Adv. Oper. Theory 4 (2019), no. 1, 284–304
https://doi.org/10.15352/aot.1802-1319
ISSN: 2538-225X (electronic)
https://projecteuclid.org/aot

ℓ_1-SUMMABILITY AND LEBESGUE POINTS OF d-DIMENSIONAL FOURIER TRANSFORMS

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Communicated by L. Molnar

Abstract. The classical Lebesgue's theorem is generalized, and it is proved that under some conditions on the summability function θ, the ℓ_1-θ-means of a function f from the Wiener amalgam space $W(L_1, \ell_\infty)(\mathbb{R}^d) \supset L_1(\mathbb{R}^d)$ converge to f at each modified strong Lebesgue point and thus almost everywhere. The θ-summability contains the Weierstrass, Abel, Picard, Bessel, Fejér, de La Vallée-Poussin, Rogosinski, and Riesz summations.

1. Introduction

For the Fejér means of an integrable function $f \in L_1(\mathbb{R})$, the classical theorem of Lebesgue [18] says that

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T s_t f(x)\, dt = f(x)$$

at each Lebesgue point of f, thus almost everywhere, where

$$s_t f(x) := \frac{1}{\sqrt{2\pi}} \int_{-t}^{t} \widehat{f}(v) e^{\imath x v}\, dv \qquad (t > 0)$$

and $\widehat{f}$ denotes the Fourier transform of the one-dimensional function f. In the present paper this result will be generalized to the ℓ_1-summability of higher dimensional functions.

Date: Received: Feb. 21, 2018; Accepted: Sep. 10, 2018.
2010 *Mathematics Subject Classification.* Primary 42B08; Secondary 42A38, 42A24, 42B25.
Key words and phrases. Fourier transforms, ℓ_1-summability, Fejér summability, θ-summability, Lebesgue points.

A general method of summation, the so called θ-summation method, which is generated by a single function θ and which includes the well known Fejér, Riesz, Weierstrass, Abel, and so on summability methods, is studied intensively in the literature (see, e.g., Butzer and Nessel [3], Christ [4], Stein and Weiss [24, 25], Lu and Yan [19], Trigub and Belinsky [27], Gát [9, 10, 11], Goginava [12, 13, 14], Simon [22, 23], Nagy, Persson, Tephnadze and Wall [20, 21], and Weisz [28, 30]).

The ℓ_1- or triangular means of d-dimensional Fourier transforms generated by the θ-summation are defined by

$$\sigma_T^\theta f(x) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \theta\left(\frac{|v|}{T}\right) \widehat{f}(v) e^{\imath x \cdot v}\, dv, \tag{1.1}$$

where $|v| = |v_1| + \cdots + |v_d|$. For $\theta(t) = \max((1-|t|), 0)$ we get back the usual Fejér means (see later). Berens, Li, and Xu [1, 2] have proved that $\sigma_T^\theta f \to f$ almost everywhere for the Riesz summability (i.e., if $\theta(v) := \max((1-|v|)^\beta, 0)$, $0 < \beta < \infty$), where $f \in L_1(\mathbb{R}^d)$. Szili and Vértesi [26] considered the ℓ_1-Fejér summability. Recently, using Hardy spaces and the boundedness of the maximal θ-operator from the Hardy space to the $L_p(\mathbb{R})$ space, in [29], we generalized this convergence result and gave a common proof for several different θ's, such as for the Weierstrass, Abel, Picard, Bessel, Fejér, de La Vallée-Poussin, Rogosinski, and Riesz summations. However, in contrary to the one-dimensional case, the set of convergence is not yet known.

In this paper, we generalize the classical Lebesgue's theorem about the Lebesgue points of one-dimensional integrable functions to multi-dimensional functions and also to the Wiener amalgam space $W(L_1, \ell_\infty)(\mathbb{R}^d)$, which is much larger than $L_1(\mathbb{R}^d)$. More exactly, we introduce the concept of modified strong Lebesgue points. It is verified in [31, Theorem 2] that almost every point is a modified strong Lebesgue point of $f \in W(L_1, \ell_\infty)(\mathbb{R}^d)$. Under some weak conditions on θ, we show that the ℓ_1-θ-means of a multidimensional function $f \in W(L_1, \ell_\infty)(\mathbb{R}^d)$ $(d \geq 2)$ converge to f at each modified strong Lebesgue point. The same results for $d = 2$ were shown in [32]. The proof for $d = 2$ in [32] is much simpler, and it differs from the present proof significantly. The difference between the proofs is that in [32], we could find a useful closed form for the kernel function in the two-dimensional case, but there is no closed form for higher dimensions. So the present proof needs essentially new ideas.

2. Wiener amalgam spaces

Let us fix $d \geq 2$, $d \in \mathbb{N}$. For a set $\mathbb{Y} \neq \emptyset$, let $\mathbb{Y}^d$ be its Cartesian product $\mathbb{Y} \times \cdots \times \mathbb{Y}$ taken with itself d times. For $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ and $u = (u_1, \ldots, u_d) \in \mathbb{R}^d$, set

$$u \cdot x := \sum_{k=1}^d u_k x_k, \qquad \|x\|_p := \left(\sum_{k=1}^d |x_k|^p\right)^{1/p}, \qquad |x| := \|x\|_1 .$$

We briefly write $L_p(\mathbb{R}^d)$ instead of the $L_p(\mathbb{R}^d,\lambda)$ space equipped with the norm

$$\|f\|_p := \left(\int_{\mathbb{R}^d} |f(x)|^p \, d\lambda(x)\right)^{1/p} \qquad (1 \le p < \infty),$$

with the usual modification for $p=\infty$, where λ is the Lebesgue measure. Integrating over $[0,1)^d$, we obtain the definition of $L_p[0,1)^d$. These spaces are generalized as follows. A measurable function f belongs to the *Wiener amalgam space* $W(L_p,\ell_q)(\mathbb{R}^d)$ $(1 \le p,q \le \infty)$ if

$$\|f\|_{W(L_p,\ell_q)} := \left(\sum_{k\in\mathbb{Z}^d} \|f(\cdot+k)\|^q_{L_p[0,1)^d}\right)^{1/q} < \infty,$$

with the obvious modification for $q=\infty$.

It is easy to see that $W(L_p,\ell_p)(\mathbb{R}^d) = L_p(\mathbb{R}^d)$ and the following continuous embeddings hold true:

$$W(L_{p_1},\ell_q)(\mathbb{R}^d) \supset W(L_{p_2},\ell_q)(\mathbb{R}^d) \qquad (p_1 \le p_2)$$

and

$$W(L_p,\ell_{q_1})(\mathbb{R}^d) \subset W(L_p,\ell_{q_2})(\mathbb{R}^d) \qquad (q_1 \le q_2),$$

$(1 \le p_1,p_2,q_1,q_2 \le \infty)$. Thus

$$W(L_\infty,\ell_1)(\mathbb{R}^d) \subset L_p(\mathbb{R}^d) \subset W(L_1,\ell_\infty)(\mathbb{R}^d) \qquad (1 \le p \le \infty).$$

For more about Wiener amalgam spaces, see Fournier and Stewart [8] and Heil [16]. Note that all homogeneous Banach space over R^d can be continuously embedded into $W(L_1,\ell_\infty)(\mathbb{R}^d)$ (see Katznelson [17]).

In this paper the constant C may vary from line to line.

3. The summability function

In this article, we will consider a general summability method, the so called ℓ_1-θ-summation defined by a function $\theta : [0,\infty) \to \mathbb{R}$. This summation contains all well known summability methods, such as the Fejér, Riesz, Weierstrass, Abel, Picard, and Bessel summations. Here we simplify the conditions on θ given in Weisz [31].

We suppose that θ is *absolutely continuous*. Suppose further that

$$\theta(0)=1, \qquad \int_0^\infty (t\vee 1)^d |\theta'(t)| \, dt < \infty, \tag{3.1}$$

where $\vee$ denotes the maximum, that is, $t \vee 1 = \max(t,1)$.

