$, we consider the operator $dH|_{(f_0,g_0)} : E \to F$, given by $dH|_{(f_0,g_0)}(f, g) = f_0 \cdot g + g_0 \cdot f$.

Proposition 4.3. *If D is a compact, connected subset of $[0,\,1]^{n+1}$ such that $[0,\,1] = \pi_0(D) \supseteq \pi_1(D) \supseteq \cdots \supseteq \pi_n(D)$, then $dH|_{(f_0,g_0)}$ is the Fréchet derivative of H at (f_0, g_0) for every $(f_0, g_0) \in E$.*

Proof. Given ω_0 and ω_1 in $C^{(n)}_{\langle D\rangle}[0,\,1]$, we have

$$\begin{aligned}
&\lim_{\|\omega\|\to 0} \frac{\|H(f_0+\omega_0, g_0+\omega_1) - H(f_0, g_0) - dH|_{(f_0,g_0)}(\omega_0,\omega_1)\|_{\langle D\rangle}}{\|(\omega_0,\omega_1)\|} \\
&\quad= \lim_{\|\omega\|\to 0} \frac{\|(f_0+\omega_0)\cdot(g_0+\omega_1) - f_0\cdot g_0 - \omega_0\cdot g_0 - \omega_1\cdot f_0\|_{\langle D\rangle}}{\max\{\|\omega_0\|_{\langle D\rangle}, \|\omega_1\|_{\langle D\rangle}\}} \\
&\quad= \lim_{\|\omega\|\to 0} \frac{\|\omega_0\cdot\omega_1\|_{\langle D\rangle}}{\max\{\|\omega_0\|_{\langle D\rangle}, \|\omega_1\|_{\langle D\rangle}\}}.
\end{aligned}$$

Since D norm is a quasi-algebra norm with constant $2^n(n+1)$, we have

$$\begin{aligned}
&\lim_{\|\omega\|\to 0} \frac{\|H(f_0+\omega_0, g_0+\omega_1) - H(f_0, g_0) - dH|_{(f_0,g_0)}(\omega_0,\omega_1)\|_{\langle D\rangle}}{\|(\omega_0,\omega_1)\|} \\
&\qquad\leq \lim_{\|\omega\|\to 0} \frac{2^n(n+1)\|\omega_0\|_{\langle D\rangle}\|\omega_1\|_{\langle D\rangle}}{\|\omega_0\|_{\langle D\rangle}} = 0.
\end{aligned}$$

This completes the proof. □

The next proposition characterizes those points (f_0, g_0) that yield a surjective Fréchet derivative $dH|_{(f_0,g_0)}$. We denote the zero set of a map $f \in C^{(n)}[0,\,1]$ by $\mathcal{Z}(f)$ (i.e. $\mathcal{Z}(f) = \{t \in [0,\,1] : \ f(t) = 0\}$).

Proposition 4.4. *The Fréchet derivative of H, $dH|_{(f_0,g_0)}$ is surjective if and only if $\mathcal{Z}(f_0)\cap\mathcal{Z}(g_0)=\emptyset$.*

Proof. We first prove that the condition on the zero sets is necessary. Suppose there exists t_0 such that $f_0(t_0)=g_0(t_0)=0$. Then the range of $dH|_{(f_0,g_0)}$ is contained in the space of all functions that vanish at t_0. This implies that $dH|_{(f_0,g_0)}$ is not surjective.

Conversely, we first assume that $f_0(0)\neq 0$ and $\mathcal{Z}(f_0)\cup\mathcal{Z}(g_0)$ is finite. Then there exists a (finite) partition of $[0,\,1]$, $0=a_0<b_0<a_1<b_1<\cdots<1$, such that $f_0|_{[a_i,\,b_i]}\neq 0$ and $g_0|_{[b_i,a_{i+1}]}\neq 0$. Let $L>0$ such that

$$L\le\min\{|f_0(t)|,\,|g_0(s)|:\ t\in\cup_i[a_i,b_i],\ \text{and}\ s\in\cup_i[b_i,a_{i+1}]\}.$$

