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COUPLED FIXED POINT, F-INVARIANT SET AND FIXED POINT OF N-ORDER

BESSEM SAMET[1]* AND CALOGERO VETRO[2]

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ABSTRACT. In this paper, we establish some new coupled fixed point theorems in complete metric spaces, using a new concept of F-invariant set. We introduce the notion of fixed point of N-order as natural extension of that of coupled fixed point. As applications, we discuss and adapt the presented results to the setting of partially ordered cone metric spaces. The presented results extend and complement some known existence results from the literature.

1. INTRODUCTION

In the last decades, a wide discussion on coupled fixed point theorems aimed the interest of many scientists because of their important role in the study of nonlinear differential equations, nonlinear integral equations and differential inclusions. We recall the following definition.

Definition 1.1. (see Bhaskar and Lakshmikantham [4].) Let X be a non-empty set and $F : X \times X \to X$ be a given mapping. An element $(x, y) \in X \times X$ is said to be a coupled fixed point of the mapping F if $F(x, y) = x$ and $F(y, x) = y$.

Now, going back in the literature on functional analysis (see [7] for some remarks), we encountered some interesting manuscripts on coupled fixed point theorems (see [4, 6, 8, 9, 11, 14, 19, 20] and the references cited therein). These works testify that the interest for establishing new coupled fixed point theorems is remained vivid until today. Motivated by these arguments, we would like to give a contribution to this research inviting also the reader to reflect on the prospect

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* Corresponding author.

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of some recent evolutions of fixed point theory. To this end, we introduce a new concept of F-invariant set, that is necessary to state and prove our main theorem. The presented results essentially extend and complement some known existence results from the literature, but at the same time consent us to focus the attention on two questions concerning uniqueness and equality between components of the coupled fixed point. We introduce also a new notion of fixed point of N-order as natural extension of the notion of coupled fixed point. Consequently, we present some results for fixed point of N-order.

Recently, there is also a trend to weaken the requirement on contractive conditions by considering metric spaces and cone metric spaces endowed with partial order [1, 2, 3, 5, 13, 15, 16, 17, 18]. In this paper, as applications, we derive results involving partially ordered metric spaces and cone metric spaces. The last argument is presented considering a recent paper of Du [10], destined to modify the interest for fixed point theorems in cone metric spaces.

2. Main results

In this section, we first introduce the concept of F-invariant.

Definition 2.1. Let (X, d) be a metric space and $F : X \times X \to X$ be a given mapping. Let M be a non-empty subset of X^4. We say that M is F-invariant subset of X^4 if and only if for all $x, y, z, w \in X$, we have

(a) $(x, y, z, w) \in M \Leftrightarrow (w, z, y, x) \in M$;
(b) $(x, y, z, w) \in M \Rightarrow (F(x, y), F(y, x), F(z, w), F(w, z)) \in M$.

Clearly, $M = X^4$ is trivially F-invariant.
The following example plays a key role in the proof of our results involving a partially ordered set.

Example 2.2. Let (X, d) be a metric space endowed with a partial order $\leq$. Let $F : X \times X \to X$ be a mapping satisfying the mixed monotone property, that is, for all $x, y \in X$, we have

$$\begin{aligned} x_1, x_2 \in X,\ x_1 \leq x_2 \quad &\Rightarrow \quad F(x_1, y) \leq F(x_2, y); \\ y_1, y_2 \in X,\ y_1 \leq y_2 \quad &\Rightarrow \quad F(x, y_2) \leq F(x, y_1). \end{aligned}$$

Define the subset $M \subseteq X^4$ by

$$M = \{(a, b, c, d) \in X^4 \,|\, c \leq a,\ b \leq d\}.$$

Then, M is F-invariant subset of X^4.

The following new theorem is our main result and concerns the existence of a coupled fixed point.

