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BESICOVITCH ALMOST AUTOMORPHIC SOLUTIONS OF NONAUTONOMOUS DIFFERENTIAL EQUATIONS OF FIRST ORDER

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ABSTRACT. The main purpose of this paper is to analyze the existence and uniqueness of Besicovitch almost automorphic solutions and weighted Besicovitch pseudo-almost automorphic solutions of nonautonomous differential equations of first order. We provide an interesting application of our abstract theoretical results.

1. INTRODUCTION AND PRELIMINARIES

Abstract linear nonautonomous parabolic equations of first order are still an active field of scientific research (see [1], [7], [23] and references cited therein for the basic information on the subject). Almost periodic and almost automorphic type solutions of nonautonomous parabolic equations have been examined, among many other research papers, in [4], [7]–[17] and [22]–[24].

As mentioned in the abstract, the main aim of this paper is to analyze the existence and uniqueness of Besicovitch almost automorphic type solutions of nonautonomous differential equations of first order. The organization and main ideas of paper are briefly described as follows. After giving some preliminary results on hyperbolic evolution systems, we collect the basic properties of Besicovitch almost automorphic type functions in subsection 1.1. Our main results are presented in section 2, where we continue our recent research studies raised

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in [17]–[20] (see also [21] and the forthcoming monograph [22]). Motivated by the consideration of Diagana [9], we apply our abstract results in the analysis of existence and uniqueness of Besicovitch almost automorphic solutions to the nonautonomous one-dimensional heat equation with Dirichlet boundary conditions (see section 3).

We use the standard notation throughout the paper. X and $L(X)$ denote a complex Banach space and the space of all continuous linear mappings from X into X, respectively. If A is a linear operator acting on X, then the domain, kernel space, and range of A will be denoted by $D(A)$, $N(A)$, and $R(A)$, respectively. Since no confusion seems likely, we will identify A with its graph. The resolvent set of A is denoted by $\rho(A)$, while the resolvent of A is denoted by $R(\cdot : A)$. Given $\alpha \in (0, \pi]$ in advance, set $\Sigma_\alpha := \{z \in \mathbb{C} \setminus \{0\} : |\arg(z)| < \alpha\}$. Let $I = [0, \infty)$ or $I = \mathbb{R}$. The space of all bounded continuous functions from I into X is denoted by $C_b(I : X)$. Equipped with the sup-norm, this space becomes one of Banach's.

We need to recall some basic definitions and results about hyperbolic evolution systems and Green's functions (see [1], [7] and [23] for further information).

Definition 1.1. A family $\{U(t, s) : t \geq s,\ t,\ s \in \mathbb{R}\} \subseteq L(X)$ is said to be an evolution system if and only if the following holds:

(a) $U(s, s) = I$, $U(t, s) = U(t, r)U(r, s)$ for $t \geq r \geq s$ and $t,\ r,\ s \in \mathbb{R}$,

(b) $\{(\tau, s) \in \mathbb{R}^2 : \tau > s\} \ni (t, s) \mapsto U(t, s)x$ is continuous for any fixed element $x \in X$.

Henceforward, we assume that the family $A(\cdot)$ satisfies the following condition introduced by Acquistapace and Terreni in [1] (with $\omega = 0$):

(H1): There is an $\omega \geq 0$ such that the family of closed linear operators $A(t)$, $t \in \mathbb{R}$ on X satisfies $\overline{\Sigma_\phi} \subseteq \rho(A(t) - \omega)$,

$$\big\|R(\lambda : A(t) - \omega)\big\| = O\Big(\big(1 + |\lambda|\big)^{-1}\Big), \quad t \in \mathbb{R},\ \lambda \in \overline{\Sigma_\phi}, \text{ and}$$

$$\Big\|(A(t) - \omega)R(\lambda : A(t) - \omega)\big[R(\omega : A(t)) - R(\omega : A(s))\big]\Big\| = O\Big(|t - s|^{\mu}|\lambda|^{-\nu}\Big),$$

for any $t,\ s \in \mathbb{R},\ \lambda \in \overline{\Sigma_\phi}$, where $\phi \in (\pi/2, \pi)$, $0 < \mu,\ \nu \leq 1$ and $\mu + \nu > 1$.

Then there exists an evolution system $U(\cdot, \cdot)$ generated by $A(\cdot)$, satisfying that $\|U(t, s)\| = O(1)$ for $t \geq s$, as well as several other important conditions and estimates; see [1] for more details. In what follows, we assume that the following condition holds true, as well:

(H2): The evolution system $U(\cdot, \cdot)$ generated by $A(\cdot)$ is hyperbolic (or, equivalently, has exponential dichotomy); that is, there exist a family of projections $(P(t))_{t \in \mathbb{R}} \subseteq L(X)$, being uniformly bounded and strongly continuous in t, and constants $M',\ \omega > 0$ such that the following holds, with $Q := I - P$ and $Q(\cdot) := I - P(\cdot)$,

(a) $U(t, s)P(s) = P(t)U(t, s)$ for all $t \geq s$,

(b) the restriction $U_Q(t, s) : Q(s)X \to Q(t)X$ is invertible for all $t \geq s$ (here we define $U_Q(s, t) = U_Q(t, s)^{-1}$),

(c) $\|U(t, s)P(s)\| \leq M'e^{-\omega(t-s)}$ and $\|U_Q(s, t)Q(t)\| \leq M'e^{-\omega(t-s)}$ for all $t \geq s$.

In the case that the choice $P(t) = I$, for all $t \in \mathbb{R}$, is possible, then we say that $U(\cdot,\cdot)$ is exponentially stable; $U(\cdot,\cdot)$ is said to be (bounded) exponentially bounded if and only if there exist real constants $M > 0$ and $(\omega = 0)$ $\omega \in \mathbb{R}$ such that $\|U(t,s)P(s)\| \leq Me^{-\omega(t-s)}$ for all $t \geq s$.

Define the associated Green's function $\Gamma(\cdot,\cdot)$ by

$$\Gamma(t,s) := \begin{cases} U(t,s)P(s), & t \geq s,\ t,s \in \mathbb{R}, \\ -U_Q(t,s)Q(s), & t < s,\ t,s \in \mathbb{R}. \end{cases}$$

Let M' be the constant from (H2). Then

$$\|\Gamma(t,s)\| \leq M'e^{-\omega|t-s|}, \quad t,s \in \mathbb{R}, \tag{1.1}$$

and the function

$$u(t) := \int_{-\infty}^{+\infty} \Gamma(t,s)f(s)\,ds, \quad t \in \mathbb{R} \tag{1.2}$$

is a unique mild solution of abstract Cauchy problem

$$u'(t) = A(t)u(t) + f(t), \quad t \in \mathbb{R}, \tag{1.3}$$

which means that $u(\cdot)$ is a unique bounded continuous function on $\mathbb{R}$ satisfying

$$u(t) = U(t,s)u(s) + \int_s^t U(t,\tau)f(\tau)\,d\tau, \quad t \geq s;$$

see [23].