Lemma 3.1. *If θ is absolutely continuous and satisfies the second condition of (3.1), then $\theta(t)$ converges to some real number A as $t\to\infty$ and*

$$\lim_{t\to\infty} t^d(\theta(t)-A) = 0. \tag{3.2}$$

Proof. Indeed,

$$|\theta(t) - \theta(T)| \le \int_t^T |\theta'(s)| \, ds < \epsilon$$

if $T > t > t_0(\epsilon) > 0$. Thus θ is Cauchy and so convergent to some real number A, as $t \to \infty$. On the other hand,

$$t^d \, |\theta(t) - \theta(T)| \le t^d \int_t^T |\theta'(s)| \, ds \le \int_t^T s^d \, |\theta'(s)| \, ds < \epsilon$$

if $T > t > t_1(\epsilon) > 0$. Letting $T \to \infty$, we have $t^d \, |\theta(t) - A| < \epsilon$, which yields (3.2). □

If $A \neq 0$, then, writing $\theta = A + (\theta - A)$, we decompose the θ-means defined later in (4.2) into two parts. The first part is the inversion formula (4.1) multiplied by the constant A, which is divergent in general. As we will see later, under some conditions, the second part converges almost everywhere. So we may suppose that

$$\lim_{t \to \infty} \theta(t) = 0, \tag{3.3}$$

which implies also that

$$\lim_{t \to \infty} t^d \theta(t) = 0.$$

Since by integration by parts,

$$\int_0^\infty t^{d-1} \theta(t) \, dt = -\frac{1}{d} \int_0^\infty t^d \theta'(t) \, dt,$$

the function $t^{d-1}\theta(t)$ is also integrable.

In addition, we will suppose also that

$$\left| \int_0^\infty t^k \theta'(t) (\operatorname{soc})^{(k)}(tu) \, dt \right| \le C u^{-\alpha} \qquad (k = 0, \ldots, d-1) \tag{3.4}$$

for some $0 < \alpha < \infty$ and all $u > 0$, where the function soc is defined by

$$\operatorname{soc} t := \begin{cases} \cos t & \text{if } d \text{ is even;} \\ \sin t & \text{if } d \text{ is odd,} \end{cases}$$

and $(\operatorname{soc})^{(k)}$ denotes its kth derivative. Since, by (3.1), the left hand side is always finite, (3.4) holds for small u, say for $0 < u \le 1$. So (3.4) is important for large u, say for $u > 1$. If (3.4) holds for some $\alpha > 1$, then it holds also for $\alpha = 1$. So we may suppose that (3.4) holds for some $0 < \alpha \le 1$ and all $u > 0$.

Lemma 3.2. *Suppose that θ is absolutely continuous and satisfies the second condition of (3.1). If*

$$\left| \int_0^\infty \theta'(t) \, e^{\imath t u} \, dt \right| \le C u^{-\alpha} \tag{3.5}$$

for some $0 < \alpha \le 1$ and all $u > 0$, then (3.4) holds.

Proof. Let us denote the integral on the left hand side by $F(u)$, that is,

$$F(u) = \int_0^\infty \theta'(t)\, e^{\imath t u}\, dt.$$

Then by (3.1) and the Lagrange mean value theorem, for any $x \geq 1$ and $0 < \epsilon \leq 1$, there exists $v \in (x, x+\epsilon)$ such that

$$\begin{aligned}\left| \int_0^\infty t\theta'(t)\, e^{\imath t v}\, dt \right| &= |F'(v)| \leq |F(x+\epsilon) - F(x)| \\ &\leq C(x+\epsilon)^{-\alpha} + Cx^{-\alpha} \leq C(1+2^\alpha)(x+\epsilon)^{-\alpha}.\end{aligned}$$

Since the second derivative of F is bounded, for any $u \in [x, x+\epsilon]$, we have

$$|F'(u) - F'(v)| = |F''(\xi)|\, |u-v| \leq C'\epsilon.$$

If $\epsilon \leq (x+\epsilon)^{-\alpha}$, in other words, $x + \epsilon \leq (1/\epsilon)^{1/\alpha}$, then

$$\begin{aligned}|F'(u)| \leq |F'(v)| + |F'(u) - F'(v)| &\leq \left(C(1+2^\alpha) + C'\right)(x+\epsilon)^{-\alpha} \\ &\leq \left(C(1+2^\alpha) + C'\right) u^{-\alpha}.\end{aligned}$$

This leads us to the inequality

$$|F'(u)| \leq \left(C(1+2^\alpha) + C'\right) u^{-\alpha} \qquad \text{on the interval } [1, (1/\epsilon)^{1/\alpha}].$$

Since ϵ is arbitrary, the inequality holds on the interval $[1, \infty)$. F' is also bounded; hence

$$\left| \int_0^\infty t\theta'(t)\, e^{\imath t u}\, dt \right| \leq C_1 u^{-\alpha} \qquad (u > 0).$$

In the same way, we can show that

$$\left| \int_0^\infty t^k\theta'(t)\, e^{\imath t u}\, dt \right| \leq C_k u^{-\alpha} \qquad (u > 0, k = 0, \ldots, d-1),$$

which implies (3.4). □

4. The kernel functions

The *Fourier transform* of $f \in L_1(\mathbb{R}^d)$ is defined by

$$\widehat{f}(x) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} f(t) e^{-\imath x \cdot t}\, dt \qquad (x \in \mathbb{R}^d),$$

where $\imath = \sqrt{-1}$. Suppose that $f \in L_p(\mathbb{R}^d)$ for some $1 \leq p \leq 2$. The Fourier inversion formula

$$f(x) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \widehat{f}(v) e^{\imath x \cdot v}\, dv \qquad (x \in \mathbb{R}^d, \widehat{f} \in L_1(\mathbb{R}^d)) \tag{4.1}$$

motivates the definition of the ℓ_1-*Dirichlet integral* $s_t f$:

$$s_t f(x) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} 1_{\{|v| \leq t\}} \widehat{f}(v) e^{\imath x \cdot v}\, dv = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} f(x-u) D_t^d(u)\, du \qquad (t > 0),$$

where $|v| = |v_1| + \cdots + |v_d|$ and the *Dirichlet kernel* is defined by

$$D_t^d(u) := \int_{\mathbb{R}^d} 1_{\{|v| \leq t\}} e^{\imath u \cdot v}\, dv.$$

The dimension d is omitted in the notation of D_t^d if it does not cause ambiguity. Obviously,

$$|D_t^d(u)| \leq Ct^d \qquad (u \in \mathbb{R}^d).$$

It is known (see, e.g., Grafakos [15]) that for $f \in L_p(\mathbb{R}^d)$, $1 < p < \infty$,

$$\lim_{T\to\infty} s_T f = f \qquad \text{in the } L_p(\mathbb{R}^d)\text{-norm and a.e.}$$

This convergence does not hold for $p = 1$. However, using a summability method, we can prove some almost everywhere results for $p = 1$.

For $T > 0$, the ℓ_1-θ-*means* of a function $f \in L_p(\mathbb{R}^d)$ $(1 \leq p \leq 2)$ are defined by

$$\sigma_T^\theta f(x) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \theta\left(\frac{|v|}{T}\right) \widehat{f}(v) e^{\imath x\cdot v}\, dv. \tag{4.2}$$

It is easy to see that

$$\sigma_T^\theta f(x) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} f(x-u) K_T^\theta(u)\, du, \tag{4.3}$$

where the ℓ_1-θ-*kernel* is given by

$$\begin{aligned} K_T^\theta(x) &:= \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \theta\left(\frac{|u|}{T}\right) e^{\imath x\cdot u}\, du \\ &= \frac{-1}{(2\pi)^{d/2} T} \int_{\mathbb{R}^d} \int_{|u|}^\infty \theta'\left(\frac{t}{T}\right) dt\, e^{\imath x\cdot u}\, du \\ &= \frac{-1}{(2\pi)^{d/2} T} \int_0^\infty \theta'\left(\frac{t}{T}\right) D_t(x)\, dt \\ &= \frac{-1}{(2\pi)^{d/2}} \int_0^\infty \theta'(t) D_{tT}(x)\, dt. \end{aligned} \tag{4.4}$$

Observe that the integrability of the kernel function K_T^θ (see Lemma 4.6) implies that the convolution in (4.3) can be extended to all $f \in W(L_1, \ell_\infty)(\mathbb{R}^d)$. Hence

$$\sigma_T^\theta f(x) = \frac{-1}{T} \int_0^\infty \theta'\left(\frac{t}{T}\right) s_t f(x)\, dt.$$

Note that for the Fejér means (i.e., for $\theta(t) = (1-t)\vee 0$) we get the usual definition

$$\sigma_T^\theta f(x) = \frac{1}{T} \int_0^T s_t f(x)\, dt.$$

It is clear that

$$|K_T^\theta| \leq CT^d. \tag{4.5}$$

The Dirichlet kernel D_t can be expressed with the help of divided differences. The nth *divided difference* of a function f at the (pairwise distinct) knots $x_1, \ldots, x_n \in \mathbb{R}$ is introduced inductively as

$$[x_1]f := f(x_1), \qquad [x_1, \ldots, x_n]f := \frac{[x_1, \ldots, x_{n-1}]f - [x_2, \ldots, x_n]f}{x_1 - x_n}.$$

One can see that the divided differences are symmetric functions of the knots. It is known (see, e.g., DeVore and Lorentz [5, p. 120]) that if f is $(n-1)$-times continuously differentiable on $[a,b]$ and $x_1,\dots,x_n \in [a,b]$, then there exists $\xi \in [a,b]$ such that

$$[x_1,\dots,x_n]f = \frac{f^{(n-1)}(\xi)}{(n-1)!}. \tag{4.6}$$

Let

$$G_t(u) := (-1)^{[d/2]} 2^d u^{(d-2)/2} \operatorname{soc}(t\sqrt{u}).$$

The following lemma is proved in Berens and Xu [2].