We choose $\epsilon_0>0$ such that $\epsilon_0<\min\{\frac{|a_0-b_0|}{3},\,\frac{|a_1-b_0|}{3},\,\dots\}$,

$$\min\{|f_0(t)|:\ t\in\cup_i(a_i-\epsilon_0,\,b_i+\epsilon_0)\}\ge\frac{L}{2},$$

and

$$\min\{|g_0(s)|:\ s\in\cup_i(b_i-\epsilon_0,\,a_{i+1}+\epsilon_0)\}\ge\frac{L}{2}.$$

For each i, we define a $C^\infty[0,\,1]$ bump function $\varphi_i|_{[a_i,b_i]}\equiv 1$ and

$$\varphi_i|_{[0,\,1]\setminus(a_i-\epsilon_0,b_i+\epsilon_0)}\equiv 0.$$

Let $\varphi=\sum_i\varphi_i$. Similarly we define ψ as the sum of ψ_i, where $\psi_i|_{[b_i,\,a_{i+1}]}=1$ and $\psi_i|_{[0,\,1]\setminus(b_i-\epsilon_0,a_{i+1}+\epsilon_0)}\equiv 0$. We notice that $\varphi+\psi$ is never equal to zero.

In order to prove surjectivity of $dH|_{(f_0,g_0)}$, we consider Z as a function in F, and we need to construct $(X,Y)\in E\times E$ such that $X\cdot f_0+Y\cdot g_0=Z$. We set $X(t)=\frac{Z(t)\cdot\varphi(t)}{f_0(t)(\varphi(t)+\psi(t))}$ for $t\notin\mathcal{Z}(f_0)$, $X(t)=0$ for $t\in\mathcal{Z}(f_0)$, $Y(t)=\frac{Z(t)\cdot\psi(t)}{g_0(t)(\varphi(t)+\psi(t))}$ for $t\notin\mathcal{Z}(g_0)$, and $Y(t)=0$ for $t\in\mathcal{Z}(g_0)$. We notice that $f_0\cdot X+g_0\cdot Y=Z$. Moreover, if $\mathcal{Z}(f_0)=\emptyset$ (or $\mathcal{Z}(g_0)=\emptyset$), then we just set $X(t)=\frac{Z(t)}{f_0(t)}$ and $Y(t)=0$ (respectively, $X(t)=0$ and $Y(t)=\frac{Z(t)}{g_0(t)}$).

It remains to show that, for arbitrary zero sets with empty intersection, we can construct a partition with the property described above. We have assumed that $f_0(0)\neq 0$ (a similar argument works if we assume that $g_0(0)\neq 0$). We set $a_0=0$. Let $x_0=\sup\{t:\ \mathcal{Z}(f_0)\cap[0,t]=\emptyset\}$. It is clear that, if $f_0(x_0)=0$, then $g_0(x_0)\neq 0$. We choose $0<\epsilon_0<\frac{x_0}{2}$ such that $\mathcal{Z}(g_0)\cap[x_0-\epsilon_0,x_0+\epsilon_0]=\emptyset$. We set $b_0=x_0-\frac{\epsilon_0}{2}$. Let $y_0=\sup\{t:\ \mathcal{Z}(g_0)\cap[x_0,t]=\emptyset\}$. Clearly, $g_0(y_0)=0$, and then $f_0(y_0)\neq 0$. There exists $\epsilon_1>0$ such that $\epsilon_1<\frac{y_0-x_0}{2}$ and $\mathcal{Z}(f_0)\cap[y_0-\epsilon_1,y_0+\epsilon_1]=\emptyset$. We set $a_1=y_0-\frac{\epsilon_1}{2}$. We continue this process to find a sequence $a_0=0<b_0<a_1<b_1\cdots$ with the desirable property. We claim that this sequence is finite, since, otherwise, a_i and b_i would converge to some point t in $[0,\,1]$; then either $f_0(t)\neq 0$ or $g_0(t)\neq 0$. We assume that $f_0(t)\neq 0$. Then f_0 does not vanish in a small neighborhood of t. This neighborhood contains all the intervals $[a_i,b_i]$ and $[b_i,a_{i+1}]$, after a certain order. This is impossible because each interval $[b_i,a_{i+1}]$ must contain a zero of f_0. This contradiction shows that $a_0=0<b_0<a_1<b_1\cdots<(a_n$ or $b_n=)\,1$. We observe that the argument above handles the case when $\mathcal{Z}(f_0)\cup\mathcal{Z}(g_0)$ is infinite, Since the intersection of these two zero sets is empty, it allows us to define the partition as explained above, and the argument follows. This completes the proof. □

We say that a closed subspace M of a normed space X is complemented if there exists a closed subspace N such that $X = M \oplus N$.