Assume that $f : [0,\infty) \to X$ is continuous. By a mild solution of the abstract Cauchy problem

$$u'(t) = A(t)u(t) + f(t), \quad t > 0 \text{ with } u(0) = x, \tag{1.4}$$

we mean the function

$$u(t) := U(t,0)x + \int_0^t U(t,s)f(s)\,ds, \quad t \geq 0.$$

1.1. **Besicovitch almost automorphic type functions in Banach spaces.** The class of Besicovitch almost automorphic functions has been introduced by Bedouhene, Challali, Mellah, Raynaud de Fitte, and Smaali in [2] (see [3] for the fundamental monograph concerning scalar-valued Besicovitch almost periodic functions). This class extends the classes of Weyl almost automorphic functions and Stepanov almost automorphic functions (see [19] and references cited therein).

Definition 1.2. Let $p \geq 1$. Then we say that a function $f \in L^p_{loc}(\mathbb{R} : X)$ is Besicovitch p-almost automorphic if and only if for every real sequence (s_n), there exist a subsequence (s_{n_k}) and a function $f^* \in L^p_{loc}(\mathbb{R} : X)$ such that

$$\lim_{k\to\infty} \limsup_{l\to+\infty} \frac{1}{2l} \int_{-l}^{l} \Big\| f\big(t + s_{n_k} + x\big) - f^*(t+x) \Big\|^p\,dx = 0 \tag{1.5}$$

and

$$\lim_{k\to\infty} \limsup_{l\to+\infty} \frac{1}{2l} \int_{-l}^{l} \Big\| f^*\big(t - s_{n_k} + x\big) - f(t+x) \Big\|^p\,dx = 0 \tag{1.6}$$

for each $t \in \mathbb{R}$. The set of all such functions are denoted by $B^pAA(\mathbb{R} : X)$.

It can be simply proved that the set $B^pAA(\mathbb{R} : X)$, equipped with the usual operations, has a linear vector structure ([19]).

The class of weighted pseudo-almost automorphic functions has been introduced by Blot, Mophou, N'Guérékata, and Pennequin in [6]. Set

$$\mathbb{U} := \{\rho \in L^1_{loc}(\mathbb{R}) : \rho(t) > 0 \text{ a.e. } t \in \mathbb{R}\},$$

$$\mathbb{U}_\infty := \{\rho \in \mathbb{U} : \inf_{x\in\mathbb{R}} \rho(x) < \infty \text{ and } \nu(T,\rho) := \lim_{T\to+\infty} \int_{-T}^{T} \rho(t)\, dt = \infty\},$$

and

$$\mathbb{U}_b := L^\infty(\mathbb{R}) \cap \mathbb{U}_\infty.$$

Then we have $\mathbb{U}_b \subseteq \mathbb{U}_\infty \subseteq \mathbb{U}$. It is said that the weights $\rho_1(\cdot)$ and $\rho_2(\cdot)$ are equivalent, $\rho_1 \sim \rho_2$ for short, if and only if $\rho_1/\rho_2 \in \mathbb{U}_b$. $\mathbb{U}_T$ denote the space consisting of all weights $\rho \in \mathbb{U}_\infty$ which are equivalent with all its translations. Unless stated otherwise, we assume henceforth that $\rho_1,\ \rho_2 \in \mathbb{U}_\infty$.

The following definition has been recently introduced in [20] (see also [5] and [8]):

Definition 1.3. A function $f \in L^p_{loc}(\mathbb{R} : X)$ is said to be weighted Besicovitch p-pseudo almost automorphic if and only if it admits a decomposition $f(t) = g(t) + q(t)$, $t \in \mathbb{R}$, where $g(\cdot)$ is B^p-almost automorphic and $q(\cdot) \in L^p_{loc}(\mathbb{R} : X)$ satisfies

$$\lim_{T\to+\infty} \frac{1}{2\int_{-T}^{T}\rho_1(t)\, dt} \int_{-T}^{T} \left[\limsup_{l\to+\infty} \frac{1}{2l} \int_{t-l}^{t+l} \|q(s)\|^p\, ds\right]^{1/p} \rho_2(t)\, dt = 0. \tag{1.7}$$

Denote by $B^pWPAA(\mathbb{R}, X, \rho_1, \rho_2)$ the set of such functions and by $B^pWPAA_0(\mathbb{R}, X, \rho_1, \rho_2)$ the set of locally p-integrable X-valued functions $q(\cdot)$ such that (1.7) holds.

Equipped with the usual operations, the sets $B^pWPAA(\mathbb{R}, X, \rho_1, \rho_2)$ and $B^pWPAA_0(\mathbb{R}, X, \rho_1, \rho_2)$ become vector spaces ([19]).

We also need the following notions. Define

$$\mathbb{U}_p := \{\rho \in L^1_{loc}([0,\infty)) : \rho(t) > 0 \text{ a.e. } t \geq 0\},$$

$$\mathbb{U}_{b,p} := \{\rho \in L^\infty([0,\infty)) : \rho(t) > 0 \text{ a.e. } t \geq 0\},$$

and

$$\mathbb{U}_{\infty,p} := \{\rho \in \mathbb{U}_p : \nu(T,\rho) := \lim_{T\to+\infty} \int_0^T \rho(t)\, dt = \infty\}.$$

Then $\mathbb{U}_{b,p} \subseteq \mathbb{U}_{\infty,p} \subseteq \mathbb{U}_p$. If $\rho_1,\ \rho_2 \in \mathbb{U}_{\infty,p}$, then we define

$$PAP_0\big([0,\infty), X, \rho_1, \rho_2\big) := \left\{ f \in C_b([0,\infty) : X) : \lim_{T\to+\infty} \frac{1}{\int_0^T \rho_1(t)\, dt} \int_0^T \|f(t)\|\rho_2(t)\, dt = 0 \right\}.$$

2. Formulation and proofs of main results

The main result of this paper reads as follows:

Theorem 2.1. *Let $f \in B^1AA(\mathbb{R} : X) \cap L^\infty(\mathbb{R} : X)$. Then the function $u(\cdot)$, defined by (1.2), is a unique mild solution of the abstract Cauchy problem (1.3). Furthermore, if*

$$\lim_{l\to+\infty} \frac{1}{2l} \int_{-l}^{l} \int_{-\infty}^{x} \Big\| \Gamma(x+\alpha, s+\alpha) - \Gamma(x,s) \Big\| \, ds \, dx = 0 \quad (\alpha \in \mathbb{R}), \tag{2.1}$$

then $u \in B^1AA(\mathbb{R} : X) \cap C_b(\mathbb{R} : X)$.