Lemma 4.1. *We have*

$$D_t(x) = [x_1^2,\dots,x_d^2]G_t.$$

Definition 4.2. We say that a sequence of index pairs $(i_l,j_l)_l = (i_l,j_l)_{l=1}^{d-1}$ is proper, if it satisfies the following properties:

- $i_1 = 1$, $j_1 = d$,
- for any $l = 1,\dots,d-2$, we have that either $i_{l+1} = i_l$ and $j_{l+1} = j_l - 1$ or $i_{l+1} = i_l + 1$ and $j_{l+1} = j_l$.

The set of all proper index sequences will be denoted by $\mathcal{I}(1,\dots,d)$.

Obviously, $(i_l)_l$ is nondecreasing and $(j_l)_l$ is nonincreasing. Note that $i_l < j_l$ for all $l = 1,\dots,d-1$. More exactly, $j_l - i_l = d - l$ $(l = 1,\dots,d-1)$. We define the set $\mathcal{I}^{(k)}(1,\dots,d)$ of index sequences as follows. For a proper index sequence $(i_l,j_l)_{l=1}^{d-1} \in \mathcal{I}(1,\dots,d)$, we say that the first k term of the sequence, that is, $(i_l,j_l)_{l=1}^{k}$ is in $\mathcal{I}^{(k)}(1,\dots,d)$. Of course $\mathcal{I}^{(d-1)}(1,\dots,d)=\mathcal{I}(1,\dots,d)$. We will use the following representation of the kernel function D_t.

Lemma 4.3. *For $k = 0,1,\dots,d-2$, we have*

$$D_t(x) = \sum_{(i_l,j_l)_{l=1}^{k+1} \in \mathcal{I}^{(k+1)}(1,\dots,d)} (-1)^{i_{k+1}-1} \left(\prod_{l=1}^{k} (x_{i_l}^2 - x_{j_l}^2)^{-1} \right) [x_{i_{k+1}}^2,\dots,x_{j_{k+1}}^2]G_t.$$

Proof. Using Lemma 4.1, we can easily prove the lemma by induction. For $k = 0$ the equation is the same as Lemma 4.1. The details are left to the reader. □

Next we estimate the kernel function K_T^θ in two different ways.

Lemma 4.4. *Suppose that the absolutely continuous function θ satisfies (3.1) and (3.3). If $x_1 > x_2 > \cdots > x_d > 0$, $1 < n_1 < \cdots < n_m < d$ and $m = 0,\dots,d-2$, then*

$$|K_T^\theta(x)| \le CT^m \sum_{(i_l,j_l)_l \in \mathcal{I}(\{1,\dots,d\} \setminus \{n_1,\dots,n_m\})} x_{i_1}^{-1} \prod_{l=1}^{d-1-m} (x_{i_l} - x_{j_l})^{-1}. \tag{4.7}$$

Proof. It follows from Lemma 4.3 with $k = d-2$ that

$$\begin{aligned} |D_t(x)| = \left|D_t^d(x)\right| &\le \sum_{(i_l,j_l)_l \in \mathcal{I}(1,\ldots,d)} \left(\prod_{l=1}^{d-1} (x_{i_l}^2 - x_{j_l}^2)^{-1}\right) \left|G_t(x_{i_{d-1}}^2) - G_t(x_{j_{d-1}}^2)\right| \\ &\le C \sum_{(i_l,j_l)_l \in \mathcal{I}(1,\ldots,d)} \left(\prod_{l=1}^{d-1} (x_{i_l}^2 - x_{j_l}^2)^{-1}\right) \left(x_{i_{d-1}}^{d-2} + x_{j_{d-1}}^{d-2}\right) \\ &\le C \sum_{(i_l,j_l)_l \in \mathcal{I}(1,\ldots,d)} x_{i_1}^{-1} \left(\prod_{l=1}^{d-1} (x_{i_l} - x_{j_l})^{-1}\right) \left(\prod_{l=2}^{d-1} (x_{i_l} + x_{j_l})^{-1}\right) x_{i_{d-1}}^{d-2} \\ &\le C \sum_{(i_l,j_l)_l \in \mathcal{I}(1,\ldots,d)} x_{i_1}^{-1} \left(\prod_{l=1}^{d-1} (x_{i_l} - x_{j_l})^{-1}\right), \end{aligned} \tag{4.8}$$

because $x_{i_1} + x_{j_1} \ge x_{i_1}$ and $x_{i_l} + x_{j_l} \ge x_{i_{d-1}}$ for $l = 2, \ldots, d-1$. Then we obtain by (4.4) that

$$\left|K_T^\theta(x)\right| \le C \int_0^\infty |\theta'(t)| \left|D_{tT}^d(x)\right| \, dt \le C \sum_{(i_l,j_l)_l \in \mathcal{I}(1,\ldots,d)} x_{i_1}^{-1} \prod_{l=1}^{d-1} (x_{i_l} - x_{j_l})^{-1},$$

which is exactly (4.7) for $m = 0$. Next we prove the result for $m = 1$. We may suppose that $n_1 = 2$. Observe that

$$\begin{aligned} &D_t^d(x) \\ &= 2^d \int_{(0,\infty)^d} 1_{\{|v| \le t\}} \cos(x_1 v_1) \ldots \cos(x_d v_d) \, dv \\ &= 2^d \int_0^t \int_0^{t-v_2} \int_0^{t-v_1-v_2} \cdots \int_0^{t-v_1-\ldots-v_{d-1}} \cos(x_1 v_1) \ldots \cos(x_d v_d) \, dv_2 dv_1 dv_3 \ldots dv_d \\ &= 2 \int_0^t \cos(x_2 v_2) D_{t-v_2}^{d-1}(x_1, x_3, \ldots, x_d) \, dv_2. \end{aligned}$$

Using (4.8) for the $(d-1)$-dimensional Dirichlet kernel $D_{t-v_2}^{d-1}(x_1, x_3, \ldots, x_d)$, we have

$$\left|D_t^d(x)\right| \le Ct \sum_{(i_l,j_l)_l \in \mathcal{I}(1,3,\ldots,d)} x_{i_1}^{-1} \prod_{l=1}^{d-2} (x_{i_l} - x_{j_l})^{-1}$$

and so

$$\left|K_T^\theta(x)\right| \le C \int_0^\infty |\theta'(t)| \left|D_{tT}^d(x)\right| \, dt \le CT \sum_{(i_l,j_l)_l \in \mathcal{I}(1,3,\ldots,d)} x_{i_1}^{-1} \prod_{l=1}^{d-2} (x_{i_l} - x_{j_l})^{-1},$$

which yields (4.7) for $m = 1$. The proof can be finished in the same way. □

Note that $x_{i_1} = x_1$ and $x_{j_1} = x_d$.

Lemma 4.5. *Suppose that the absolutely continuous function θ satisfies (3.1), (3.3), and (3.5) for some $0 < \alpha \le 1$ and all $u > 0$. If $x_1 > x_2 > \cdots > x_d > 1/T$, then*

$$|K_T^\theta(x)| \le CT^{m-\alpha} \sum_{(i_l,j_l)_l \in \mathcal{I}(1,\dots,d)} x_{i_1}^{-1} x_{j_1}^{-\alpha} \prod_{l=1}^{d-1-m} (x_{i_l} - x_{j_l})^{-1} \qquad (m = 0, \dots, d-2) \tag{4.9}$$

