MOMENT PROPERTY OF EXPONENTIAL MONOMIALS ON STURM–LIOUVILLE HYPERGROUPS

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ABSTRACT. In this paper we will characterize the translates of exponential monomials on an arbitrary Sturm–Liouville hypergroup.

1. INTRODUCTION

In the theory of hypergroups Sturm–Liouville hypergroups play an important role. This important case of hypergroups is based on the classical Sturm–Liouville boundary value problem on nonnegative reals. For the definition of Sturm–Liouville hypergroups see [1, 2, 3]. If we consider the Sturm–Liouville differential equation as an eigenvalue problem of a second order differential operator and we fix the eigenvalue then the eigenfunction of this operator has a multiplicative property with respect to convolution defined by the hypergroup, this fact can be proved. Hence we say that the eigenfunctions (or in other words the solution of the differential equation for different eigenvalues) are exponential functions over the actual hypergroup. Specifically, exponential functions satisfy Cauchy's functional equation; briefly, let $K = (\mathbb{R}_0, A)$ be a Sturm–Liouville hypergroup, where A is a positive and continuously differentiable function on positive reals, then the Sturm–Liouville equation is

$$\partial_1^2 \Phi(x, \lambda) + \frac{A'(x)}{A(x)} \partial_1 \Phi(x, \lambda) = \lambda \, \Phi(x, \lambda) \qquad \Phi(0, \lambda) = 1, \ \partial_1 \Phi(0, \lambda) = 0,$$

furthermore, for any y in K let τ_y denote a translation operator on the space of all complex valued functions on K which are integrable with respect to $\delta_x * \delta_y$ for any x, y in K. Here δ_x denotes the Dirac-measure. With this in hand Cauchy's equation can be written in the form $\tau_y \, \Phi(x, \lambda) = \Phi(x * y, \lambda) = \Phi(x, \lambda)\Phi(y, \lambda)$.

Actually, we can say that the function Φ denotes an exponential family. Using the exponential family we define *exponential monomials* on K as functions of the form $x \mapsto P(\partial_2)\varphi(x, \lambda)$, where P is a complex polynomial, λ is a complex number and ∂_2 denotes the partial differential operator with respect to the second variable. In this sense we can formulate differential equations of exponential monomials. The class of special exponential monomials has a linear independence property, this result can be found in [4]. Let N be a nonnegative integer. If we derivate the Sturm–Liouville equation N-times with respect to its "eigenvalue" we get the equation of an exponential monomial. For instance, the equation of the Nth exponential monomial is

$$\partial_1^2 \partial_2^N \Phi(x, \lambda) + \frac{A'(x)}{A(x)} \partial_1 \partial_2^N \Phi(x, \lambda) = N \partial_2^{N-1} \Phi(x, \lambda) + \lambda\, \partial_2^N \Phi(x, \lambda).$$

With the translation operator moment functions can be introduced. For any nonnegative integer N the complex valued continuous function φ on K is called a *moment function of order N*, if there are complex valued continuous functions φ_k on K for $k = 0, 1, \ldots, N$ such that $\varphi_0 = 1$, $\varphi_N = \varphi$, and

$$\tau_y \varphi(x) = \varphi_k(x * y) = \sum_{i=0}^{k} \binom{k}{i} \varphi_i(x)\varphi_{k-i}(y)$$

holds for $k = 0, 1, \ldots, N$ and for all x, y in K. In this case we say that the functions φ_k $(k = 0, 1, \ldots, N)$ form *a moment sequence of order N*. In the main result of this paper we will show that the translates of exponential monomials seem to have the characterization property of moment functions. This means that there is another close connection between exponential monomials and moment functions. The idea of the proof is based on the exchangeability of translation and derivation. We give the description in the following section.