Proof. Due to (1.1), we have

$$\int_{-\infty}^{x} \|\Gamma(x+\alpha, s+\alpha) - \Gamma(x,s)\| \, ds \leq 2M' \int_{-\infty}^{x} e^{-\omega(x-s)} \, ds = 2M'/\omega, \; x \in \mathbb{R} \; (\alpha \in \mathbb{R}).$$

Writing $\int_{-l+t}^{l+t} \cdot = \int_{-l}^{l} \cdot + \int_{l}^{l+t} \cdot - \int_{-l}^{-l+t} \cdot$, and using (2.1) after that, we get that

$$\lim_{l\to+\infty} \frac{1}{2l} \int_{-l+t}^{l+t} \int_{-\infty}^{x} \Big\| \Gamma(x+\alpha, s+\alpha) - \Gamma(x,s) \Big\| \, ds \, dx = 0, \; t \in \mathbb{R} \quad (\alpha \in \mathbb{R}). \tag{2.2}$$

Furthermore, we have $u(t) = u_1(t) + u_2(t)$, $t \in \mathbb{R}$, where $u_1(t) := \int_{-\infty}^{t} \Gamma(t,s) f(s) \, ds$ and $u_2(t) := \int_{t}^{\infty} \Gamma(t,s) f(s) \, ds$ ($t \in \mathbb{R}$). It can be easily shown that u_1, $u_2 \in L^\infty(\mathbb{R} : X)$. We will first prove that $u_1 \in C(\mathbb{R} : X)$. Define, for every $k \in \mathbb{N}$, $\Psi_k(t) := \int_{t-k+1}^{t-k} \Gamma(t,s) f(s) \, ds$, $t \in \mathbb{R}$. Since $\sum_{k=1}^{\infty} \Psi_k(t) = u_1(t)$ uniformly for $t \in \mathbb{R}$ (see e.g. the proof of [11, Theorem 2.3]), it is sufficient to show that for any fixed $k \in \mathbb{N}$ one has $\Psi_k \in C(\mathbb{R} : X)$. Since, due to (1.1),

$$\begin{aligned} &\left\| \int_{t-k+1}^{t-k} \Gamma(t,s) f(s) \, ds - \int_{t'-k+1}^{t'-k} \Gamma(t',s) f(s) \, ds \right\| \\ &\quad \leq \left\| \int_{t'-k}^{t-k} \Gamma(t,s) f(s) \, ds \right\| + \left\| \int_{t'-k+1}^{t-k+1} \Gamma(t',s) f(s) \, ds \right\| \\ &\quad\quad + \int_{t'-k}^{t-k+1} \left\| \Big[\Gamma(t,s) - \Gamma(t',s) \Big] f(s) \right\| ds \\ &\quad \leq 2M' \|f\|_\infty |t - t_k| + \int_{t'-k}^{t-k+1} \left\| \Big[\Gamma(t,s) - \Gamma(t',s) \Big] f(s) \right\| ds, \end{aligned} \tag{2.3}$$

the right continuity of $\Psi_k(\cdot)$ follows by applying the dominated convergence theorem and the fact that $\Gamma(t',s)f(s) = U(t',s)P(s)f(s)$ converges to $\Gamma(t,s)f(s) = U(t,s)P(s)f(s)$ as $t' \to t$. For the left continuity of $\Psi_k(\cdot)$, we can apply the same argument as above, by observing the following consequence of (2.3):

$$\begin{aligned}
&\Bigg\| \int_{t-k+1}^{t-k} \Gamma(t,s)f(s)\,ds - \int_{t'-k+1}^{t'-k} \Gamma(t',s)f(s)\,ds \Bigg\| \\
&\quad \leq M'\|f\|_{\infty}\big|t - -t_k\big| + \int_{t'-k}^{t'-k+1} \Big\| \Big[\Gamma(t,s) - \Gamma(t',s)\Big] f(s) \Big\|\,ds \\
&\qquad + \Bigg\| \int_{t-k+1}^{t'-k+1} \Big[\Gamma(t,s) - \Gamma(t',s)\Big] f(s)\,ds \Bigg\| \\
&\quad \leq 3M'\|f\|_{\infty}\big|t - -t_k\big| + \int_{t'-k}^{t'-k+1} \Big\| \Big[\Gamma(t,s) - \Gamma(t',s)\Big] f(s) \Big\|\,ds.
\end{aligned}$$

Let us prove that $u_1 \in B^1AA(\mathbb{R} : X)$. Fix a real sequence (s_n) and a number $t \in \mathbb{R}$. Then there exist a subsequence (s_{n_k}) and a function $f^* \in L^1_{loc}(\mathbb{R} : X)$ such that (1.5)–(1.6) hold with $p = 1$. First of all, we will prove that there exists $l_0 > 0$ such that

$$\frac{1}{2l} \int_{-l}^{l} \big\|f^*(s)\big\|\,ds \leq \|f\|_{\infty} + 1, \quad l > l_0. \tag{2.4}$$

To see this, it suffices to observe that there exist $k_0 \in \mathbb{N}$ and $l_0' > 0$ such that, for each $l \geq l_0'$, we have

$$\frac{1}{2l} \int_{-l}^{l} \Big\| f\big(s_{n_{k_0}} + s\big) - f^*(s) \Big\|\,ds < 1/2;$$

see (1.5) with $\epsilon = 1/3$ and $t = 0$. This yields

$$\begin{aligned}
\frac{1}{2l} \int_{-l}^{l} \|f^*(s)\|\,ds &\leq \frac{1}{2l} \int_{-l}^{l} \Big\| f\big(s_{n_{k_0}} + s\big) - f^*(s) \Big\|\,ds + \frac{1}{2l} \int_{-l}^{l} \Big\| f\big(s_{n_{k_0}} + s\big) \Big\|\,ds \\
&\leq \frac{1}{2} + \|f\|_{\infty}, \quad l \geq l_0',
\end{aligned}$$

so that (2.4) holds with $l_0 = l_0'$. As a simple consequence of (2.4), we have that

$$\int_0^s \big\|f^*(t+r)\big\|\,dr \leq \text{Const.} \cdot (|t| + |s|), \quad s \geq 0. \tag{2.5}$$

Next, we will prove that for each $s \geq 0$ we have

$$\lim_{l \to +\infty} \frac{1}{l} \int_{-l+t-s}^{-l+t} \big\|f^*(s)\big\|\,ds = 0. \tag{2.6}$$

Let $\epsilon > 0$ be fixed. Then, owing to (1.5), there exist $k(\epsilon) \in \mathbb{N}$ and $l_1(\epsilon) > 0$ such that

$$\begin{aligned}
&\frac{1}{l}\int_{-l+t-s}^{-l+t} \big\|f^*(s)\big\|\,ds\\
&\quad\leq \frac{1}{l}\int_{-l+t-s}^{-l+t} \Big\|f\big(s_{n_k}+s\big)-f^*(s)\Big\|\,ds+\frac{1}{l}\int_{-l+t-s}^{-l+t} \Big\|f\big(s_{n_k}+s\big)\Big\|\,ds\\
&\quad\leq \frac{1}{l}\int_{-l+t-s}^{l+s+t} \Big\|f\big(s_{n_k}+s\big)-f^*(s)\Big\|\,ds+\frac{s}{l}\|f\|_\infty \leq \epsilon\frac{l+s}{l}+\frac{s}{l}\|f\|_\infty, \quad l\geq l_1(\epsilon).
\end{aligned}$$