and

$$|K_T^\theta(x)| \le CT^{d-1-\alpha} x_{j_1}^{-1-\alpha}. \tag{4.10}$$

Proof. By Lemma 4.3 with $k = d-2$ and (4.4), we have

$$\begin{aligned}
&|K_T^\theta(x)| \\
&\le C \sum_{(i_l,j_l)\in\mathcal{I}} \left(\prod_{l=1}^{d-1} (x_{i_l}^2 - x_{j_l}^2)^{-1}\right) \left| \int_0^\infty \theta'(t) \left(G_{tT}(x_{i_{d-1}}^2) - G_{tT}(x_{j_{d-1}}^2)\right) dt \right| \\
&\le C \sum_{(i_l,j_l)\in\mathcal{I}} \left(\prod_{l=1}^{d-1} (x_{i_l}^2 - x_{j_l}^2)^{-1}\right) \left| \int_0^\infty \theta'(t) \left(x_{i_{d-1}}^{d-2} \operatorname{soc}(tTx_{i_{d-1}}) - x_{j_{d-1}}^{d-2} \operatorname{soc}(tTx_{j_{d-1}})\right) dt \right| \\
&\le CT^{-\alpha} \sum_{(i_l,j_l)\in\mathcal{I}} \left(\prod_{l=1}^{d-1} (x_{i_l}^2 - x_{j_l}^2)^{-1}\right) \left(x_{i_{d-1}}^{d-2-\alpha} + x_{j_{d-1}}^{d-2-\alpha}\right) \\
&\le CT^{-\alpha} \sum_{(i_l,j_l)\in\mathcal{I}} x_{i_1}^{-1} \left(\prod_{l=1}^{d-1} (x_{i_l} - x_{j_l})^{-1}\right) \left(\prod_{l=2}^{d-1} (x_{i_l} + x_{j_l})^{-1}\right) \left(x_{i_{d-1}}^{d-2-\alpha} + x_{j_{d-1}}^{d-2-\alpha}\right) \\
&\le CT^{-\alpha} \sum_{(i_l,j_l)\in\mathcal{I}} x_{i_1}^{-1} \left(\prod_{l=1}^{d-1} (x_{i_l} - x_{j_l})^{-1}\right) \left(x_{i_{d-1}}^{-\alpha} + x_{j_{d-1}}^{-\alpha}\right) \\
&\le CT^{-\alpha} \sum_{(i_l,j_l)\in\mathcal{I}} x_{i_1}^{-1} x_{j_1}^{-\alpha} \prod_{l=1}^{d-1} (x_{i_l} - x_{j_l})^{-1},
\end{aligned}$$

because $x_{i_1} + x_{j_1} > x_{i_1}$ and $x_{i_l} + x_{j_l} > x_{i_{d-1}} > x_{j_{d-1}} > x_{j_1}$ for $l = 2, \dots, d-1$. This shows (4.9) for $m = 0$. We can easily prove by induction that

$$G_t^{(n)}(u) = \sum_{j=0}^{n} c_j t^j u^{\frac{d-2n-2+j}{2}} \operatorname{soc}^{(j)}(tu^{1/2}),$$

where $c_j \in \mathbb{R}$ $(j = 0, \dots, n)$ and $c_0 = 0$ if d is even and $2n+2 > d$. Using this formula with $n = m$ as well as (4.6), (4.4), (3.4), and Lemma 4.3 for $k = d-1-m$

$(m = 1, \ldots, d-1)$, we infer

$$|K_T^\theta(x)| \tag{4.11}$$

$$\leq C \sum_{(i_l,j_l)_{l=1}^{d-m} \in \mathcal{I}^{(d-m)}(1,\ldots,d)} \left(\prod_{l=1}^{d-1-m} (x_{i_l}^2 - x_{j_l}^2)^{-1} \right) \left| \int_0^\infty \theta'(t) [x_{i_{d-m}}^2, \ldots, x_{j_{d-m}}^2] G_{tT} \, dt \right|$$

$$\leq C \sum_{(i_l,j_l)_{l=1}^{d-m} \in \mathcal{I}^{(d-m)}(1,\ldots,d)} \left(\prod_{l=1}^{d-1-m} (x_{i_l}^2 - x_{j_l}^2)^{-1} \right) \left| \int_0^\infty \theta'(t) \frac{G_{tT}^{(m)}(\xi_{(i_l,j_l)_l})}{m!} \, dt \right|$$

$$\leq C \sum_{(i_l,j_l)_{l=1}^{d-m} \in \mathcal{I}^{(d-m)}(1,\ldots,d)} \left(\prod_{l=1}^{d-1-m} (x_{i_l}^2 - x_{j_l}^2)^{-1} \right) \sum_{j=0}^{m}$$

$$\left| \int_0^\infty \theta'(t) (Tt)^j \xi_{(i_l,j_l)_l}^{\frac{d-2m-2+j}{2}} \operatorname{soc}^{(j)}(tT\xi_{(i_l,j_l)_l}^{1/2}) \, dt \right|$$

$$\leq C \sum_{(i_l,j_l)_{l=1}^{d-m} \in \mathcal{I}^{(d-m)}(1,\ldots,d)} \left(\prod_{l=1}^{d-1-m} (x_{i_l}^2 - x_{j_l}^2)^{-1} \right) \sum_{j=0}^{m} T^{j-\alpha} \xi_{(i_l,j_l)_l}^{\frac{d-2m-2+j-\alpha}{2}}, \tag{4.12}$$

where $x_{j_{d-m}}^2 \leq \xi_{(i_l,j_l)_l} \leq x_{i_{d-m}}^2$. If $m = 1, \ldots, d-2$, then

$$|K_T^\theta(x)| \leq C \sum_{(i_l,j_l)_{l=1}^{d-m} \in \mathcal{I}^{(d-m)}(1,\ldots,d)} x_{i_1}^{-1} \left(\prod_{l=1}^{d-1-m} (x_{i_l} - x_{j_l})^{-1} \right) \sum_{j=0}^{m} T^{j-\alpha} \xi_{(i_l,j_l)_l}^{\frac{-m+j-\alpha}{2}}$$

$$\leq C \sum_{(i_l,j_l)_{l=1}^{d-m} \in \mathcal{I}^{(d-m)}(1,\ldots,d)} x_{i_1}^{-1} \left(\prod_{l=1}^{d-1-m} (x_{i_l} - x_{j_l})^{-1} \right) \sum_{j=0}^{m} T^{j-\alpha} x_{j_1}^{-m+j-\alpha}$$

$$\leq CT^{m-\alpha} \sum_{(i_l,j_l)_{l=1}^{d-m} \in \mathcal{I}^{(d-m)}(1,\ldots,d)} x_{i_1}^{-1} \left(\prod_{l=1}^{d-1-m} (x_{i_l} - x_{j_l})^{-1} \right) x_{j_1}^{-\alpha},$$

which is exactly (4.9). If $m = d-1$, then (4.11) yields that

$$|K_T^\theta(x)| \leq C \sum_{j=0}^{d-1} T^{j-\alpha} \xi_{(i_1,j_1)}^{\frac{-d+j-\alpha}{2}} \leq C \sum_{j=0}^{d-1} T^{d-1-\alpha} x_{j_1}^{-1-\alpha} \leq CT^{d-1-\alpha} x_{j_1}^{-1-\alpha},$$

which proves (4.10). □

We have proved the next lemma in Weisz [29, Theorem 1].

Lemma 4.6. *Suppose that the absolutely continuous function θ satisfies* (3.1), (3.3), *and* (3.5) *for some $0 < \alpha \leq 1$ and all $u > 0$. Then*

$$\int_{\mathbb{R}^d} \left| K_T^\theta \right| \, d\lambda \leq C \qquad (T > 0).$$

Now we can extend the definition of the ℓ_1-θ-means $\sigma_T^\theta f$ by formula (4.3) to all $f \in W(L_1, \ell_\infty)(\mathbb{R}^d)$.

5. Modified Lebesgue points

$L_p^{loc}(\mathbb{R}^d)$ $(1 \le p < \infty)$ denotes the space of measurable functions f for which $|f|^p$ is locally integrable. For $f \in L_1^{loc}(\mathbb{R}^d)$ the *Hardy–Littlewood maximal function* is defined by

$$Mf(x) := \sup_{h>0} \frac{1}{(2h)^d} \int_{-h}^{h} \cdots \int_{-h}^{h} |f(x-s)|\, ds,$$

while the *strong Hardy–Littlewood maximal function* by

$$M_s f(x) := \sup_{h_1,\dots,h_d>0} \frac{1}{2^d \prod_{j=1}^d h_j} \int_{-h_1}^{h_1} \cdots \int_{-h_d}^{h_d} |f(x-s)|\, ds.$$

It is known that M is of weak type $(1,1)$ while M_s is not. We introduce a new maximal function which is also of weak type $(1,1)$. In this article, by a *diagonal*, we understand any diagonal of the two-dimensional faces of the unit cube $[0,1]^d$. Let us denote by $P_{2^{i_1}h,\dots,2^{i_d}h}$ a parallelepiped, whose center is the origin and whose sides are parallel to the axes and/or to the diagonals and whose kth side length is $2^{i_k+1}h$ if the kth side is parallel to an axis and $\sqrt{2}2^{i_k+1}h$ if the kth side is parallel to a diagonal $(i \in \mathbb{N}^d, h > 0, k = 1,\dots,d)$. For some $\tau > 0$ and $f \in L_1^{loc}(\mathbb{R}^d)$, we define the *modified Hardy–Littlewood maximal function* by

$$\mathcal{M}^\tau f(x) := \sup_{P_{2^{i_1}h,\dots,2^{i_d}h}, i\in\mathbb{N}^d, h>0} 2^{-\tau|i|} \frac{1}{\left|P_{2^{i_1}h,\dots,2^{i_d}h}\right|} \int_{P_{2^{i_1}h,\dots,2^{i_d}h}} |f(x-s)|\, ds,$$

where the supremum is taken over all parallelepipeds $P_{2^{i_1}h,\dots,2^{i_d}h}$ $(i \in \mathbb{N}^d, h > 0)$ just defined. Obviously, $\mathcal{M}^{\tau_1} f \le \mathcal{M}^{\tau_2} f$ for $\tau_1 > \tau_2 > 0$. If the supremum is taken over all parallelepipeds whose sides are parallel to the axes and $\tau = 0$, we get back the definition of the strong Hardy–Littlewood maximal function $M_s f$, and, moreover, if in addition $i_1 = \cdots = i_d$, we get back the Hardy–Littlewood maximal function Mf.