2. Translates of exponential monomials

Let us turn to describe what we get if we consider the equation of the Nth monomial and substitute $x * y$ for x. We easily get on one hand the following equation

$$\partial_1^2 \partial_2^N \Phi(x*y, \lambda) + \frac{A'(x)}{A(x)} \partial_1 \partial_2^N \Phi(x*y, \lambda) = N \partial_2^{N-1} \Phi(x*y, \lambda) + \lambda\, \partial_2^N \Phi(x*y, \lambda). \quad (2.1)$$

On the other hand, if we focus on the multiplicative property of the exponential function, then we have that equation

$$\partial_1^2 \Phi(x * y, \lambda) + \frac{A'(x)}{A(x)} \partial_1 \Phi(x * y, \lambda) = \lambda\, \Phi(x * y, \lambda) = \lambda \Phi(x, \lambda)\Phi(y, \lambda) \quad (2.2)$$

holds. That is, first we execute the convolution then we start to differentiate equation (2.2). It is easy to see that if we derivate equation (2.2) N-times with respect to λ, we have

$$\partial_1^2 \partial_2^N \Phi(x * y, \lambda) + \frac{A'(x)}{A(x)} \partial_1 \partial_2^N \Phi(x * y, \lambda) = \partial_2^N \big(\lambda \Phi(x, \lambda)\Phi(y, \lambda)\big), \quad (2.3)$$

where ∂_2^N clearly stands for $\frac{d^N}{d\lambda^N}$. We introduce the function $\nu : \mathbb{R}_0 \times \mathbb{C} \to \mathbb{C}$ of the form $\nu(x, \lambda) = 1 \cdot \lambda = \lambda$ so in the equation (2.3) the second order partial derivate can be written in the form

$$\partial_2^N \big(\lambda \Phi(x, \lambda) \Phi(y, \lambda) \big) = \partial_2^N \big[\nu(x, \lambda) \ \Phi(x * y, \lambda) \big]. \tag{2.4}$$

From equation (2.1), (2.3) and (2.4) we get

$$N \partial_2^{N-1} \Phi(x * y, \lambda) + \lambda \ \partial_2^N \Phi(x * y, \lambda) = \partial_2^N \big[\nu(x, \lambda) \ \Phi(x * y, \lambda) \big].$$

This makes the connection between exponential monomials and their translates clear. The following theorem presents the exact form of the Nth derivatives of the function $(x, \lambda) \mapsto \nu(x, \lambda) \ \Phi(x * y, \lambda)$ with respect to λ.

Theorem 2.1. *Let* $K = (\mathbb{R}_0, A)$ *be a Sturm–Liouville hypergroup,* λ *a fixed complex number,* N *a nonnegative integer,* y *an arbitrary nonnegtive real number,* $\Phi : \mathbb{R}_0 \times \mathbb{C} \to \mathbb{C}$ *an exponential function and* $\nu : \mathbb{R}_0 \times \mathbb{C} \to \mathbb{C}$ *a function with the the property* $\nu(x, \lambda) = 1 \cdot \lambda = \lambda$. *Then the second order partial derivatives of the function* $\nu(x, \lambda) \ \Phi(x * y, \lambda)$ *can be written in the form*

$$\partial_2^N \big[\nu(x, \lambda) \ \Phi(x * y, \lambda) \big] = \lambda \partial_2^N \Phi(x, \lambda) \Phi(y, \lambda) +$$

$$+ \sum_{i=0}^{N-1} \partial_2^i \Phi(x, \lambda) \Big(\lambda \binom{N}{i} \partial_2^{N-i} \Phi(y, \lambda) + N \binom{N-1}{i} \partial_2^{N-i-1} \Phi(y, \lambda) \Big).$$