This gives (2.6). Define $u_1^*(t) := \int_{-\infty}^t \Gamma(t,s)f^*(s)\,ds$, $t \in \mathbb{R}$. Then $u_1^*(\cdot)$ is well-defined and locally bounded since the partial integration in combination with (1.1) and (2.5) implies that

$$\left\|\int_{-\infty}^t \Gamma(t,s)f^*(s)\,ds\right\| = \left\|\int_0^\infty \Gamma(t,t+s)f^*(t+s)\,ds\right\|$$

$$\begin{aligned}
&\leq \int_0^\infty \Gamma(t,t+s)\big\|f^*(t+s)\big\|\,ds \leq M'\int_0^\infty e^{-\omega s}\big\|f^*(t+s)\big\|\,ds\\
&\leq M'\int_0^\infty \omega e^{-\omega s}\int_0^s \big\|f^*(t+r)\big\|\,dr\,ds \leq \text{Const.}\cdot(1+|t|).
\end{aligned}$$

Furthermore, by the Fubini theorem and a straightforward calculation, we have

$$\begin{aligned}
&\frac{1}{2l}\int_{-l+t}^{l+t}\left\|\int_{-\infty}^{x+s_{n_k}} \Gamma(x+s_{n_k},s)f(s)\,ds-\int_{-\infty}^x \Gamma(x,s)f^*(s)\,ds\right\|dx\\
&\leq \frac{1}{2l}\int_{-l+t}^{l+t}\int_{-\infty}^x \Big\|\Gamma(x+s_{n_k},s+s_{n_k})-\Gamma(x,s)\Big\|\|f(s+s_{n_k})\|\,ds\,dx\\
&\quad+\frac{1}{2l}\int_{-l+t}^{l+t}\int_{-\infty}^x \|\Gamma(x,s)\|\big\|f\big(s_{n_k}+s\big)-f^*(s)\big\|\,ds\,dx\\
&\leq \frac{\|f\|_\infty}{2l}\int_{-l+t}^{l+t}\int_{-\infty}^x \Big\|\Gamma(x+s_{n_k},s+s_{n_k})-\Gamma(x,s)\Big\|\,ds\,dx\\
&\quad+\frac{1}{2l}\int_{-l+t}^{l+t}\int_{-\infty}^x \|\Gamma(x,s)\|\big\|f\big(s_{n_k}+s\big)-f^*(s)\big\|\,ds\,dx\\
&= \frac{\|f\|_\infty}{2l}\int_{-l+t}^{l+t}\int_{-\infty}^x \Big\|\Gamma(x+s_{n_k},s+s_{n_k})-\Gamma(x,s)\Big\|\,ds\,dx\\
&\quad+\frac{1}{2l}\int_{-\infty}^{-l+t}\int_{-l+t}^{l+t} \|\Gamma(x,s)\|\big\|f\big(s_{n_k}+s\big)-f^*(s)\big\|\,dx\,ds\\
&\quad+\frac{1}{2l}\int_{-l+t}^{l+t}\int_{s}^{l+t} \|\Gamma(x,s)\|\big\|f\big(s_{n_k}+s\big)-f^*(s)\big\|\,dx\,ds.
\end{aligned}$$

Keeping in mind (1.1) and (2.2), it suffices to show that

$$\lim_{k\to\infty}\limsup_{l\to+\infty}\frac{1}{2l}\int_{-\infty}^{-l+t}\int_{-l+t}^{l+t}e^{-\omega|x-s|}\big\|f\big(s_{n_k}+s\big)-f^*(s)\big\|\,dx\,ds=0 \tag{2.7}$$

and

$$\lim_{k\to\infty}\limsup_{l\to+\infty}\frac{1}{2l}\int_{-l+t}^{l+t}\int_{s}^{l+t}e^{-\omega|x-s|}\big\|f\big(s_{n_k}+s\big)-f^*(s)\big\|\,dx\,ds=0. \tag{2.8}$$

The proof of (2.8) is almost trivial since (1.5) holds with $p=1$ and

$$\begin{aligned}\frac{1}{2l}\int_{-l+t}^{l+t}\int_{s}^{l+t}&e^{-\omega|x-s|}\big\|f\big(s_{n_k}+s\big)-f^*(s)\big\|\,dx\,ds\\&\leq\bigg[\int_0^\infty e^{-\omega r}\,dr\bigg]\bigg[\frac{1}{2l}\int_{-l+t}^{l+t}\big\|f\big(s_{n_k}+s\big)-f^*(s)\big\|\,ds\bigg].\end{aligned}$$

To prove (2.7), we can argue as follows:

$$\begin{aligned}&\frac{1}{2l}\int_{-l+t}^{l+t}\int_{s}^{l+t}e^{-\omega|x-s|}\big\|f\big(s_{n_k}+s\big)-f^*(s)\big\|\,dx\,ds\\&\leq\text{Const.}\cdot\frac{1}{2l}\int_{-\infty}^{-l+t}e^{-\omega(-l+t-s)}\big\|f\big(s_{n_k}+s\big)-f^*(s)\big\|\,ds\\&\leq\text{Const.}\cdot\|f\|_\infty\frac{1}{2l}\int_{-\infty}^{-l+t}e^{-\omega(-l+t-s)}\,ds\\&\quad+\text{Const.}\cdot\frac{1}{2l}\int_{-\infty}^{-l+t}e^{-\omega(-l+t-s)}\big\|f^*(s)\big\|\,ds\\&\leq\text{Const.}\cdot\|f\|_\infty\frac{1}{2l}\int_0^\infty e^{-\omega s}\,ds+\text{Const.}\cdot\frac{1}{2l}\int_0^\infty e^{-\omega s}\big\|f^*(-l+t-s)\big\|\,ds.\end{aligned}$$

The first addend in the last estimate clearly tends to zero as $l\to+\infty$. But, the situation is similar with the second addend since the reverse Fatou's lemma and (2.6) together imply that

$$\begin{aligned}0\leq\limsup_{l\to+\infty}\frac{1}{2l}\int_0^\infty&e^{-\omega s}\big\|f^*(-l+t-s)\big\|\,ds\\&=\text{Const.}\cdot\limsup_{l\to+\infty}\frac{1}{2l}\int_0^\infty e^{-\omega s}\int_0^s\big\|f^*(-l+t-r)\big\|\,dr\,ds\\&=\text{Const.}\cdot\limsup_{l\to+\infty}\frac{1}{2l}\int_0^\infty e^{-\omega s}\int_{-l+t-s}^{-l+t}\big\|f^*(r)\big\|\,dr\,ds\\&\leq\text{Const.}\cdot\frac{1}{2l}\int_0^\infty e^{-\omega s}\limsup_{l\to+\infty}\int_{-l+t-s}^{-l+t}\big\|f^*(r)\big\|\,dr\,ds=0,\end{aligned}$$

so that actually

$$\lim_{l\to+\infty}\frac{1}{2l}\int_0^\infty e^{-\omega s}\big\|f^*(-l+t-s)\big\|\,ds=0.$$

Hence, $u_1 \in B^1AA(\mathbb{R} : X)$, as claimed; we can similarly prove that $u_2 \in C(\mathbb{R} : X) \cap B^1AA(\mathbb{R} : X)$ (for the proof of continuity of $u_2(\cdot)$, it is only worth noting that the mapping $(t,s) \mapsto U_Q(t,s)Q(s)$ is strongly continuous for $t < s$; see [23]). The proof of the theorem is thereby completed. □

Before proceeding further, we would like to observe that the condition (1.1) has not been well explored within the existing theory of nonautonomous differential equations of first order (see [4, Lemma 3.2, Theorem 3.3] for two broad-sense results in this direction).