A point $x \in \mathbb{R}^d$ is called a *Lebesgue point* of $f \in L_1^{loc}(\mathbb{R}^d)$ if

$$\lim_{h\to 0} \frac{1}{(2h)^d} \int_{-h}^{h} \cdots \int_{-h}^{h} |f(x-s) - f(x)|\, ds = 0,$$

while it is called a *strong Lebesgue point* if

$$\lim_{h_1\dots,h_d\to 0} \frac{1}{2^d \prod_{j=1}^d h_j} \int_{-h_1}^{h_1} \cdots \int_{-h_d}^{h_d} |f(x-s) - f(x)|\, ds = 0.$$

It was proved in Feichtinger and Weisz [6, 7] that almost every point $x \in \mathbb{R}^d$ is a Lebesgue point of $f \in W(L_1, \ell_\infty)(\mathbb{R}^d)$. This does not extend to strong Lebesgue points, even if $f \in L_1(\mathbb{R}^d)$. However, it holds true for $f \in L_1(\log L)^{d-1}(\mathbb{R}^d)$.

Starting from the modified Hardy–Littlewood maximal function $\mathcal{M}^\tau f$, we introduce

$$U_r^\tau f(x) := \sup_{\substack{P_{2^{i_1}h,\dots,2^{i_d}h}, i\in\mathbb{N}^d, h>0 \\ 2^{i_k}h<r, k=1,\dots d}} 2^{-\tau|i|} \frac{1}{\left|P_{2^{i_1}h,\dots,2^{i_d}h}\right|} \int_{P_{2^{i_1}h,\dots,2^{i_d}h}} |f(x-s) - f(x)|\, ds,$$

where the supremum is taken over all parallelepipeds whose sides are parallel to the axes and/or to the diagonals. We say that a point $x \in \mathbb{R}^d$ is a *modified strong Lebesgue point* of $f \in L_1^{loc}(\mathbb{R}^d)$ if for all $\tau > 0$,

$$\lim_{r \to 0} U_r^\tau f(x) = 0. \tag{5.1}$$

It is equivalent if we suppose that (5.1) holds for arbitrarily small numbers $\tau > 0$, because $\lim_{r\to 0} U_r^{\tau_2} f(x) = 0$ implies $\lim_{r\to 0} U_r^{\tau_1} f(x) = 0$ for all $\tau_1 > \tau_2 > 0$. More exactly, we need that (5.1) holds for some $\tau < \alpha/d$, where $0 < \alpha \le 1$ is given in (3.4).

If f is continuous at x, then x is a modified strong Lebesgue point of f. We have proved in [31, Theorems 1,2] that almost every point $x \in \mathbb{R}^2$ is a modified strong Lebesgue point of $f \in W(L_1, \ell_\infty)(\mathbb{R}^2)$ and, moreover, $\mathcal{M}^\tau f$ with $\tau > 0$ is almost everywhere finite for such functions. We can generalize this result to the d-dimensional case with the same proof. The details are left to the reader.

Theorem 5.1. *If $f \in W(L_1, \ell_\infty)(\mathbb{R}^d)$ and $\tau > 0$, then almost every point is a modified strong p-Lebesgue point of f and $\mathcal{M}^\tau f$ is almost everywhere finite.*

6. Pointwise convergence of the summability means

Now we prove that the ℓ_1-summability means $\sigma_T^\theta f$ converge to f at each modified strong Lebesgue points, where the modified Hardy–Littlewood maximal function $\mathcal{M}^\tau f$ is finite for small numbers $\tau > 0$.

Theorem 6.1. *Suppose that the absolutely continuous function θ satisfies (3.1), (3.3), and (3.5) for some $0 < \alpha \le 1$ and all $u > 0$. If $f \in W(L_1, \ell_\infty)(\mathbb{R}^d)$, x is a modified strong Lebesgue point of f, and $\mathcal{M}^\tau f(x)$ is finite for some $\tau < \alpha/d$, then*

$$\lim_{T\to\infty} \sigma_T^\theta f(x) = f(x).$$

Proof. If $\theta_0(s) := \theta(|s|)$, then

$$K_T^\theta(s) := T^d \widehat{\theta_0}(Ts)$$

by (4.4). The function θ_0 is integrable. Indeed, in the next integral, we substitute $s_1 + \cdots + s_d = x_1$, $s_2 = x_2, \ldots, s_d = x_d$, where $(s_1, \ldots, s_d) \in (0,\infty)^d$. Then we have to integrate over the set $\{(x_1, \ldots, x_d) \in (0,\infty)^d : x_1 > x_2 + \cdots + x_d\}$ and the Jacobian is 1. Hence

$$\begin{aligned}
2^{-d} \int_{\mathbb{R}^d} |\theta(|s|)| \, ds &= \int_{(0,\infty)^d} |\theta(s_1 + \cdots + s_d)| \, ds \\
&= \int_0^\infty \int_0^{x_1} \ldots \int_0^{x_1 - x_2 - \cdots - x_{d-1}} |\theta(x_1)| \, dx_d \ldots dx_1 \\
&\le \int_0^\infty \int_0^{x_1} \ldots \int_0^{x_1} |\theta(x_1)| \, dx_d \ldots dx_1 \\
&= \int_0^\infty t^{d-1} |\theta(t)| \, dt < \infty.
\end{aligned}$$

Since $\widehat{\theta_0} \in L_1(\mathbb{R}^2)$ by Lemma 4.6, the Fourier inversion formula yields that

$$\frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} K_T^\theta(s)\, ds = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \widehat{\theta_0}(s)\, ds = \theta(0) = 1.$$

Thus

$$\left|\sigma_T^\theta f(x) - f(x)\right| \le \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} |f(x-s) - f(x)| \left|K_T^\theta(s)\right| \, ds. \tag{6.1}$$

Instead of $\mathbb{R}^d$, it is enough to integrate on $(0,\infty)^d$. The set $(0,\infty)^d$ can be decomposed into $d!$ simplices according to the ordering of the variables. Apart from a set of measure zero of possible equalities, we may even assume strict inequalities. So it is enough to integrate over one of these simplices, say on

$$\left\{s \in (0,\infty)^d : s_1 > \cdots > s_d > 0\right\}.$$

The integrals on the other simplexes can be estimated similarly.

Let us introduce the following sets:

$$A_0 := \{s : 8/T > s_1 > \cdots > s_d > 0\},$$

$$A_1 := \{s : s_1 > \cdots > s_d > 0, s_1 > 8/T, s_k - s_{k+1} > 2/T, k = 1, \ldots, d-1\},$$

$$A_i := \{s : s_1 > \cdots > s_d > 0, s_1 > 8/T, s_k - s_{k+i} > 2/T, k = 1, \ldots, d-i \text{ and there exists } 1 \le j \le d-i+1 \text{ such that } s_j - s_{j+i-1} \le 2/T\}$$

for $i = 2, \ldots, d-1$,

$$A_d := \{s : s_1 > \cdots > s_d > 0, s_1 > 8/T, s_1 - s_d \le 2/T\}$$

and

$$B := \{s : 0 < s_d \le 1/T\}.$$

Observe that if a point s is in A_i, then $s_1 > 8/T$ and $s_{i_l} - s_{j_l} > 2/T$ for all $l = 1, \ldots, d-i$ ($i = 1, \ldots, d$). Since x is a modified strong Lebesgue point of f, we can fix a number $r < 1$ such that $U_r^\tau f(x) < \epsilon$. Let us denote the cube $[0, r/2]^d$ by $S_{r/2}$, and let $8/T < r/2$. We will integrate the right hand side of (6.1) on the sets A_0, $A_d \cap B$ and

$$\bigcup_{j=1}^{d-1} (A_j \cap B \cap S_{r/2}), \quad \bigcup_{j=1}^{d-1} (A_j \cap B \cap S_{r/2}^c), \quad \bigcup_{j=1}^{d} (A_j \cap B^c \cap S_{r/2}), \quad \bigcup_{j=1}^{d} (A_j \cap B^c \cap S_{r/2}^c),$$

where S^c denotes the complement of the set S.

