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LINEAR PRESERVERS OF TWO-SIDED RIGHT MATRIX MAJORIZATION ON $\mathbb{R}_n$

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Abstract. A nonnegative real matrix $R \in \mathbf{M}_{n,m}$ with the property that all its row sums are one is said to be row stochastic. For $x, y \in \mathbb{R}_n$, we say x is right matrix majorized by y (denoted by $x \prec_r y$) if there exists an n-by-n row stochastic matrix R such that $x = yR$. The relation $\sim_r$ on $\mathbb{R}_n$ is defined as follows. $x \sim_r y$ if and only if $x \prec_r y \prec_r x$. In the present paper, we characterize the linear preservers of $\sim_r$ on $\mathbb{R}_n$, and answer the question raised by F. Khalooei [Wavelet Linear Algebra **1** (2014), no. 1, 43–50].

1. Introduction and preliminaries

Let $\mathbf{M}_{n,m}$ be the set of all n-by-m real matrices. We denote by $\mathbb{R}_n$ ($\mathbb{R}^n$) the set of 1-by-n (n-by-1) real vectors. A matrix $R = [r_{ij}] \in \mathbf{M}_{n,m}$ with nonnegative entries is called a row stochastic matrix if $\sum_{j=1}^{n} r_{ij} = 1$ for all i. For vectors $x, y \in \mathbb{R}_n$ (resp. $\mathbb{R}^n$), it is said that x is right (resp. left) matrix majorized by y (denoted by $x \prec_r y$ (resp. $x \prec_l y$)) if $x = yR$ (resp. $x = Ry$) for some n-by-n row stochastic matrix R. A linear function $T : \mathbf{M}_{n,m} \to \mathbf{M}_{n,m}$ preserves an order relation $\prec$ in $\mathbf{M}_{n,m}$, if $TX \prec TY$ whenever $X \prec Y$.

In [3] and [4], the authors obtained all linear preservers of $\prec_r$ and $\prec_l$ on $\mathbb{R}_n$ and $\mathbb{R}^n$, respectively. Let $x, y \in \mathbb{R}_n$ (resp. $\mathbb{R}^n$). We write $x \sim_r y$ (resp. $x \sim_l y$) if and only if $x \prec_r y \prec_r x$ (resp. $x \prec_l y \prec_l x$).

In [6], the author characterized all linear preservers of $\sim_l$ from $\mathbb{R}^p$ to $\mathbb{R}^n$. Here, by specifying linear preservers of $\sim_r$ we will answer the question raised in [6]. For

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more information about linear preservers of majorization, we refer the reader to [1, 2, 5]. Also, the reference [7] is precious book in this regard.

In this paper, we characterize all linear preservers of two-sided right matrix majorization on $\mathbb{R}_n$.

A nonnegative real matrix D is called doubly stochastic if the sum of entries of every row and column of D is one.

The following conventions will be fixed throughout the paper.
We will denote by $\mathcal{P}(n)$ the collection of all n-by-n permutation matrices. The collection of all n-by-n row stochastic matrices is denoted by $\mathcal{RS}(n)$. Also, the collection of all n-by-n doubly stochastic matrices is denoted by $\mathcal{DS}(n)$. The standard basis of $\mathbb{R}_n$ is denoted by $\{e_1, \ldots, e_n\}$, and $e = (1, 1, \ldots, 1) \in \mathbb{R}_n$. Also, let A^t be the transpose of a given matrix A. Let $[X_1/\ldots/X_n]$ be the n-by-m matrix with rows $X_1, \ldots, X_n \in \mathbb{R}_m$. We denote by $\mid A \mid$ the absolute of a given matrix A.
For $u \in \mathbb{R}$, let $u^+ = \begin{cases} u & \text{if} \quad u \geq 0 \\ 0 & \text{if} \quad u < 0 \end{cases}$, and $u^- = \begin{cases} 0 & \text{if} \quad u \geq 0 \\ u & \text{if} \quad u < 0 \end{cases}$.
For every $x = (x_1, \ldots, x_n) \in \mathbb{R}_n$ we set $\mathrm{Tr}(x) := \sum_{i=1}^n x_i$, $\mathrm{Tr}_+(x) := \sum_{i=1}^n x_i^+$, and $\mathrm{Tr}_-(x) := \sum_{i=1}^n x_i^-$.
For each $x \in \mathbb{R}_n$ let $x^* = \mathrm{Tr}_+(x)e_1 + \mathrm{Tr}_-(x)e_2$, and $\|x\|_1 = \sum_{i=1}^n \mid x_i \mid$.
Let $[T]$ be the matrix representation of a linear function $T : \mathbb{R}_n \to \mathbb{R}_n$ with respect to the standard basis. In this case, $Tx = xA$, where $A = [T]$.