We have already proved that any Besicovitch p-vanishing function is Besicovich-p-almost periodic on $[0, \infty)$ (Besicovich-Doss-p-almost periodic on $[0, \infty)$), as well as that its extension by zero outside the interval $[0, \infty)$ is a Besicovitch p-almost automorphic (see [22] for the notion and precise formulations of these results). Therefore, in some sense, it is ridiculous to introduce the notion of an asymptotically Besicovitch p-almost automorphic function. Concerning the existence and uniqueness of mild solutions of the abstract Cauchy problem (1.4) belonging to the class of Besicovitch p-almost automorphic functions, we can state the following result with $p = 1$ (see also [17, Theorem 4.1]).

Theorem 2.2. *Let $f \in B^1AA(\mathbb{R} : X) \cap L^\infty(\mathbb{R} : X)$, and let (2.1) hold. Suppose that $x \in P(0)X$. Define $u(t) := U(t,0)x + \int_0^t U(t,s)f(s)\,ds$, $t \geq 0$. If the mapping $t \mapsto \int_0^t U(t,s)Q(s)f(s)\,ds$, $t \geq 0$ is in the class $B^1AA(\mathbb{R} : X)$, then $u(\cdot)$ is in the same class, as well.*

Proof. Clearly, we have the following decomposition:

$$u(t) = U(t,0)x + \int_{-\infty}^t \Gamma(t,s)f(s)\,ds - \int_{-\infty}^0 \Gamma(t,s)f(s)\,ds + \int_0^t U(t,s)Q(s)f(s)\,ds,$$

for any $t \geq 0$. Since $x \in P(0)X$, the function $U(\cdot,0)x$ is exponentially decaying due to (1.1). Keeping in mind the prescribed assumption that the mapping $t \mapsto \int_0^t U(t,s)Q(s)f(s)\,ds$, $t \geq 0$ is in the class $B^1AA(\mathbb{R} : X)$, as well as Theorem 2.1, it suffices to show that $\int_{-\infty}^0 \Gamma(t,s)f(s)\,ds \to 0$ as $t \to +\infty$. But, this simply follows from the next estimates ($t \geq 0$),

$$\left\| \int_{-\infty}^0 \Gamma(t,s)f(s)\,ds \right\| \leq M'\|f\|_\infty \int_{-\infty}^0 e^{-\omega|t-s|}\,ds \leq M'\|f\|_\infty e^{-\omega t} \int_{-\infty}^0 e^{\omega s}\,ds.$$

□

Remark 2.3. If $x \in P(0)X \cap \overline{D(A(0))}$, then the mapping $t \mapsto U(t,0)x$, $t \geq 0$ is continuous (see e.g. [23]). Since the mapping $t \mapsto \int_{-\infty}^t \Gamma(t,s)f(s)\,ds - \int_{-\infty}^0 \Gamma(t,s)f(s)\,ds$, $t \geq 0$ is bounded and continuous, by the proof of Theorem 2.1, we have that the assumption that the mapping $t \mapsto \int_0^t U(t,s)Q(s)f(s)\,ds$, $t \geq 0$ is in the class $B^1AA(\mathbb{R} : X) \cap C(\mathbb{R} : X)$ ($B^1AA(\mathbb{R} : X) \cap C_b(\mathbb{R} : X)$), which implies that $u(\cdot)$ is in the same class. It is clear that the above condition holds if the evolution system $U(\cdot,\cdot)$ is exponentially stable.

In the following theorem, we examine the weighted Besicovitch p-pseudo almost automorphic solutions of abstract Cauchy problem (1.3); see also [20, Proposition 4.2] for a similar result in this direction.

Theorem 2.4. *Suppose that* $g \in B^1AA(\mathbb{R} : X) \cap L^\infty(\mathbb{R} : X)$, *as well as* $q \in B^1WPAA_0(\mathbb{R}, X, \rho_1, \rho_2) \cap L^\infty(\mathbb{R} : X)$. *Set* $f(t) := g(t) + q(t)$, $t \in \mathbb{R}$. *Then the function* $u(\cdot)$, *defined by* (1.2), *is a unique mild solution of the abstract Cauchy problem* (1.3). *Suppose that*

$$\lim_{T\to+\infty} \frac{\int_{-T}^{T} \rho_2(t)\, dt}{\int_{-T}^{T} \rho_1(t)\, dt} = 0, \tag{2.9}$$

and (2.1) *holds. Then we have* $u \in B^1WPAA(\mathbb{R}, X, \rho_1, \rho_2) \cap C_b(\mathbb{R} : X)$.

Proof. By Theorem 2.1 and its proof, it suffices to show that the mapping $t \mapsto u_q(t) := \int_{-\infty}^{\infty} \Gamma(t, s)q(s)\, ds$, $t \in \mathbb{R}$ belongs to the class $B^1WPAA_0(\mathbb{R}, X, \rho_1, \rho_2)$. It is clear that $u_q(t) = u_{q1}(t) + u_{q2}(t)$, $t \in \mathbb{R}$, where $u_{q1}(t) := \int_{-\infty}^{t} \Gamma(t, s)q(s)\, ds$ and $u_{q2}(t) := \int_{t}^{\infty} \Gamma(t, s)q(s)\, ds$ ($t \in \mathbb{R}$). We have

$$\begin{aligned}
&\frac{1}{2\int_{-T}^{T}\rho_1(t)\,dt}\int_{-T}^{T}\left[\limsup_{l\to+\infty}\frac{1}{2l}\int_{t-l}^{t+l}\|u_{q1}(s)\|\,ds\right]\rho_2(t)\,dt\\
&= \frac{1}{2\int_{-T}^{T}\rho_1(t)\,dt}\int_{-T}^{T}\left[\limsup_{l\to+\infty}\frac{1}{2l}\int_{t-l}^{t+l}\left\|\int_0^\infty \Gamma(s,s-v)q(s-v)\,dv\right\|ds\right]\rho_2(t)\,dt\\
&\leq \frac{1}{2\int_{-T}^{T}\rho_1(t)\,dt}\int_{-T}^{T}\left[\limsup_{l\to+\infty}\frac{1}{2l}\int_{t-l}^{t+l}\int_0^\infty \|\Gamma(s,s-v)\|\|q(s-v)\|\,dv\,ds\right]\rho_2(t)\,dt\\
&\leq \frac{1}{2\int_{-T}^{T}\rho_1(t)\,dt}\int_{-T}^{T}\left[\limsup_{l\to+\infty}\frac{1}{2l}\int_0^\infty\int_{t-l}^{t+l} \|\Gamma(s,s-v)\|\|q(s-v)\|\,ds\,dv\right]\rho_2(t)\,dt\\
&\leq \frac{M'}{2\int_{-T}^{T}\rho_1(t)\,dt}\int_{-T}^{T}\left[\limsup_{l\to+\infty}\frac{1}{2l}\int_0^\infty e^{-\omega v}\int_{t-l-v}^{t+l-v} \|q(r)\|\,dr\,dv\right]\rho_2(t)\,dt\\
&\leq \frac{M'\|q\|_\infty}{2\omega\int_{-T}^{T}\rho_1(t)\,dt}\int_{-T}^{T}\rho_2(t)\,dt, \quad T > 0.
\end{aligned}$$