Proof. In the proof we use induction by N, first we analyze the case $N = 1$. Here we get

$$\frac{d}{d\lambda} (\lambda \ \Phi(x * y, \lambda)) = \frac{d}{d\lambda} (\lambda \ \Phi(x, \lambda) \Phi(y, \lambda)) =$$

$$= \Phi(x, \lambda) \Phi(y, \lambda) + \lambda \ \partial_2 \Phi(x, \lambda) \Phi(y, \lambda) + \lambda \ \Phi(x, \lambda) \partial_2 \Phi(y, \lambda) =$$

$$= \lambda \ \partial_2 \Phi(x, \lambda) \Phi(y, \lambda) + \Phi(x, \lambda) \big(\lambda \ \partial_2 \Phi(y, \lambda) + \Phi(y, \lambda) \big),$$

which shows the theorem holds for $N = 1$. Let us suppose that the formula is true for N, and we will show that it is valid for $N + 1$. In the first step of this case we derivate the equation with respect to λ. We get the following:

$$\partial_2^{N+1} \big[\nu(x, \lambda) \ \Phi(x * y, \lambda) \big] =$$

$$= \partial_2^N \Phi(x, \lambda) \Phi(y, \lambda) + \lambda \ \partial_2^{N+1} \Phi(x, \lambda) \Phi(y, \lambda) + \lambda \ \partial_2^N \Phi(x, \lambda) \partial_2 \Phi(y, \lambda) +$$

$$+ \sum_{i=0}^{N-1} \partial_2^{i+1} \Phi(x, \lambda) \Big(\lambda \binom{N}{i} \partial_2^{N-i} \Phi(y, \lambda) + N \binom{N-1}{i} \partial_2^{N-i-1} \Phi(y, \lambda) \Big) +$$

$$+ \sum_{i=0}^{N-1} \partial_2^i \Phi(x, \lambda) \Big(\binom{N}{i} \partial_2^{N-i} \Phi(y, \lambda) + \lambda \binom{N}{i} \partial_2^{N-i+1} \Phi(y, \lambda) \Big) +$$

$$+ \sum_{i=0}^{N-1} \partial_2^i \Phi(x, \lambda) \Big(N \binom{N-1}{i} \partial_2^{N-i} \Phi(y, \lambda) \Big).$$

In the next step we will focus on the index range of the sums

$$\partial_2^{N+1} \big[\nu(x, \lambda) \ \Phi(x * y, \lambda) \big] =$$

$$= \partial_2^N \Phi(x, \lambda) \Phi(y, \lambda) + \lambda \ \partial_2^{N+1} \Phi(x, \lambda) \Phi(y, \lambda) + \lambda \ \partial_2^N \Phi(x, \lambda) \partial_2 \Phi(y, \lambda) +$$

$$+\partial_2^N\Phi(x,\lambda)\big(\lambda\ N\partial_2\Phi(y,\lambda)+N\Phi(y,\lambda)\big)+$$

$$+\sum_{i=0}^{N-2}\partial_2^{i+1}\Phi(x,\lambda)\big(\lambda\binom{N}{i}\partial_2^{N-i}\Phi(y,\lambda)+N\binom{N-1}{i}\partial_2^{N-i-1}\Phi(y,\lambda)\big)+$$

$$+\sum_{i=0}^{N-1}\partial_2^i\Phi(x,\lambda)\binom{N}{i}\partial_2^{N-i}\Phi(y,\lambda)+\Phi(x,\lambda)\big(\lambda\partial_2^{N+1}\Phi(y,\lambda)+N\partial_2^N\Phi(y,\lambda)\big)+$$

$$+\sum_{i=1}^{N-1}\partial_2^i\Phi(x,\lambda)\big(\lambda\binom{N}{i}\partial_2^{N-i+1}\Phi(y,\lambda)+N\binom{N-1}{i}\partial_2^{N-i}\Phi(y,\lambda)\big).$$