2. Main results

In this section, we pay attention to the two-sided right matrix majorization on $\mathbb{R}_n$. We obtain an equivalent condition for two-sided right matrix majorization on $\mathbb{R}_n$, and we characterize all of the linear functions $T : \mathbb{R}_n \to \mathbb{R}_n$ preserving $\sim_r$.

We need the following lemma for finding some equivalent conditions for two-sided right matrix majorization on $\mathbb{R}_n$.

Lemma 2.1. *Let $x \in \mathbb{R}_n$. Then $x \sim_r x^*$.*

Proof. We prove that $x \sim_r x^*$, for each $x \in \mathbb{R}_n$. Suppose that $x = (x_1, \ldots, x_n) \in \mathbb{R}_n$. We define the matrices $R = [R_1, \ldots, R_n]$ and $S = [S_1, \ldots, S_n]$ as follows.

For each i $(1 \leq i \leq n)$

$$R_i := \begin{cases} e_1 & x_i \geq 0 \\ e_2 & x_i < 0 \end{cases}, \quad \text{and}$$

$$S_i := \begin{cases} \dfrac{1}{\mathrm{Tr}_+(x)} \displaystyle\sum_{x_j > 0} x_j e_j & x_i > 0 \\ e_1 & x_i = 0 \\ \dfrac{1}{\mathrm{Tr}_-(x)} \displaystyle\sum_{x_j < 0} x_j e_j & x_i < 0 \end{cases}.$$

It is clear that $R, S \in \mathcal{RS}(n)$, $x^* = xR$, and $x = x^*S$. Therefore, $x \sim_r x^*$. $\square$

The following proposition gives some equivalent conditions for two-sided right matrix majorization on $\mathbb{R}_n$.

Proposition 2.2. *Let x, $y \in \mathbb{R}_n$. Then the following conditions are equivalent.*
(a) $x \sim_r y$,
(b) $\mathrm{Tr}_+(x) = \mathrm{Tr}_+(y)$, *and* $\mathrm{Tr}_-(x) = \mathrm{Tr}_-(y)$,
(c) $\mathrm{Tr}(x) = \mathrm{Tr}(y)$, *and* $\|x\|_1 = \|y\|_1$.

Proof. Let $x, y \in \mathbb{R}_n$. First, we prove that (a) is equivalent to (b). We use Lemma 2.1.

If $x \sim_r y$, then $x^* \sim_r y^*$, and so $x^* = y^*$. This follows that $\mathrm{Tr}_+(x) = \mathrm{Tr}_+(y)$, and $\mathrm{Tr}_-(x) = \mathrm{Tr}_-(y)$.

If $\mathrm{Tr}_+(x) = \mathrm{Tr}_+(y)$, and $\mathrm{Tr}_-(x) = \mathrm{Tr}_-(y)$, then $x^* = y^*$. Set $z = x^* = y^*$. Lemma 2.1 ensures $x \sim_r z$ and $y \sim_r z$. It implies that $\mathrm{Tr}_+(x) = \mathrm{Tr}_+(y)$, and $\mathrm{Tr}_-(x) = \mathrm{Tr}_-(y)$.
So (a) is equivalent to (b).