Therefore, (2.9) yields that $u_{q1} \in B^1WPAA_0(\mathbb{R}, X, \rho_1, \rho_2)$. We can similarly prove that $u_{q2} \in B^1WPAA_0(\mathbb{R}, X, \rho_1, \rho_2)$, finishing the proof of theorem. □

Concerning the abstract Cauchy problem (1.4), we have the following result (see also [17, Proposition 4.5]).

Theorem 2.5. *Suppose that* $g \in B^1AA(\mathbb{R} : X) \cap L^\infty(\mathbb{R} : X)$, *as well as* $q \in B^1WPAA_0([0, \infty), X, \rho_1, \rho_2)$, $x \in P(0)X$, (2.1), *and the following conditions hold:*

(i) *The mapping* $t \mapsto \int_0^t U(t, s)Q(s)f(s)\, ds$, $t \geq 0$ *is in the class* $B^1AA(\mathbb{R} : X) + B^1WPAA_0([0, \infty), X, \rho_1, \rho_2)$.

(ii) *There exist finite numbers* $M'' > 0$ *and* $\gamma > 1$ *such that*

$$\|U(t,s)Q(s)\| \leq \frac{M''}{1+(t-s)^{\gamma}} \quad \text{for} \quad t \geq s \geq 0.$$

(iii) *There exists a non-negative measurable function* $g : [0,\infty) \to [0,\infty)$ *such that* $\rho_2(t) \leq g(s)\rho_2(t-s)$ *for* $0 \leq s \leq t < \infty$, *and*

$$\int_0^{\infty} \frac{g(s)}{1+s^{\gamma}}\, ds < \infty.$$

Define $u(t) := U(t,0)x + \int_0^t U(t,s)[g(s)+q(s)]\, ds$, $t \geq 0$. *Then* $u(\cdot)$ *is essentially bounded and belongs to the class* $B^1AA(\mathbb{R} : X) + B^1WPAA_0([0,\infty), X, \rho_1, \rho_2)$.

Proof. The mapping $t \mapsto U(t,0)x$, $t \geq 0$ is bounded since $U(\cdot,\cdot)$ is bounded. This is also clear for the mapping $t \mapsto \int_0^t U(t,s)[g(s)+q(s)]\, ds$, $t \geq 0$ since, due to (1.1) and (iii),

$$\begin{aligned}
&\left\| \int_0^t U(t,s)[g(s)+q(s)]\, ds \right\| \\
&\quad \leq \int_0^t \|\Gamma(t,s)\| \|g(s)+q(s)\|\, ds + \int_0^t \|U(t,s)Q(s)\| \|g(s)+q(s)\|\, ds \\
&\quad \leq \left(\|g\|_\infty + \|q\|_\infty\right) \int_0^t \Big[\|\Gamma(t,s)\| + \|U(t,s)Q(s)\|\Big]\, ds \\
&\quad \leq \left(\|g\|_\infty + \|q\|_\infty\right) \int_0^t \left[M' e^{-\omega(t-s)} + \frac{M''}{1+(t-s)^{\gamma}}\right] ds \\
&\quad \leq \text{Const.} \cdot \left(\|g\|_\infty + \|q\|_\infty\right), \quad t \geq 0.
\end{aligned}$$

Summa summarum, $u(\cdot)$ is essentially bounded. As in the proof of [17, Theorem 4.1], we have the following decomposition:

$$\begin{aligned}
u(t) = U(t,0)x &+ \int_{-\infty}^t \Gamma(t,s)g(s)\, ds - \int_{-\infty}^0 \Gamma(t,s)g(s)\, ds + \int_0^t \Gamma(t,s)q(s)\, ds \\
&+ \int_0^t U(t,s)Q(s)g(s)\, ds + \int_0^t U(t,s)Q(s)q(s)\, ds, \quad t \geq 0.
\end{aligned}$$

Since $x \in P(0)X$, the function $U(\cdot,0)x$ is exponentially decaying due to (1.1). Keeping in mind (i) and the proof of Theorem 2.1, it suffices to show that the mappings $t \mapsto \int_0^t \Gamma(t,s)q(s)\, ds$, $t \geq 0$, and $t \mapsto \int_0^t U(t,s)Q(s)q(s)\, ds$, $t \geq 0$, belong to the class $B^1WPAA_0([0,\infty), X, \rho_1, \rho_2)$. We will prove this only for the second mapping. By the foregoing, $t \mapsto \int_0^t U(t,s)Q(s)q(s)\, ds$, $t \geq 0$, is a bounded continuous mapping. Furthermore, we have

$$\begin{aligned}
&\frac{1}{\int_0^T \rho_1(t)\,dt}\int_0^T\left\|\int_0^t U(t,s)Q(s)q(s)\,ds\right\|\rho_2(t)\,dt\\
&\quad\leq \frac{1}{\int_0^T \rho_1(t)\,dt}\int_0^T\left[\int_0^t \|U(t,t-s)Q(t-s)\|\|q(t-s)\|\,ds\right]\rho_2(t)\,dt\\
&\quad= \frac{1}{\int_0^T \rho_1(t)\,dt}\int_0^T\int_s^T \|U(t,t-s)Q(t-s)\|\|q(t-s)\|\rho_2(t)\,dt\,ds\\
&\quad\leq \frac{1}{\int_0^T \rho_1(t)\,dt}\int_0^T\frac{g(s)}{1+s^\gamma}\left[\int_s^T \|q(t-s)\|\rho_2(t-s)\,dt\right]ds\\
&\quad= \frac{1}{\int_0^T \rho_1(t)\,dt}\int_0^T\frac{g(s)}{1+s^\gamma}\left[\int_0^{T-s} \|q(r)\|\rho_2(r)\,dr\right]ds\\
&\quad\leq \left[\int_0^\infty \frac{g(s)}{1+s^\gamma}\,ds\right]\cdot\left[\frac{1}{\int_0^T \rho_1(t)\,dt}\int_0^T \|q(r)\|\rho_2(r)\,dr\right],\quad T>0.
\end{aligned}$$

Since (iii) holds and $q\in B^1WPAA_0([0,\infty),X,\rho_1,\rho_2)$, we have that the mapping $t\mapsto\int_0^t U(t,s)Q(s)q(s)\,ds$, $t\geq 0$, is in the same class, as claimed. The proof of the theorem is thereby completed. □

Remark 2.6. As in Remark 2.3, the continuity of mapping $u(\cdot)$ is ensured by the continuity of mapping $t\mapsto\int_0^t U(t,s)Q(s)f(s)\,ds$, $t\geq 0$ and the validity of condition $x\in P(0)X\cap\overline{D(A(0))}$.