To change the index range of the first sum we introduce a substitution

$$\partial_2^{N+1}\big[\nu(x,\lambda)\ \Phi(x*y,\lambda)\big]=$$

$$=\partial_2^N\Phi(x,\lambda)\Phi(y,\lambda)+\lambda\ \partial_2^{N+1}\Phi(x,\lambda)\Phi(y,\lambda)+\lambda\ \partial_2^N\Phi(x,\lambda)\partial_2\Phi(y,\lambda)+$$
$$+\partial_2^N\Phi(x,\lambda)\big(\lambda\ N\partial_2\Phi(y,\lambda)+N\Phi(y,\lambda)\big)+$$

$$+\sum_{i=1}^{N-1}\partial_2^i\Phi(x,\lambda)\big(\lambda\binom{N}{i-1}\partial_2^{N-i+1}\Phi(y,\lambda)+N\binom{N-1}{i-1}\partial_2^{N-i}\Phi(y,\lambda)\big)+$$

$$+\sum_{i=0}^{N-1}\partial_2^i\Phi(x,\lambda)\binom{N}{i}\partial_2^{N-i}\Phi(y,\lambda)+\Phi(x,\lambda)\big(\lambda\partial_2^{N+1}\Phi(y,\lambda)+N\partial_2^N\Phi(y,\lambda)\big)+$$

$$+\sum_{i=1}^{N-1}\partial_2^i\Phi(x,\lambda)\big(\lambda\binom{N}{i}\partial_2^{N-i+1}\Phi(y,\lambda)+N\binom{N-1}{i}\partial_2^{N-i}\Phi(y,\lambda)\big).$$

We simplify the expression

$$\partial_2^{N+1}\big[\nu(x,\lambda)\ \Phi(x*y,\lambda)\big]=$$

$$=\partial_2^N\Phi(x,\lambda)\Phi(y,\lambda)+\lambda\ \partial_2^{N+1}\Phi(x,\lambda)\Phi(y,\lambda)+\lambda\ \partial_2^N\Phi(x,\lambda)\partial_2\Phi(y,\lambda)+$$

$$+\sum_{i=0}^{N-1}\binom{N}{i}\partial_2^i\Phi(x,\lambda)\partial_2^{N-i}\Phi(y,\lambda)+$$

$$+\binom{N}{N}\partial_2^N\Phi(x,\lambda)\big(\lambda\ N\partial_2\Phi(y,\lambda)+N\Phi(y,\lambda)\big)+$$

$$+\sum_{i=1}^{N-1}\partial_2^i\Phi(x,\lambda)\big(\lambda\big[\binom{N}{i-1}+\binom{N}{i}\big]\partial_2^{N-i+1}\Phi(y,\lambda)+$$

$$\sum_{i=1}^{N-1}\partial_2^i\Phi(x,\lambda)\big(N\big[\binom{N-1}{i-1}+\binom{N-1}{i}\big]\partial_2^{N-i}\Phi(y,\lambda)\big)+$$

$$+\binom{N}{0}\Phi(x,\lambda)\big(\partial_2^{N+1}\Phi(y,\lambda)+N\partial_2^N\Phi(y,\lambda)\big),$$

and using the identity for binomial coefficients, we have

$$\partial_2^{N+1}\big[\nu(x,\lambda)\ \Phi(x*y,\lambda)\big]=$$

$$=\partial_2^N\Phi(x,\lambda)\Phi(y,\lambda)+\lambda\ \partial_2^{N+1}\Phi(x,\lambda)\Phi(y,\lambda)+\lambda\ \partial_2^N\Phi(x,\lambda)\partial_2\Phi(y,\lambda)+$$

$$+ \sum_{i=0}^{N-1} \binom{N}{i} \partial_2^i \Phi(x,\lambda) \partial_2^{N-i} \Phi(y,\lambda) +$$

$$+ \binom{N}{N} \partial_2^N \Phi(x,\lambda) \big(\lambda \ N \partial_2 \Phi(y,\lambda) + N \Phi(y,\lambda) \big) +$$

$$+ \sum_{i=1}^{N-1} \partial_2^i \Phi(x,\lambda) \Big(\lambda \binom{N+1}{i} \partial_2^{N-i+1} \Phi(y,\lambda) +$$

$$\sum_{i=1}^{N-1} \partial_2^i \Phi(x,\lambda) \Big(N \binom{N}{i} \partial_2^{N-i} \Phi(y,\lambda) \Big) +$$

$$+ \binom{N}{0} \Phi(x,\lambda) \big(\partial_2^{N+1} \Phi(y,\lambda) + N \partial_2^N \Phi(y,\lambda) \big).$$