Now, the relations

$$\mathrm{Tr}(x) = \mathrm{Tr}_+(x) + \mathrm{Tr}_-(x), \quad \text{and} \quad \|x\|_1 = \mathrm{Tr}_+(x) - \mathrm{Tr}_-(x)$$

ensure that (b) is equivalent to (c), too. □

Now, we express the non-invertible linear preservers of two-sided right matrix majorization on $\mathbb{R}_n$. In the case $n = 1$, any linear function can be a linear preserver of $\sim_r$.

Theorem 2.3. *Let T be a non-invertible linear function on $\mathbb{R}_n$. Then T preserves $\sim_r$ if and only if there exists some $\boldsymbol{a} \in \mathbb{R}_n$ such that $Tx = \mathrm{Tr}(x)\boldsymbol{a}$ for all $x \in \mathbb{R}_n$.*

Proof. First, assume that $x, y \in \mathbb{R}_n$ and $x \sim_r y$. Proposition 2.2 ensures that $\mathrm{Tr}(x) = \mathrm{Tr}(y)$, and hence $Tx \sim_r Ty$. It implies that T preserve $\sim_r$.

Next, let T preserve $\sim_r$. The case $n = 1$ is clear. Assume that $n \geq 2$, and $[T] = A = [A_1/\dots/A_n]$. There exists some $C \in \mathbb{R}_n \setminus \{0\}$ such that $TC = 0$, since T is not invertible. From $C^* \sim_r C$, we see $TC^* = 0$. We know that $C^* = \alpha e_1 + \beta e_2$, where $\beta \leq 0 \leq \alpha$.

For $r \neq s$, it follows from $\alpha e_r + \beta e_s \sim_r C^*$ that $T(\alpha e_r + \beta e_s) \sim_r TC^*$. Hence, $T(\alpha e_r + \beta e_s) = 0$. Let us consider two cases.

Case 1. Let $\alpha + \beta \neq 0$. Then

$$2(\alpha + \beta)Te_1 = T(\alpha e_1 + \beta e_2) + T(\beta e_1 + \alpha e_2) = 0.$$

This shows that $Te_1 = 0$. From $Te_i \sim_r Te_1$, for each i ($1 \leq i \leq n$), we conclude that $Te_i = 0$, and so $A = 0$. In this case, the vector $\mathbf{a}$ is zero.

Case 2. Let $\alpha + \beta = 0$. Then $\alpha = -\beta$. Since $C \in \mathbb{R}_n \setminus \{0\}$, we deduce $\alpha \neq 0$. From

$$0 = T(\alpha e_r + \beta e_s) = T(\alpha e_r - \alpha e_s) = \alpha(A_r - A_s),$$

we have $A_r = A_s$, for each ($r \neq s$). Here, we put $\mathbf{a} := A_1 = \cdots = A_n$.

Therefore, in any cases there exists some $\mathbf{a} \in \mathbb{R}_n$ such that $Tx = \mathrm{Tr}(x)\mathbf{a}$ for all $x \in \mathbb{R}_n$. □

Theorem 2.4. *Let $T : \mathbb{R}_2 \to \mathbb{R}_2$ be an invertible linear function. Then T preserves $\sim_r$ if and only if there exist some $\alpha \in \mathbb{R} \setminus \{0\}$, and some invertible matrix $D \in \mathcal{DS}(2)$ such that $Tx = \alpha x D$ for all $x \in \mathbb{R}_n$.*

Proof. As the sufficiency of the condition is easy to be verified, we only prove the necessity of the condition. Assume that T preserves $\sim_r$, and $[T] = A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We know that $Tx = xA$, for each $x \in \mathbb{R}_2$.