Remark 2.7. All established results continue to hold in the case that the operator family $(A(t))_{t\in\mathbb{R}}$ generates an exponentially stable evolution family $(U(t,s))_{t\geq s}$ in the sense of [9, Definition 3.1] (therefore, the condition (H1) need not be necessarily satisfied and (H2) holds with $P(t)=I$ and $Q(t)=0$, $t\in\mathbb{R}$; $\Gamma(t,s)\equiv U(t,s)$). The only thing worth noting here is that, with the notation already employed, the condition (iii) from the formulation of Theorem 2.5 can be slightly weakened by assuming that $\int_0^\infty e^{-\omega s}g(s)\,ds<\infty$, which follows from the fact that we can estimate the term $\|U(t,t-s)\|$ in the proof of this theorem by $M'e^{-\omega s}$ $(t\geq s\geq 0)$.

3. An application

In a series of his research papers, Diagana has examined the existence and uniqueness of almost periodic and almost automorphic type solutions of nonautonomous differential equations of first order (see e.g. [7] and references cited therein). In the following example, we continue the analysis carried out in [9, Section 4], which nicely fit into consideration of Besicovitch almost automorphic solutions.

Example 3.1. Assume that $X:=L^2[0,\pi]$ and that Δ denotes the Dirichlet Laplacian in X, acting with the domain $H^2[0,\pi]\cap H_0^1[0,\pi]$. Then Δ generates a strongly continuous semigroup $(T(t))_{t\geq 0}$ on X, satisfying the estimate $\|T(t)\| \leq e^{-t}$, $t \geq 0$. Consider the following problem

$$u_t(t,x) = u_{xx}(t,x) + q(t,x)u(t,x) + f(t,x),\ t \geq 0,\ x \in [0,\pi]; \tag{3.1}$$

$$u(0) = u(\pi) = 0,\ u(0,x) = u_0(x) \in X, \tag{3.2}$$

where $q : \mathbb{R} \times [0,\pi] \to \mathbb{R}$ is a jointly continuous function satisfying that $q(t,x) \leq -\gamma_0$, $(t,x) \in \mathbb{R} \times [0,\pi]$, for some number $\gamma_0 > 0$. Define

$$A(t)\varphi := \Delta\varphi + q(t,\cdot)\varphi,\ \varphi \in D(A(t)) := D(\Delta) = H^2[0,\pi] \cap H_0^1[0,\pi],\ t \in \mathbb{R}.$$

Then $(A(t))_{t\in\mathbb{R}}$ generates an exponentially stable evolution family $(U(t,s))_{t\geq s}$ in the sense of [9, Definition 3.1], which is given by

$$U(t,s)\varphi := T(t-s)e^{\int_s^t q(r,\cdot)\,dr}\varphi, \quad t \geq s.$$

It is clear that we can rewrite the initial value problem (3.1)–(3.2) in the following form:

$$u'(t) = A(t)u(t) + f(t),\ t \geq 0; \quad u(0) = u_0.$$

Hence, Theorems 2.2 and 2.5 are susceptible to applications provided that the following condition holds:

$$\lim_{l\to+\infty} \frac{1}{2l} \int_{-l}^{l} \int_{-\infty}^{v} \sup_{x\in[0,\pi]} \left| e^{\int_{s+\alpha}^{v+\alpha} q(r,x)\,dr} - e^{\int_s^v q(r,x)\,dr} \right| ds\,dv = 0 \quad (\alpha \in \mathbb{R}); \tag{3.3}$$

see (2.1). The condition (3.3) holds for a substantially large class of functions $q(\cdot,\cdot)$, and we will prove here, for the sake of completeness, that this condition particularly holds for the function $q(t,x) = -\gamma_0 - 3t^2 - f(x)$, $t \in \mathbb{R}$, $x \in [0,\pi]$, where $f : \mathbb{R} \to [0,\infty)$ is a continuous function. Let $\alpha \in \mathbb{R}$ be fixed. In our concrete situation, we have the following estimate of the integrand:

$$\sup_{x\in[0,\pi]} \left| e^{\int_{s+\alpha}^{v+\alpha} q(r,x)\,dr} - e^{\int_s^v q(r,x)\,dr} \right| \leq \text{Const.} \cdot \left[e^{s^3-v^3} + e^{s^3-v^3+3|\alpha|(v-s)^2} \right].$$

Using substitution $x = v - s$, the only thing we need to prove is that

$$\lim_{l\to+\infty} \frac{1}{2l} \int_{-l}^{l} \int_0^{\infty} e^{3x^2v-3xv^2-x^3+3|\alpha|x^2}\,dx\,dv = 0.$$

For $v \in [-l,0]$, we have $e^{3x^2v-3xv^2-x^3+3|\alpha|x^2} \leq e^{-3xv^2-x^3+3|\alpha|x^2}$, so that the Fubini theorem yields that

$$\begin{aligned}
&\int_{-l}^{0} \int_0^{\infty} e^{3x^2v-3xv^2-x^3+3|\alpha|x^2}\,dx\,dv \\
&\quad \leq \int_{-l}^{0} \int_0^{\infty} e^{-3xv^2-x^3+3|\alpha|x^2}\,dx\,dv \\
&\quad \leq \int_0^{\infty} e^{-x^3+3|\alpha|x^2} \left[\int_{-\infty}^{0} e^{-3xv^2}\,dv \right] dx \\
&\quad = \left[\int_0^{\infty} \frac{1}{\sqrt{3x}} e^{-x^3+3|\alpha|x^2}\,dx \right] \left[\int_0^{\infty} e^{-y^2}\,dy \right], \quad l > 0.
\end{aligned}$$

Since the integrand is always bounded by the proof of Theorem 2.1, the above computation shows that we only need to prove yet that

$$\lim_{l\to+\infty}\frac{1}{2l}\int_1^l\int_0^\infty e^{3x^2v-3xv^2-x^3+3|\alpha|x^2}\,dx\,dv=0.$$

For $x\in[0,1]$, we have $e^{3x^2v-3xv^2-x^3+3|\alpha|x^2}\leq \text{Const.}\cdot e^{3xv(1-v)}$, so that

$$\begin{aligned}
&\int_1^l\int_0^1 e^{3x^2v-3xv^2-x^3+3|\alpha|x^2}\,dx\,dv\\
&\qquad\leq \text{Const.}\cdot\int_1^l\int_0^1 e^{3xv(1-v)}dv\\
&\qquad= \text{Const.}\int_0^1\int_0^{l-1} e^{-3xy(1+y)}\,dy\,dx\\
&\qquad\leq \text{Const.}\int_0^1\int_0^\infty e^{-3xy^2}\,dy\,dx\\
&\qquad= \left[\int_0^1\frac{dx}{\sqrt{3x}}\right]\left[\int_0^\infty e^{-y^2}\,dy\right]\quad (l\geq 1),
\end{aligned}$$

so that we need to prove that

$$\lim_{l\to+\infty}\frac{1}{2l}\int_1^l\int_1^\infty e^{3x^2v-3xv^2-x^3+3|\alpha|x^2}\,dx\,dv=0. \tag{3.4}$$