Since $A_0 \subset S_{r/2}$, we have by (4.5),

$$\begin{aligned} &\int_{A_0} |f(x-s) - f(x)| \left|K_T^\theta(s)\right| \, ds \\ &\le \; CT^d \int_0^{8/T} \cdots \int_0^{8/T} |f(x-s) - f(x)| \, ds \le C U_r^\tau f(x) < C\epsilon. \end{aligned}$$

Observe that $A_d \cap B = \emptyset$.

Let us denote by $r_0 = r_0(T)$ the unique natural number i, for which $r/2 \le 2^{i+1}/T < r$. On the set $A_1 \cap B$ we use (4.7) with $m = 0$ to obtain

$$\begin{aligned}\int_{A_1\cap B\cap S_{r/2}} &|f(x-s)-f(x)|\left|K_T^\theta(s)\right| ds \\ &\le C \sum_{(i_l,j_l)_l\in\mathcal{I}(1,\ldots,d)} \int_{A_1\cap B\cap S_{r/2}} |f(x-s)-f(x)|\, s_{i_1}^{-1} \prod_{l=1}^{d-1}(s_{i_l}-s_{j_l})^{-1}\, ds \\ &\le C \sum_{(i_l,j_l)_l\in\mathcal{I}(1,\ldots,d)} \int_{A_1\cap B\cap S_{r/2}} |f(x-s)-f(x)|\, s_{i_1}^{-2} \prod_{l=2}^{d-1}(s_{i_l}-s_{j_l})^{-1}\, ds.\end{aligned} \tag{6.2}$$

For a given proper index sequence $(i_l, j_l)_l \in \mathcal{I}(1,\ldots,d)$, we introduce a permutation $i'_1,\ldots,i'_d$ of $1,\ldots,d$ and then we integrate with respect to $s_{i'_d}, s_{i'_{d-1}},\ldots,s_{i'_1}$, in this order. Let $i'_1 = i_1 = 1$ and we will consider the integral

$$\sum_{k_1=3}^{r_0} \int_{2^{k_1}/T}^{2^{k_1+1}/T} \cdots s_{i_1}^{-2}\, ds_{i'_1}.$$

Next let $i'_2 = j_1 = d$ and the integral $\int_0^{1/T} \cdots ds_{i'_2}$ will be computed. In the next step we consider two cases.

- If $i_2 = i_1$ and $j_2 = j_1 - 1$, then let $i'_3 = j_2$ and we consider the integral $\sum_{k_2=0}^{k_1} \int_{s_{i_2}-2^{k_2+1}/T}^{s_{i_2}-2^{k_2}/T} \cdots (s_{i_2}-s_{j_2})^{-1}\, ds_{i'_3}$.
- If $i_2 = i_1 + 1$ and $j_2 = j_1$, then let $i'_3 = i_2$ and we consider the integral $\sum_{k_2=0}^{k_1} \int_{s_{j_2}+2^{k_2}/T}^{s_{j_2}+2^{k_2+1}/T} \cdots (s_{i_2}-s_{j_2})^{-1}\, ds_{i'_3}$.

We define i'_4 as follows.

- If $i_3 = i_2$ and $j_3 = j_2 - 1$, then let $i'_4 = j_3$ and we consider the integral $\sum_{k_3=0}^{k_1} \int_{s_{i_3}-2^{k_3+1}/T}^{s_{i_3}-2^{k_3}/T} \cdots (s_{i_3}-s_{j_3})^{-1}\, ds_{i'_4}$.
- If $i_3 = i_2 + 1$ and $j_3 = j_2$, then let $i'_4 = i_3$ and we consider the integral $\sum_{k_3=0}^{k_1} \int_{s_{j_3}+2^{k_3}/T}^{s_{j_3}+2^{k_3+1}/T} \cdots (s_{i_3}-s_{j_3})^{-1}\, ds_{i'_4}$.

Continuing this process, we estimate (6.2) by an expression, where we integrate over a parallelepiped $P_{2^{k_1}/T,\dots,2^{k_d}/T}$, with $k_d = 0$. We conclude

$$\begin{aligned}
&\int_{A_1\cap B\cap S_{r/2}} |f(x-s)-f(x)| \left|K_T^\theta(s)\right| \, ds \\
&\le C \sum_{(i_l,j_l)_l\in\mathcal{I}(1,\dots,d)} \sum_{k_1=3}^{r_0} \sum_{k_2=0}^{k_1} \cdots \sum_{k_{d-1}=0}^{k_1} \left(\frac{2^{k_1}}{T}\right)^{-2} \prod_{l=2}^{d-1} \left(\frac{2^{k_l}}{T}\right)^{-1} \\
&\qquad \int_{P_{2^{k_1}/T,\dots,2^{k_d}/T}} |f(x-s)-f(x)| \, ds \\
&\le C \sum_{(i_l,j_l)_l\in\mathcal{I}(1,\dots,d)} \sum_{k_1=3}^{r_0} \sum_{k_2=0}^{k_1} \cdots \sum_{k_{d-1}=0}^{k_1} 2^{(\tau-1/(d-1))|k|} \\
&\qquad 2^{-\tau|k|} \frac{1}{\left|P_{2^{k_1}/T,\dots,2^{k_d}/T}\right|} \int_{P_{2^{k_1}/T,\dots,2^{k_d}/T}} |f(x-s)-f(x)| \, ds \\
&\le C \sum_{(i_l,j_l)_l\in\mathcal{I}(1,\dots,d)} \sum_{k_1=3}^{r_0} \sum_{k_2=0}^{k_1} \cdots \sum_{k_{d-1}=0}^{k_1} 2^{(\tau-1/(d-1))|k|} U_r^\tau f(x) < C\epsilon.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\int_{A_1\cap B\cap S_{r/2}^c} |f(x-s)-f(x)| \left|K_T^\theta(s)\right| \, ds \\
&\le \ C \sum_{(i_l,j_l)_l\in\mathcal{I}(1,\dots,d)} \sum_{k_1=r_0}^{\infty} \sum_{k_2=0}^{k_1} \cdots \sum_{k_{d-1}=0}^{k_1} \left(\frac{2^{k_1}}{T}\right)^{-2} \prod_{l=2}^{d-1} \left(\frac{2^{k_l}}{T}\right)^{-1} \\
&\qquad \int_{P_{2^{k_1}/T,\dots,2^{k_d}/T}} |f(x-s)-f(x)| \, ds \\
&\le \ C \sum_{(i_l,j_l)_l\in\mathcal{I}(1,\dots,d)} \sum_{k_1=r_0}^{\infty} \sum_{k_2=0}^{k_1} \cdots \sum_{k_{d-1}=0}^{k_1} 2^{(\tau-1/(d-1))|k|} \\
&\qquad 2^{-\tau|k|} \frac{1}{\left|P_{2^{k_1}/T,\dots,2^{k_d}/T}\right|} \int_{P_{2^{k_1}/T,\dots,2^{k_d}/T}} |f(x-s)-f(x)| \, ds,
\end{aligned}$$

which can be further estimated by

$$\int_{A_1\cap B\cap S_{r/2}^c} |f(x-s)-f(x)| \left|K_T^\theta(s)\right| \, ds$$

$$
\begin{aligned}
&\leq C \sum_{(i_l,j_l)_l \in \mathcal{I}(1,\ldots,d)} \sum_{k_1=r_0}^{\infty} \sum_{k_2=0}^{k_1} \cdots \sum_{k_{d-1}=0}^{k_1} 2^{(\tau-1/(d-1))|k|} \mathcal{M}^\tau f(x) \\
&\quad + C \sum_{(i_l,j_l)_l \in \mathcal{I}(1,\ldots,d)} \sum_{k_1=r_0}^{\infty} \sum_{k_2=0}^{k_1} \cdots \sum_{k_{d-1}=0}^{k_1} 2^{-|k|/(d-1)} |f(x)| \\
&\leq C \sum_{k_1=r_0}^{\infty} 2^{(\tau-1/(d-1))k_1} \mathcal{M}^\tau f(x) + C \sum_{k_1=r_0}^{\infty} 2^{-k_1/(d-1)} |f(x)| \\
&\leq C 2^{(\tau-1/(d-1))r_0} \mathcal{M}^\tau f(x) + C 2^{-r_0/(d-1)} |f(x)| \\
&\leq C (Tr)^{\tau-1/(d-1)} \mathcal{M}^\tau f(x) + C (Tr)^{-1/(d-1)} |f(x)| \to 0
\end{aligned}
$$

as $T \to \infty$. Recall that $k_d = 0$ here.