If $ab < 0$; then

$$\{\mathrm{Tr}_+(a,b), \mathrm{Tr}_-(a,b)\} = \{a,b\}.$$

As $e_1 \sim_r e_2$ and T preserves $\sim_r$, we have $Te_1 \sim_r Te_2$. This follows that

$$\{\mathrm{Tr}_+(c,d), \mathrm{Tr}_-(c,d)\} = \{a,b\}.$$

We conclude that $a = d$ and $b = c$, since T is invertible. This means that $[T] = A = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$.

The relation $e_1 \sim_r \frac{1}{2}e$ shows that

$$(a,b) \sim_r (\frac{a+b}{2}, \frac{a+b}{2}),$$

whence

$$\begin{aligned} \{a,b\} &= \{\mathrm{Tr}_+(a,b), \mathrm{Tr}_-(a,b)\} \\ &= \{\mathrm{Tr}_+(\frac{a+b}{2}, \frac{a+b}{2}), \mathrm{Tr}_-(\frac{a+b}{2}, \frac{a+b}{2})\} \\ &= \{0, a+b\}. \end{aligned}$$

So $a = 0$ or $b = 0$, which is a contradiction, and thus $ab \geq 0$.

Since $-T$ preserves $\sim_r$, without loss of generality, we may assume that $a, b \geq 0$. From $e_1 \sim_r e_2$, we observe that $Te_1 \sim_r Te_2$, and hence $(a,b) \sim_r (c,d)$. This implies that

$$\mathrm{Tr}_-(c,d) = \mathrm{Tr}_-(a,b) = 0,$$

and hence $c, d \geq 0$. Thus, the entries of A are nonnegative.

If $ad = 0$ and $bc = 0$, then from the invertibility of T we get

$$A = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}, \text{ or } A = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}.$$

Now, $Te_1 \sim_r Te_2$ ensures that $a = d$, or $b = c$, and hence,

$$A = a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \text{ or } A = b\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

as desired.

If $ad \neq 0$ and $bc \neq 0$; since PT preserves $\sim_r$ for each $P \in \mathcal{P}(2)$, without loss of generality, we may assume that $bc \neq 0$. To complete the proof, we show that

$\frac{a}{c} = \frac{d}{b}$. In this case, since $a + b = c + d$, we have $a = d$ and $b = c$. Hence,

$$A = \begin{pmatrix} a & b \\ b & a \end{pmatrix} = \alpha D,$$

where

$$D = \begin{pmatrix} \frac{a}{a+b} & \frac{b}{a+b} \\ \frac{b}{a+b} & \frac{a}{a+b} \end{pmatrix} \in \mathcal{DS}(2), \text{ and } \alpha = a + b \in \mathbb{R} \setminus \{0\}.$$

We observe that D is invertible, since T is invertible.

If $\frac{a}{c} < \frac{d}{b}$; we conclude that $\frac{d}{b} < 1$, since $a + b = c + d$. So for each $x \in \mathbb{R}$ that $\frac{a}{c} < x < \frac{d}{b}$ we have

$$0 = \{\mathrm{Tr}_+(T(x, -1))\} = \{\mathrm{Tr}_+(T(-1, x))\} = cx - a > 0,$$

a contradiction.

Similarly, by assuming $\frac{a}{c} > \frac{d}{b}$ we will be contradictory, and it completes this proof. □

Now, we state the previous theorem for $n \geq 3$.

Theorem 2.5. *Let $T : \mathbb{R}_n \to \mathbb{R}_n$ $(n \geq 3)$ be an invertible linear function. Then T preserves $\sim_r$ if and only if there exist some $\alpha \in \mathbb{R} \setminus \{0\}$ and a permutation matrix $P \in \mathcal{P}(n)$ such that $Tx = \alpha x P$, $\forall x \in \mathbb{R}_n$.*

Proof. We only need to prove the necessity of the condition. Assume that T is invertible and T preserves $\sim_r$ for $n \geq 3$. First, we prove that the linear function $\mid T \mid$ which is defined as $[\mid T \mid] = \mid A \mid$ preserves $\sim_r$. We show that each column of A is either nonnegative or non-positive. For this purpose, we prove

$$\mid a_{rj} + a_{sj} \mid = \mid a_{rj} \mid + \mid a_{sj} \mid, \quad \text{for each} \quad r, s, j \quad (1 \leq r, s, j \leq n).$$