Let $\epsilon\in(0,1)$ be given. For a given $v\geq 1$, the function $F(x):=3x^2v-3xv^2-(1-\epsilon)x^3$, $x\geq 1$, attains its local minimum (resp., maximum) at the point $x=v\frac{1-\sqrt{\epsilon}}{1-\epsilon}$ (resp., $x=v\frac{1+\sqrt{\epsilon}}{1-\epsilon}$). We have

$$\begin{aligned}
&\int_1^l\int_1^\infty e^{3x^2v-3xv^2-x^3+3|\alpha|x^2}\,dx\,dv\\
&\quad=\int_1^l\int_1^{v\frac{1-\sqrt{\epsilon}}{1-\epsilon}} e^{3x^2v-3xv^2-x^3+3|\alpha|x^2}\,dx\,dv+\int_1^l\int_{v\frac{1-\sqrt{\epsilon}}{1-\epsilon}}^\infty e^{3x^2v-3xv^2-x^3+3|\alpha|x^2}\,dx\,dv\\
&\quad\leq\left[\int_1^\infty e^{-3v^2\left(\frac{1-\sqrt{\epsilon}}{1-\epsilon}-1\right)}\,dv\right]\left[\int_0^\infty e^{-x^3+3|\alpha|x^2}\,dx\right]\\
&\qquad+\left[\int_1^\infty e^{-c_\epsilon v^3}\,dv\right]\left[\int_0^\infty e^{-\epsilon x^3+3|\alpha|x^2}\,dx\right],
\end{aligned}$$

where $\lim_{\epsilon\to 0+}c_\epsilon=1$. Hence, the mapping

$$l\mapsto\int_1^l\int_1^\infty e^{3x^2v-3xv^2-x^3+3|\alpha|x^2}\,dx\,dv,\quad l\geq 1,$$

is bounded, finishing the proof of (3.4).

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References

1. P. Acquistapace and B. Terreni, *A uniffied approach to abstract linear nonautonomous parabolic equations*, Rend. Sem. Mat. Univ. Padova **78** (1987), 47–107.
2. F. Bedouhene, N. Challali, O. Mellah, P. Raynaud de Fitte, and M. Smaali, *Almost automorphy and various extensions for stochastic processes*, J. Math. Anal. Appl. **429** (2015), no. 2, 1113–1152.
3. A. S. Besicovitch, *Almost periodic functions*, Dover Publications Inc., New York, 1954.
4. M. Baroun, S. Boulite, G. M. N'Guérékata, and L. Maniar, *Almost automorphy of semilinear parabolic evolution equations*, Electron. J. Differential Equations, **2008**, no. 60, 9pp.
5. J. Blot, P. Cieutat, and K. Ezzinbi, *New approach for weighted pseudo-almost periodic functions under the light of measure theory, basic results and applications*, Appl. Anal. **92** (2011), no. 3, 493–526.
6. J. Blot, G. M. Mophou, G. M. N'Guérékata, and D. Pennequin, *Weighted pseudo almost automorphic functions and applications to abstract differential equations*, Nonlinear Anal. **71** (2009), no. 3-4, 903–909.
7. T. Diagana, *Almost automorphic type and almost periodic type functions in abstract spaces*, Springer–Verlag, New York, 2013.
8. T. Diagana, K. Ezzinbi, and M. Miraoui, *Pseudo-almost periodic and pseudo-almost automorphic solutions to some evolution equations involving theoretical measure theory*, Cubo, **16** (2014), no. 2, 01–31.
9. T. Diagana, *Stepanov-like pseudo-almost periodicity and its applications to some nonautonomous differential equations*, Nonlinear Anal. **69** (2008), no. 12, 4277–4285.
10. T. Diagana, *Existence of pseudo-almost automorphic mild solutions to some nonautonomous partial evolution equations*, Adv. Difference Equ. **2011**, Art. ID 895079, 23 pp.
11. H.-S. Ding, W. Long, and G. M. N'Guérékata, *Almost periodic solutions to abstract semilinear evolution equations with Stepanov almost periodic coefficients*, J. Comput. Anal. Appl. **13** (2011), no. 2, 231–242.
12. H.-S. Ding, W. Long, and G. M. N'Guérékata, *Almost automorphic solutions of nonautonomous evolution equations*, Nonlinear Anal. **70** (2009), no. 12, 4158–4164.
13. H.-S. Ding, J. Liang, and T.-J. Xiao, *Almost automorphic solutions to nonautonomous semilinear evolution equations in Banach spaces*, Nonlinear Anal. **73** (2010), no. 5, 1426–1438.
14. S. Fatajou, N. V. Minh, G. M. N'Guérékata, and A. Pankov, *Stepanov-like almost automorphic solutions for nonautonomous evolution equations*, Electron. J. Differential Equations **2007**, no. 121 (2007), 11 pp.
15. G. M. N'Guérékata, *Almost automorphic and almost periodic functions in abstract spaces*, Kluwer Academic/Plenum Publishers, New York, 2001.
16. G. M. N'Guérékata, *Topics in almost automorphy*, Springer–Verlag, New York, 2005.
17. M. Kostić, *Generalized almost periodic solutions and generalized asymptotically almost periodic solutions of inhomogenous evolution equations*, Sarajevo J. Math. (to appear).
18. M. Kostić, *On Besicovitch-Doss almost periodic solutions of abstract Volterra integro-differential equations*, Novi Sad J. Math. **47** (2017), no. 2, 187–200.
19. M. Kostić, *Generalized almost automorphic and generalized asymptotically almost automorphic solutions of abstract Volterra integro-differential inclusions*, preprint.
20. M. Kostić, *Generalized weighted pseudo-almost periodic solutions and generalized weighted pseudo-almost automorphic solutions of abstract Volterra integro-differential inclusions*, preprint.
21. M. Kostić, *Almost periodicity of abstract Volterra integro-differential equations*, Adv. Oper. Theory **2** (2017), no. 3-5, 353–382.
22. M. Kostić, *Almost periodic and almost Aautomorphic type solutions of abstract Volterra integro-differential equations*, Book Manuscript, 2017.

23. L. Maniar and R. Schnaubelt, *Almost periodicity of inhomogeneous parabolic evolution equations*, Lecture Notes in Pure and Appl. Math. **234**, Dekker, New York, 2003, 299–318.
24. R. Zhang, Y.-K. Chang, and G. M. N'Guérékata, *New composition theorems of Stepanov-like weighted pseudo almost automorphic functions and applications to nonautonomous evolution equations*, Nonlinear Anal. Real World Appl. **13** (2012), no. 6, 2866–2879.

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