Lemma 4.5. *Let D be a compact, connected subset of $[0,\,1]^{n+1}$ such that $[0,\,1] = \pi_0(D) \supseteq \pi_1(D) \supseteq \cdots \supseteq \pi_n(D)$. Then, for every $(f_0, g_0) \in E$, such that $\mathcal{Z}(f_0) = \emptyset$ or $\mathcal{Z}(g_0) = \emptyset$, the kernel of the Fréchet derivative $dH|_{(f_0,g_0)}$ is complemented in E.*

Proof. We assume that $\mathcal{Z}(f_0) = \emptyset$. We show that $W = \ker dH|_{(f_0,g_0)}$ is a complemented subspace. To this end, we show that P, given by $P(f,g) = (f, -\frac{g_0}{f_0} \cdot f)$, is a bounded projection: Hence, we will have $W \oplus Ran(P) = E$. It is clear that P is a projection. We show that P is bounded,

$$\begin{aligned}\left\|\left(f, -\frac{g_0}{f_0}\cdot f\right)\right\| &= \max\left\{\|f\|_{\langle D\rangle}, \left\|\frac{g_0}{f_0}\cdot f\right\|_{\langle D\rangle}\right\}\\ &\le \max\left\{\|f\|_{\langle D\rangle}, 2^n(n+1)\left\|\frac{g_0}{f_0}\right\|_{\langle D\rangle}\cdot\|f\|_{\langle D\rangle}\right\}\\ &\le 2^n(n+1)\max\left\{1, \left\|\frac{g_0}{f_0}\right\|_{\langle D\rangle}\right\}\cdot\max\{\|f\|_{\langle D\rangle}\cdot\|g\|_{\langle D\rangle}\}.\end{aligned}$$

Since $(f,g) - P(f,g) \in W$ and $P(f,g) \in Ran(P)$, it follows that $W \oplus Ran(P) = E$. □

Remark 4.6. We observe that the proof provided for the Lemma 4.5 also implies that the kernel of the Fréchet derivative $dH|_{(f_0,g_0)}$ is complemented in E if either the ratio $\frac{f_0}{g_0}$ or $\frac{g_0}{f_0}$ has continuous extension to the interval $[0,\,1]$.

We now state a version of the submersion theorem for normed spaces.

Theorem 4.7 (see [15] and [16])**.** *Let E, F be normed spaces, let $U \subset E$ be open, and let $\phi \in C^{(k)}(U, F)$ with $k \ge 1$. Assume that there exist $a \in U$ and a subspace E_1 of E such that $\ker d\phi|_a$ is an isomorphism between E_1 and F. Moreover, assume also that $E = E_1 \oplus \ker d\phi|_a$ (i.e. $\ker d\phi|_a$ is a complemented subspace in E). Then there exist $U' \subset U$, an open set containing a, $W \subset F$, an open set containing $\phi(a)$, and $\tilde{U} \subset \ker d\phi|_a$, an open set containing 0 such that the map*

$$\begin{aligned} g : &U' \longrightarrow W \times \tilde{U}\\ &x \longmapsto (\phi(x), \pi(x-a))\end{aligned}$$

is a $C^{(k)}$-diffeomorphism from U' onto $g(U')$, where $\pi := E_1 \oplus \ker d\phi|_a \to \ker d\phi|_a$ denotes the projection onto $\ker d\phi|_a$.

This allows us to derive the following result.