Let $1 \leq r, s \leq n$. From $e_r \sim_r e_s$, as T preserves $\sim_r$, we have $Te_r \sim_r Te_s$, and so $\|Te_r\|_1 = \|Te_s\|_1$. Since $2e_r \sim_r e_r + e_s$, this follows that $T(2e_r) \sim_r T(e_r + e_s)$. Therefore, $2\|Te_r\|_1 = \|Te_r + Te_s\|_1$. We observe that

$$\mathrm{Tr}(\mid T \mid (x)) = \mathrm{Tr}(x)\mathrm{Tr}(A), \tag{2.1}$$

and

$$\| \mid T \mid (x)\|_1 = \|T(x)\|_1. \tag{2.2}$$

Observe that

$$\begin{aligned} 2\|Te_r\|_1 &= \|Te_r + Te_s\|_1 \\ &= \textstyle\sum_{j=1}^n \mid a_{rj} + a_{sj} \mid \\ &\leq \textstyle\sum_{j=1}^n \mid a_{rj} \mid + \sum_{j=1}^n \mid a_{sj} \mid \\ &= \|Te_r\|_1 + \|Te_s\|_1 \\ &= 2\|Te_r\|_1. \end{aligned}$$

This implies that

$$\sum_{j=1}^{n} | a_{rj} + a_{sj} |= \sum_{j=1}^{n} | a_{rj} | + \sum_{j=1}^{n} | a_{sj} |,$$

and hence for each j $(1 \leq j \leq n)$

$$| a_{rj} + a_{sj} |=| a_{rj} | + | a_{sj} | .$$

Fix

$$C^+ = \{1 \leq j \leq n \mid e_j A^t \geq 0\},$$

and

$$C^- = \{1 \leq j \leq n \mid e_j A^t \leq 0\}.$$

Also, as $Te_r \sim_r Te_s$, we see that $\mathrm{Tr}_+(Te_r) = \mathrm{Tr}_+(Te_s)$, $\mathrm{Tr}_-(Te_r) = \mathrm{Tr}_-(Te_s)$, and $\mathrm{Tr}(Te_r) = \mathrm{Tr}(Te_s)$. So we can choose $\mathrm{Tr}_+(A) = \mathrm{Tr}_+(Te_1)$, $\mathrm{Tr}_-(A) = \mathrm{Tr}_-(Te_1)$, and $\mathrm{Tr}(A) = \mathrm{Tr}(Te_1)$. Now, we show that for each $x \in \mathbb{R}_n$ we have

$$\mathrm{Tr}(| T | (x)) = \mathrm{Tr}(x)\mathrm{Tr}(A), \tag{2.3}$$

and

$$\| \, | T | (x)\|_1 = \|T(x)\|_1. \tag{2.4}$$

Observe that

$$\begin{aligned} \mathrm{Tr}(| T | (x)) &= \textstyle\sum_{j=1}^{n} x. | e_j A^t | \\ &= \sum_{j\in C^+(A)} x. | e_j A^t | + \sum_{j\in C^-(A)} x. | e_j A^t | \\ &= \sum_{j\in C^+(A)} x.e_j A^t - \sum_{j\in C^-(A)} x.e_j A^t \\ &= x. \sum_{j\in C^+(A)} e_j A^t - x. \sum_{j\in C^-(A)} e_j A^t \\ &= (\textstyle\sum_{i=1}^{n} x_i) \displaystyle\sum_{j\in C^+(A)} a_{ij} - (\sum_{i=1}^{n} x_i) \sum_{j\in C^-(A)} a_{ij} \\ &= \mathrm{Tr}(x)(\mathrm{Tr}_+(A) - \mathrm{Tr}_-(A)) \\ &= \mathrm{Tr}(x)\mathrm{Tr}(A), \end{aligned}$$

and this proves the relation (2.3).