All this can be written as

$$\partial_2^{N+1} \big[\nu(x,\lambda) \ \Phi(x*y,\lambda) \big] =$$

$$= \lambda \ \partial_2^{N+1} \Phi(x,\lambda) \Phi(y,\lambda) +$$

$$+ \sum_{i=0}^{N} \binom{N}{i} \partial_2^i \Phi(x,\lambda) \partial_2^{N-i} \Phi(y,\lambda) +$$

$$+ N \sum_{i=0}^{N} \binom{N}{i} \partial_2^i \Phi(x,\lambda) \partial_2^{N-i} \Phi(y,\lambda) +$$

$$+ \sum_{i=0}^{N} \partial_2^i \Phi(x,\lambda) \Big(\lambda \binom{N+1}{i} \partial_2^{N-i+1} \Phi(y,\lambda).$$

It is easy to see that the previous and the following equations are equal to

$$\partial_2^{N+1} \big[\nu(x,\lambda) \ \Phi(x*y,\lambda) \big] =$$

$$= \lambda \ \partial_2^{N+1} \Phi(x,\lambda) \Phi(y,\lambda) +$$

$$+ (N+1) \sum_{i=0}^{N} \binom{N}{i} \partial_2^i \Phi(x,\lambda) \partial_2^{N-i} \Phi(y,\lambda) +$$

$$+ \sum_{i=0}^{N} \partial_2^i \Phi(x,\lambda) \Big(\lambda \binom{N+1}{i} \partial_2^{N-i+1} \Phi(y,\lambda).$$

This final step completes the proof of the expected formula

$$\partial_2^{N+1} \big[\nu(x,\lambda) \ \Phi(x*y,\lambda) \big] = \lambda \ \partial_2^{N+1} \Phi(x,\lambda) \Phi(y,\lambda) +$$

$$+ \sum_{i=0}^{N} \partial_2^i \Phi(x,\lambda) \Big(\lambda \binom{N+1}{i} \partial_2^{N-i+1} \Phi(y,\lambda) + (N+1) \binom{N}{i} \partial_2^{N-i} \Phi(y,\lambda) \Big),$$

which means that the statement holds and the theorem is true. □

Remark 2.2. The proof above shows a strong analogy with that of the classical binomial theorem.

3. Main result

The following theorem is the main result of this paper and we will show that the translates of exponential monomials can be written in a compact form. The previous theorem will be utilized to prove the statement.

Theorem 3.1. *Let $K = (\mathbb{R}_0, A)$ be a Sturm–Liouville hypergroup, λ a complex number, N a nonnegative integer and the function $x \mapsto \partial_2^N \Phi(x, \lambda)$ is a special exponential monomial. Then*

$$\tau_y \, \partial_2^N \Phi(x, \lambda) = \sum_{i=0}^{N} \binom{N}{i} \partial_2^i \Phi(x, \lambda) \partial_2^{N-i} \Phi(y, \lambda)$$

for any y in $\mathbb{R}_0$.