Proposition 4.8. *Let D be a compact, connected subset of $[0,\,1]^{n+1}$ such that $[0,\,1] = \pi_0(D) \supseteq \pi_1(D) \supseteq \cdots \supseteq \pi_n(D)$. If $(f_0, g_0) \in E \times E$ such that $\mathcal{Z}(f_0) = \emptyset$ or $\mathcal{Z}(g_0) = \emptyset$, then (f_0, g_0) is a point of local openness for the multiplication.*

Proof. The statement follows from an application of Theorem 4.7. Lemma 4.5 and Proposition 4.4 imply the hypotheses of the proposition. Then H is a submersion by Theorem 4.7, and there exist open neighborhoods of f_0 and g_0 in $C^{(n)}_{\langle D\rangle}[0,\,1]$, U and V, and a neighborhood in $C^{(n)}_{\langle D\rangle}[0,\,1]$ of 0, W, such that W is $C^{(n)}$-diffeomorphic to a subset of $U \times V$. This implies the statement. □

We now give a necessary condition to assure that a pair of functions in $C^{(n)}_{\langle D\rangle}[0,\,1]$ has the property $(*)$.

Proposition 4.9. *Let D be a connected and compact subset of $[0,\,1]^{n+1}$ such that $\pi_0(D) = [0,\,1]$. Let f_0 and g_0 be functions in $C^{(n)}_{\langle D\rangle}[0,\,1]$. If (f_0, g_0) is a point of local openness for the multiplication, then $\gamma(t) = (f_0(t), g_0(t))$ has only acceptable crossings.*

Proof. We assume that γ has a positive crossing. This means that there exist $t_0 \in (0,1)$ and a small interval $(t_0 - \epsilon,\, t_0 + \epsilon_0)$ around t_0 such that $f_0(t)$ and $g_0(t)$ are both positive over the interval $(t_0,\, t_0 + \epsilon_0)$ and both negative over the interval $(t_0 - \epsilon_0,\, t_0)$. Therefore $(f \cdot g)|_{(t_0-\epsilon_0,\, t_0+\epsilon_0)}$ is nonnegative. Given $\delta > 0$, the function $f \cdot g + \frac{\delta}{2}$ is in $\mathcal{B}_{\langle D\rangle}(f \cdot g, \delta)$ and strictly positive over the open interval $(t_0 - \epsilon,\, t_0 + \epsilon_0)$. Let $\epsilon = \frac{1}{2}\min\{|f(t_0 - \epsilon_0)|,\, |f(t_0 + \epsilon_0)|\}$. Given h in $\mathcal{B}_{\langle D\rangle}(f_0, \epsilon)$, we have $\|f_0 - h\|_\infty \le \|f_0 - h\|_{\langle D\rangle} < \epsilon$. Therefore, h must vanish at some point in the interval $(t_0 - \epsilon,\, t_0 + \epsilon_0)$. This implies that every function in the product $\mathcal{B}(f_0, \epsilon)\cdot\mathcal{B}(g_0, \epsilon)$ must vanish at some point in the interval $(t_0-\epsilon,\, t_0+\epsilon_0)$. Therefore $f \cdot g + \frac{\delta}{2}$ is not in $\mathcal{B}(f_0, \epsilon) \cdot \mathcal{B}(g_0, \epsilon)$. This completes the proof. □

In conclusion, we mention that a problem of potential interest is the study of the openness of maps with dense domain. More precisely we consider a product of p-integrable functions on the interval $[0,\,1]$. It is clear that the product P, $P(f,g) = f \cdot g$, of two p-integrable functions is not necessarily p-integrable. However such a product is densely defined (i.e. $P : Dom(P) \to L_p([0,\,1])$, with $Dom(P)$ a dense subset of $L_p([0,\,1])$). The domain of P contains $L_p([0,\,1]) \times B([0,\,1]) \cup B([0,\,1]) \times L_p([0,\,1])$, where $B([0,\,1])$ denotes the set of all bounded functions. We formulate the following result.

Theorem 4.10. *Let $1 \le p < \infty$, and let P be the multiplication on $L_p([0,\,1])$ with domain $Dom(P)$. Then P is uniformly open.*

The proof follows the same approach provided by Balcerzak, Majchrzycki, and Wachowicz in [5] for the openness of maps of two variables on spaces of integrable functions, where $T : L_p([0,\,1]) \times L_q([0,\,1]) \to L_1[0,1]$ with $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$.

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