To prove the relation (2.4) we have

$$\begin{aligned} \| \, | T | (x)\|_1 &= \textstyle\sum_{j=1}^{n} | x. | e_j A^t || \\ &= \sum_{j\in C^+(A)} | x.e_j A^t | + \sum_{j\in C^-(A)} | -x.e_j A^t | \\ &= \sum_{j\in C^+(A)} | x.e_j A^t | + \sum_{j\in C^-(A)} | x.e_j A^t | \\ &= \textstyle\sum_{j=1}^{n} | x.e_j A^t | \\ &= \|T(x)\|, \end{aligned}$$

as desired.

Now, let $x, y \in \mathbb{R}_n$, and let $x \sim_r y$. In this case, $\mathrm{Tr}(x) = \mathrm{Tr}(y)$, and since T preserves $\sim_r$, we deduce $Tx \sim_r Ty$. Therefore, $\|T(x)\|_1 = \|T(y)\|_1$. We conclude from (2.3) and (2.4) that $\mathrm{Tr}(\mid T \mid (x)) = \mathrm{Tr}(\mid T \mid (y))$ and $\| \mid T \mid (x)\|_1 = \| \mid T \mid (y)\|_1$, hence $\mid T \mid (x) \sim_r \mid T \mid (y)$, and finally that $\mid T \mid$ preserves $\sim_r$. So, without loss of generality, we can assume that entries of $[T]$ are nonnegative.

Now, we claim that in each column of A there exists at most a nonzero entry. Since T is invertible, if in a column, for example the j^{th} column, there exists more than a nonzero entry, then without loss of generality, we may assume $a_{1j} \neq a_{2j}$ and $a_{3j} \neq 0$. Let us consider

$$\alpha^* = \min\{\frac{a_{3k}}{a_{1k} + +a_{2k}} \mid a_{1k} \neq a_{2k}, \quad a_{3k} \neq 0, \quad \forall 1 \leq k \leq n\},$$

and suppose j_0 $(1 \leq j_0 \leq n)$ is such that $\alpha^* = \dfrac{a_{3j_0}}{a_{1j_0} + +a_{2j_0}}$. We set the vectors $\mathbf{c}, \mathbf{d} \in \mathbb{R}_n$ as follows.

$$\begin{aligned} \mathbf{c} : &= \begin{cases} 2\alpha^* e_2 - e_3 & \text{if} \quad a_{1j_0} < a_{2j_0} \\ 2\alpha^* e_1 - e_3 & \text{if} \quad a_{1j_0} > a_{2j_0} \end{cases}, \quad \text{and} \\ \mathbf{d} : &= \alpha^*(e_1 + e_2) - e_3. \end{aligned}$$

From $\mathbf{c} \sim_r \mathbf{d}$, we deduce $T\mathbf{c} \sim_r T\mathbf{d}$, then $\mathrm{Tr}_+(T\mathbf{c}) = \mathrm{Tr}_+(T\mathbf{d})$. For each $x \in \mathbb{R}$, we have $xe_1 - e_3 \sim_r xe_2 - e_3$. This gives

$$T(xe_1 - e_3) \sim_r T(xe_2 - e_3),$$

and consequently,

$$\mathrm{Tr}_+ T(xe_1 - e_3) \sim_r \mathrm{Tr}_+ T(xe_2 - e_3).$$

We choose x small enough such that

$$\mathrm{Tr}_+ T(xe_1 - e_3) = x \sum_{a_{3j}=0} a_{1j},$$

and

$$\mathrm{Tr}_+ T(xe_2 - e_3) = x \sum_{a_{3j}=0} a_{2j},$$

and so

(i) $\quad \sum_{a_{3j}=0} a_{1j} = \sum_{a_{3j}=0} a_{2j}$.

We also have the following statements.