A point $s \in A_2$ is in $A_2^{\alpha_1,\ldots,\alpha_m}$ for some $1 \leq \alpha_1 < \cdots < \alpha_m \leq d-1$ and $1 \leq m \leq d-1$ if $s_{\alpha_j} - s_{\alpha_j+1} \leq 2/T$ $(j = 1, \ldots, m)$ and $s_k - s_{k+2} > 2/T$ for all $k = 1, \ldots, d-2$. Obviously, instead of A_2, it is enough to integrate over $A_2^{\alpha_1,\ldots,\alpha_m}$. If $m = d-1$, then the integral over $A_2^{\alpha_1,\ldots,\alpha_m} \cap B$ is similar to that over A_0. Suppose that $1 \leq m \leq d-2$ and $\alpha_m + 1 < d$. Then we choose $n_j = \alpha_j + 1$, $j = 1, \ldots, m$. It is easy to see that if $\alpha_m + 1 = d$, then we can also choose $1 < n_1 < \cdots < n_m < d$ such that $n_j = \alpha_j$ or $n_j = \alpha_j + 1$, $j = 1, \ldots, m$ and we can estimate the next integral in the same way. On the set $A_2^{\alpha_1,\ldots,\alpha_m} \cap B$, we use (4.7) to obtain

$$
\begin{aligned}
&\int_{A_2^{\alpha_1,\ldots,\alpha_m} \cap B \cap S_{r/2}} |f(x-s) - f(x)| \left| K_T^\theta(s) \right| ds \\
&\leq C T^m \sum_{(i_l,j_l)_l \in \mathcal{I}(\{1,\ldots,d\} \setminus \{n_1,\ldots,n_m\})} \\
&\qquad \int_{A_2^{\alpha_1,\ldots,\alpha_m} \cap B \cap S_{r/2}} |f(x-s) - f(x)| \, s_{i_1}^{-1} \prod_{l=1}^{d-1-m} (s_{i_l} - s_{j_l})^{-1} \, ds \\
&\leq C T^m \sum_{(i_l,j_l)_l \in \mathcal{I}(\{1,\ldots,d\} \setminus \{n_1,\ldots,n_m\})} \\
&\qquad \int_{A_2^{\alpha_1,\ldots,\alpha_m} \cap B \cap S_{r/2}} |f(x-s) - f(x)| \, s_{i_1}^{-2} \prod_{l=2}^{d-1-m} (s_{i_l} - s_{j_l})^{-1} \, ds. \qquad (6.3)
\end{aligned}
$$

Here, we consider first the integral $\int_{s_{n_m-1}-2/T}^{s_{n_m-1}} \cdots ds_{n_m}$, and then the integrals $\int_{s_{n_{m-1}-1}-2/T}^{s_{n_{m-1}-1}} \cdots ds_{n_{m-1}}$, $\ldots$, $\int_{s_{n_1-1}-2/T}^{s_{n_1-1}} \cdots ds_{n_1}$. Then we integrate as before in (6.2) with $\mathcal{I}(\{1,\ldots,d\} \setminus \{n_1,\ldots,n_m\})$ instead of $\mathcal{I}(\{1,\ldots,d\})$. Inequality (6.3) implies that we integrate over a parallelepiped $P_{2^{k_1}/T,\ldots,2^{k_d}/T}$ with $k_{n_1} = \cdots =$

$k_{n_m}=1,\ k_d=0$:

$$\int_{A_2^{\alpha_1,\ldots,\alpha_m}\cap B\cap S_{r/2}} |f(x-s)-f(x)|\left|K_T^\theta(s)\right| ds$$
$$\leq CT^m \sum_{(i_l,j_l)_l\in\mathcal{I}(\{1,\ldots,d\}\setminus\{n_1,\ldots,n_m\})} \sum_{k_1=3}^{r_0}\sum_{k_{\beta_2}=0}^{k_1}\cdots\sum_{k_{\beta_{d-1-m}}=0}^{k_1}\left(\frac{2^{k_1}}{T}\right)^{-2}\prod_{l=2}^{d-1-m}\left(\frac{2^{k_{\beta_l}}}{T}\right)^{-1}$$
$$\int_{P_{2^{k_1}/T,\ldots,2^{k_d}/T}} |f(x-s)-f(x)|\, ds$$
$$\leq C \sum_{(i_l,j_l)_l\in\mathcal{I}(\{1,\ldots,d\}\setminus\{n_1,\ldots,n_m\})} \sum_{k_1=3}^{r_0}\sum_{k_{\beta_2}=0}^{k_1}\cdots\sum_{k_{\beta_{d-1-m}}=0}^{k_1} 2^{(\tau-1/(d-1-m))|k|}$$
$$2^{-\tau|k|}\frac{1}{\left|P_{2^{k_1}/T,\ldots,2^{k_d}/T}\right|}\int_{P_{2^{k_1}/T,\ldots,2^{k_d}/T}} |f(x-s)-f(x)|\, ds$$
$$\leq C \sum_{(i_l,j_l)_l\in\mathcal{I}(\{1,\ldots,d\}\setminus\{n_1,\ldots,n_m\})} \sum_{k_1=3}^{r_0}\sum_{k_{\beta_2}=0}^{k_1}\cdots\sum_{k_{\beta_{d-1-m}}=0}^{k_1} 2^{(\tau-1/(d-1-m))|k|} U_r^\tau f(x) < C\epsilon,$$

where the indices $1<\beta_2<\cdots<\beta_{d-1-m}<d$ are all different from $n_1,\ldots,n_m$.

The integrals on the sets $A_2^{\alpha_1,\ldots,\alpha_m}\cap B\cap S_{r/2}^c$ and on $A_j\cap B$ $(j=3,\ldots,d-1)$ can be estimated in the same way.

Now let us consider the set B^c, that is, when $s_d>1/T$. Obviously, $s_k>1/T$ $(k=1,\ldots,d-1)$. On the set $A_1\cap B^c$ we will use the inequality (4.9) with $m=0$. We introduce the set

$$E:=\{s: s_d>s_1/2\}.$$

Obviously, $s_{j_1}>s_{i_1}/2$ on the set E. Then

$$\int_{A_1\cap B^c\cap E\cap S_{r/2}} |f(x-s)-f(x)|\left|K_T^\theta(s)\right| ds$$
$$\leq CT^{-\alpha}\sum_{(i_l,j_l)_l\in\mathcal{I}(1,\ldots,d)}\int_{A_1\cap B^c\cap E\cap S_{r/2}} |f(x-s)-f(x)|\, s_{i_1}^{-1}s_{j_1}^{-\alpha}\prod_{l=1}^{d-1}(s_{i_l}-s_{j_l})^{-1}\, ds$$
$$\leq CT^{-\alpha}\sum_{(i_l,j_l)_l\in\mathcal{I}(1,\ldots,d)}\int_{A_1\cap B^c\cap E\cap S_{r/2}} |f(x-s)-f(x)|\, s_{i_1}^{-1-\alpha}\prod_{l=1}^{d-1}(s_{i_l}-s_{j_l})^{-1}\, ds. \tag{6.4}$$

We integrate as in (6.2). The only difference is that, with respect to $s_{i_2'}$, we consider the integral $\sum_{k_d=0}^{k_1} \int_{s_{i_1}-2^{k_d+1}/T}^{s_{i_1}-2^{k_d}/T} \cdots (s_{i_1}-s_{j_1})^{-1}\, ds_{i_2'}$. We obtain

$$\begin{aligned}
&\int_{A_1\cap B^c\cap E\cap S_{r/2}} |f(x-s)-f(x)| \left|K_T^\theta(s)\right| \, ds \\
&\leq CT^{-\alpha} \sum_{(i_l,j_l)_l\in\mathcal{I}(1,\ldots,d)} \sum_{k_1=3}^{r_0} \sum_{k_2=0}^{k_1} \cdots \sum_{k_d=0}^{k_1} \left(\frac{2^{k_1}}{T}\right)^{-1-\alpha} \prod_{l=2}^{d} \left(\frac{2^{k_l}}{T}\right)^{-1} \\
&\quad \int_{P_{2^{k_1}/T,\ldots,2^{k_d}/T}} |f(x-s)-f(x)| \, ds \\
&\leq C \sum_{(i_l,j_l)_l\in\mathcal{I}(1,\ldots,d)} \sum_{k_1=3}^{r_0} \sum_{k_2=0}^{k_1} \cdots \sum_{k_d=0}^{k_1} 2^{(\tau-\alpha/d)|k|} \\
&\quad 2^{-\tau|k|} \frac{1}{\left|P_{2^{k_1}/T,\ldots,2^{k_d}/T}\right|} \int_{P_{2^{k_1}/T,\ldots,2^{k_d}/T}} |f(x-s)-f(x)| \, ds \\
&\leq C \sum_{(i_l,j_l)_l\in\mathcal{I}(1,\ldots,d)} \sum_{k_1=3}^{r_0} \sum_{k_2=0}^{k_1} \cdots \sum_{k_d=0}^{k_1} 2^{(\tau-\alpha/d)|k|} U_r^\tau f(x) < C\epsilon. \qquad (6.5)
\end{aligned}$$