Proof. The proof is based on induction by N. In the first step, if $N = 1$ on one hand we get that

$$\partial_1^2 \partial_2 \Phi(x * y, \lambda) + p(x)\partial_1 \partial_2 \Phi(x * y, \lambda) = \Phi(x * y, \lambda) + \lambda \partial_2 \Phi(x * y, \lambda)$$

and on the other hand the following is valid

$$\partial_1^2 \partial_2 \Phi(x * y, \lambda) + p(x)\partial_1 \partial_2 \Phi(x * y, \lambda) =$$
$$= \Phi(x, \lambda)\Phi(y, \lambda) + \lambda \partial_2 \Phi(x, \lambda)\Phi(y, \lambda) + \lambda \Phi(x, \lambda)\partial_2 \Phi(y, \lambda).$$

Since the left hand side of these equations are equal we have

$$\Phi(x * y, \lambda) + \lambda \partial_2 \Phi(x * y, \lambda) =$$
$$\Phi(x, \lambda)\Phi(y, \lambda) + \lambda \partial_2 \Phi(x, \lambda)\Phi(y, \lambda) + \lambda \Phi(x, \lambda)\partial_2 \Phi(y, \lambda)$$

and if we use the multiplicative property of exponential functions and simplify the equation we get

$$\tau_y \partial_2 \Phi(x, \lambda) = \partial_2 \Phi(x, \lambda)\Phi(y, \lambda) + \Phi(x, \lambda)\partial_2 \Phi(y, \lambda).$$

It means that the statement holds for the case of $N = 1$. Now we can suppose that

$$\tau_y \, \partial_2^{N-1} \Phi(x, \lambda) = \sum_{i=0}^{N-1} \binom{N-1}{i} \partial_2^i \Phi(x, \lambda) \partial_2^{N-i-1} \Phi(y, \lambda)$$

holds. In the final step of proof we use the characterization equation of exponential monomials and the previous statement. Let us consider the characterization equation which is the following

$$\partial_1^2 \partial_2^N \Phi(x * y, \lambda) + p(x)\partial_1 \partial_2^N \Phi(x * y, \lambda) = N \partial_2^{N-1} \Phi(x * y, \lambda) + \lambda \partial_2^N \Phi(x * y, \lambda)$$

and according to the previous theorem

$$\partial_1^2 \partial_2^N \Phi(x * y, \lambda) + p(x)\partial_1 \partial_2^N \Phi(x * y, \lambda) = \lambda \partial_2^N \Phi(x, \lambda)\Phi(y, \lambda)+$$
$$+ \sum_{i=0}^{N-1} \partial_2^i \Phi(x, \lambda) \Big(\lambda \binom{N}{i} \partial_2^{N-i} \Phi(y, \lambda) + N \binom{N-1}{i} \partial_2^{N-i-1} \Phi(y, \lambda) \Big).$$

This is similar than the previous, specifically from the equality of left hand sides of the previous two equations we get

$$N \partial_2^{N-1} \Phi(x * y, \lambda) + \lambda \partial_2^N \Phi(x * y, \lambda) =$$

$$= \lambda \partial_2^N \Phi(x,\lambda) \Phi(y,\lambda) + \sum_{i=0}^{N-1} \lambda \binom{N}{i} \partial_2^i \Phi(x,\lambda) \partial_2^{N-i} \Phi(y,\lambda) +$$

$$+ \sum_{i=0}^{N-1} N \binom{N-1}{i} \partial_2^i \Phi(x,\lambda) \partial_2^{N-i-1} \Phi(y,\lambda) =$$

$$= \lambda \sum_{i=0}^{N} \binom{N}{i} \partial_2^i \Phi(x,\lambda) \partial_2^{N-i} \Phi(y,\lambda) +$$

$$+ N \sum_{i=0}^{N-1} \binom{N-1}{i} \partial_2^i \Phi(x,\lambda) \partial_2^{N-i-1} \Phi(y,\lambda).$$

At this point using the assumption and simplifying the equation we can see that

$$\tau_y \, \partial_2^N \Phi(x,\lambda) = \sum_{i=0}^{N} \binom{N}{i} \partial_2^i \Phi(x,\lambda) \partial_2^{N-i} \Phi(y,\lambda)$$

holds, which means that our statement follows. $\qquad\square$

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