(ii) If $a_{1j} = a_{2j}$, then $(T\mathbf{c})_j = (T\mathbf{d})_j = 2\alpha a_{1j} - a_{3j}$,

(iii) If $a_{1j} \neq a_{2j}$, and $a_{3j} \neq 0$, then $(T\mathbf{d})_j \leq 0$. Because

$$\alpha^*(a_{1j} + a_{2j}) - a_{3j} \leq \frac{a_{3j}}{a_{1j} + a_{2j}}(a_{1j} + a_{2j}) - a_{3j} = 0.$$

On the other hand, $(T\mathbf{c})_{j_0} > 0$. From $(i), (ii)$, and (iii) we conclude $\mathrm{Tr}_+(T\mathbf{c}) - \mathrm{Tr}_+(T\mathbf{d}) > 0$, which is a contradiction. Therefore, in each column of A there is at most one nonzero entry. As A is invertible, this implies that each column of A has exactly one nonzero entry. Also, in each row of A, there should be exactly one nonzero entry. Suppose a_i is the only nonzero entry (positive) in the i^{th} row, where i $(1 \leq i \leq n)$.

For each i, j $(1 \leq i, j \leq n)$ from $Te_i \sim_r Te_j$ it may be conclude that

$$\mathrm{Tr}_+(Te_i) = \mathrm{Tr}_+(Te_j),$$

and so

$$a_i = \mathrm{Tr}_+(Te_i) = \mathrm{Tr}_+(Te_j) = a_j.$$

Set $\alpha := a_1 = \cdots = a_n$. Therefore, there exists some $P \in \mathcal{P}(n)$ such that $A = \alpha P$, as required.

□

We can summarize the theorems below. Remember that for $n = 1$ any linear function can be a linear preserver of $\sim_r$.

Theorem 2.6. *Let $T : \mathbb{R}_n \to \mathbb{R}_n$ $(n \geq 2)$ be a linear function. Then T preserves $\sim_r$ if and only if one of the following conditions occur.*

(a) T is non-invertible and there exists some $\boldsymbol{a} \in \mathbb{R}_n$ such that $Tx = \mathrm{Tr}(x)\boldsymbol{a}$ for all $x \in \mathbb{R}_n$.

(b) T is invertible and $Tx = \alpha x D$, for some $\alpha \in \mathbb{R} \setminus \{0\}$, and some invertible doubly stochastic matrix $D \in \mathcal{DS}(2)$, whenever $n = 2$.

(c) T is invertible and there exist some $\alpha \in \mathbb{R} \setminus \{0\}$ and a permutation matrix $P \in \mathcal{P}(n)$ such that $Tx = \alpha x P$, $\forall x \in \mathbb{R}_n$, whenever $n \geq 3$.

The question that comes up here is getting the linear preservers of this relation on matrices.

References

1. T. Ando, *Majorization, doubly stochastic matrices, and comparision of eigenvalues*, Linear Algebra Appl. **118** (1989), 163–248.
2. H. Chiang and C.-K. Li, *Generalized doubly stochastic matrices and linear preservers*, Linear Multilinear Algebra **53** (2005), 1–11.
3. A. M. Hasani and M. Radjabalipour, *The structure of linear operators strongly preserving majorizations of matrices*, Electron. J. Linear Algebra **15** (2006), 260–268.
4. A. M. Hasani and M. Radjabalipour, *On linear preservers of (right) matrix majorization*, Linear Algebra Appl **423** (2007), 255–261.
5. A. Ilkhanizadeh Manesh, *Right gut-Majorization on $\boldsymbol{M}_{n,m}$*, Electron. J. Linear Algebra **31** (2016), no. 1, 13–26.
6. F. Khalooei, *Linear preservers of two-sided matrix majorization*, Wavelet Linear Algebra **1** (2014), no. 1, 43–50.
7. A. W. Marshall, I. Olkin, and B. C. Arnold, *Inequalities: theory of majorization and its applications*, Second edition. Springer Series in Statistics, Springer, New York, 2011.

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