On the set E^c, we have $s_{i_1}-s_{j_1} \geq s_{i_1}/2$ and so

$$\begin{aligned}
&\int_{A_1\cap B^c\cap E^c\cap S_{r/2}} |f(x-s)-f(x)| \left|K_T^\theta(s)\right| \, ds \\
&\leq CT^{-\alpha} \sum_{(i_l,j_l)_l\in\mathcal{I}(1,\ldots,d)} \int_{A_1\cap B^c\cap E^c\cap S_{r/2}} |f(x-s)-f(x)|\, s_{i_1}^{-2} s_{j_1}^{-\alpha} \prod_{l=2}^{d-1} (s_{i_l}-s_{j_l})^{-1} \, ds \\
&\leq CT^{-\alpha} \sum_{(i_l,j_l)_l\in\mathcal{I}(1,\ldots,d)} \int_{A_1\cap B^c\cap\cap E^c S_{r/2}} |f(x-s)-f(x)|\, s_{i_1}^{-1-\alpha} s_{j_1}^{-1} \prod_{l=2}^{d-1} (s_{i_l}-s_{j_l})^{-1} \, ds.
\end{aligned}$$

Note that $0<\alpha\leq 1$. We integrate again in the same order than in (6.2). With respect to i_2', here we consider the integral $\sum_{k_d=0}^{k_1} \int_{2^{k_d}/T}^{2^{k_d+1}/T} \cdots s_{j_1}^{-1}\, ds_{i_2'}$. Similarly to (6.5),

$$\begin{aligned}
&\int_{A_1\cap B^c\cap E^c\cap S_{r/2}} |f(x-s)-f(x)| \left|K_T^\theta(s)\right| \, ds \\
&\leq CT^{-\alpha} \sum_{(i_l,j_l)_l\in\mathcal{I}(1,\ldots,d)} \sum_{k_1=3}^{r_0} \sum_{k_2=0}^{k_1} \cdots \sum_{k_d=0}^{k_1} \left(\frac{2^{k_1}}{T}\right)^{-1-\alpha} \prod_{l=2}^{d} \left(\frac{2^{k_l}}{T}\right)^{-1} \\
&\quad \int_{P_{2^{k_1}/T,\ldots,2^{k_d}/T}} |f(x-s)-f(x)| \, ds \\
&\leq C \sum_{(i_l,j_l)_l\in\mathcal{I}(1,\ldots,d)} \sum_{k_1=3}^{r_0} \sum_{k_2=0}^{k_1} \cdots \sum_{k_d=0}^{k_1} 2^{(\tau-\alpha/d)|k|} U_r^\tau f(x) < C\epsilon.
\end{aligned}$$

The integral on the set $A_1 \cap B^c \cap S_{r/2}^c$ can be handled similarly.

On the set $A_2 \cap B^c$, we will use the inequality (4.9) with $m=1$. If $s \in A_2$, then there exists $1 \le n \le d-1$ such that $s_n - s_{n+1} \le 2/T$ and $s_k - s_{k+2} > 2/T$ for all $k=1,\dots,d-2$. Then

$$\begin{aligned}\int_{A_2\cap B^c\cap E\cap S_{r/2}} &|f(x-s)-f(x)| \left|K_T^\theta(s)\right| ds \\ \le \; & CT^{1-\alpha} \sum_{(i_l,j_l)_l \in \mathcal{I}(1,\dots,d)} \int_{A_1\cap B^c\cap S_{r/2}} |f(x-s)-f(x)|\, s_{i_1}^{-1-\alpha} \prod_{l=1}^{d-2} (s_{i_l}-s_{j_l})^{-1}\, ds.\end{aligned}$$

We integrate in the same order and in the way as in (6.4). The difference is that, if in the given order we integrate first with respect to s_{n+1} and then later with respect to s_n, then we consider the integral $\int_{s_n-2/T}^{s_n} \cdots ds_{n+1}$. (In the other case, if we integrate first with respect to s_n and then with respect to s_{n+1}, then we compute the integral $\int_{s_{n+1}}^{s_{n+1}+2/T} \dots ds_n$.) Then let $k_{n+1}=1$ and so we have

$$\begin{aligned}\int_{A_2\cap B^c\cap E\cap S_{r/2}} &|f(x-s)-f(x)| \left|K_T^\theta(s)\right| ds \\ \le CT^{1-\alpha} & \sum_{(i_l,j_l)_l \in \mathcal{I}(1,\dots,d)} \sum_{k_1=3}^{r_0} \sum_{k_2=0}^{k_1} \cdots \sum_{k_n=0}^{k_1} \sum_{k_{n+2}=0}^{k_1} \cdots \sum_{k_d=0}^{k_1} \left(\frac{2^{k_1}}{T}\right)^{-1-\alpha} \prod_{\substack{l=2\\ l\neq m+1}}^{d} \left(\frac{2^{k_l}}{T}\right)^{-1} \\ & \int_{P_{2^{k_1}/T,\dots,2^{k_d}/T}} |f(x-s)-f(x)|\, ds \\ \le C & \sum_{(i_l,j_l)_l \in \mathcal{I}(1,\dots,d)} \sum_{k_1=3}^{r_0} \sum_{k_2=0}^{k_1} \cdots \sum_{k_n=0}^{k_1} \sum_{k_{n+2}=0}^{k_1} \cdots \sum_{k_d=0}^{k_1} 2^{(\tau-\alpha/(d-1))|k|} U_r^\tau f(x) < C\epsilon.\end{aligned}$$

The integral on $A_2 \cap B^c \cap E^c \cap S_{r/2}$ as well as the integral on $A_2 \cap B^c \cap S_{r/2}^c$ can be handled similarly.

Similarly, on the set $A_i \cap B^c$ $(i=3,\dots,d-1)$, we apply inequality (4.9) with $m=i-1$. If $s \in A_i$, then there exists $1 \le n \le d-1$ such that $s_n - s_{n+i-1} \le 2/T$ and $s_k - s_{k+i} > 2/T$ for all $k=1,\dots,d-i$. We integrate in the same order as in (6.4). We may suppose that in this order we integrate first with respect to s_{n+i-1} and then with respect to $s_{n+i-2},\dots,s_n$. Then we consider the integrals $\int_{s_{n+i-2}-2/T}^{s_{n+i-2}} \cdots ds_{n+i-1}$, $\int_{s_{n+i-3}-2/T}^{s_{n+i-3}} \cdots ds_{n+i-2}$, $\dots$, $\int_{s_n-2/T}^{s_n} \cdots ds_{n+1}$. In this case $k_{n+i-1} = \cdots = k_{n+1} = 1$.

For the last case,that is, for the set $A_d \cap B^c$ we use inequality (4.10). If $s \in A_d$, then $s_1 - s_d \le 2/T$. We integrate in the following order: $s_1, s_2, \dots, s_d$ and we consider the integrals $\int_{s_d}^{s_d+2/T} \cdots ds_i$, $i=1,\dots,d-1$ and $\sum_{k_d=1}^{r_0} \int_{2^{k_d}/T}^{2^{k_d+1}/T} \dots s_d^{-1-\alpha}\, ds_d$. Here $k_1 = \cdots = k_{d-1} = 1$. The proof can be finished as above. □

Since by Theorem 5.1 almost every point is a modified strong Lebesgue point and the maximal operator $\mathcal{M}^\tau f$ is almost everywhere finite for $f \in W(L_1,\ell_\infty)(\mathbb{R}^d)$, Theorem 6.1 implies the following.

Corollary 6.2. *Suppose that the absolutely continuous function θ satisfies (3.1), (3.3), and (3.5) for some $0 < \alpha \leq 1$ and all $u > 0$. If $f \in W(L_1, \ell_\infty)(\mathbb{R}^d)$, then*

$$\lim_{T\to\infty} \sigma_T^\theta f(x) = f(x) \qquad a.e.$$

We have seen in [32] that as special cases, we can consider the Fejér, de La Vallée-Poussin, Jackson-de La Vallée-Poussin, Rogosinski, Weierstrass, Abel, Picard, Bessel, and Riesz summability methods.

Acknowledgments. We would like to thank the referees for reading the paper carefully and for their useful comments and suggestions. This research was supported by the Hungarian Scientific Research Funds (OTKA) No. K115804.

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