Theorem 2.3. *Let (X, d) be a complete metric space, $F : X \times X \to X$ be a continuous mapping and M be a non-empty subset of X^4. We assume that*
(i) *M is F-invariant;*
(ii) *there exists $(x_0, y_0) \in X^2$ such that $(F(x_0, y_0), F(y_0, x_0), x_0, y_0) \in M$;*

(iii) *for all* $(x,y,u,v)\in M$, *we have*

$$d(F(x,y),F(u,v))\le \frac{\alpha}{2}[d(x,F(x,y))+d(y,F(y,x))]$$
$$+\frac{\beta}{2}[d(u,F(u,v))+d(v,F(v,u))]+\frac{\theta}{2}[d(x,F(u,v))+d(y,F(v,u))]$$
$$+\frac{\gamma}{2}[d(u,F(x,y))+d(v,F(y,x))]+\frac{\delta}{2}[d(x,u)+d(y,v)],$$

where $\alpha,\beta,\theta,\gamma,\delta$ *are nonnegative constants such that* $\alpha+\beta+\theta+\gamma+\delta<1$. *Then* F *has a coupled fixed point, i.e., there exists* $(x^*,y^*)\in X\times X$ *such that* $F(x^*,y^*)=x^*$ *and* $F(y^*,x^*)=y^*$.

Proof. Denote

$$x_1=F(x_0,y_0),\ y_1=F(y_0,x_0),\ x_2=F(x_1,y_1),\ y_2=F(y_1,x_1).$$

Since $(x_1,y_1,x_0,y_0)\in M$, using the contractive condition in (iii), we get

$$\begin{aligned} d(x_2,x_1) &\le \frac{\alpha}{2}[d(x_1,x_2)+d(y_1,y_2)]+\frac{\beta}{2}[d(x_0,x_1)+d(y_0,y_1)] \\ &\quad +\frac{\gamma}{2}[d(x_0,x_2)+d(y_0,y_2)]+\frac{\delta}{2}[d(x_1,x_0)+d(y_1,y_0)]. \end{aligned}$$

Now, by the triangular inequality, we obtain

$$d(x_2,x_1)\le \frac{\alpha+\gamma}{2}[d(x_1,x_2)+d(y_1,y_2)]+\frac{\beta+\gamma+\delta}{2}[d(x_0,x_1)+d(y_0,y_1)]. \quad (2.1)$$

Since $(x_1,y_1,x_0,y_0)\in M$, then $(y_0,x_0,y_1,x_1)\in M$. Again, using the contractive condition, we get

$$\begin{aligned} d(y_1,y_2) &\le \frac{\alpha}{2}[d(y_0,y_1)+d(x_0,x_1)]+\frac{\beta}{2}[d(y_1,y_2)+d(x_1,x_2)] \\ &\quad +\frac{\theta}{2}[d(x_0,x_2)+d(y_0,y_2)]+\frac{\delta}{2}[d(x_1,x_0)+d(y_1,y_0)]. \end{aligned}$$

By the triangular inequality, we obtain

$$d(y_1,y_2)\le \frac{\beta+\theta}{2}[d(x_1,x_2)+d(y_1,y_2)]+\frac{\alpha+\theta+\delta}{2}[d(x_0,x_1)+d(y_0,y_1)]. \quad (2.2)$$

Now, combining (2.1) and (2.2), we get

$$d(x_1,x_2)+d(y_1,y_2)\le r[d(x_0,x_1)+d(y_0,y_1)], \quad (2.3)$$

where

$$r=\frac{\beta+\gamma+\alpha+\theta+2\delta}{2-(\alpha+\gamma+\beta+\theta)}<1. \quad (2.4)$$

Denote

$$x_3=F(x_2,y_2),\ y_3=F(y_2,x_2).$$

Since M is F-invariant and $(x_1,y_1,x_0,y_0)\in M$, then $(x_2,y_2,x_1,y_1)\in M$. Using a similar argument to the one above, we get

$$d(x_2,x_3)+d(y_2,y_3)\le r[d(x_1,x_2)+d(y_1,y_2)].$$

Then, from (2.3), it follows that

$$d(x_2, x_3) + d(y_2, y_3) \leq r^2[d(x_0, x_1) + d(y_0, y_1)].$$

Hence, we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that for all $n \in \mathbb{N}$, we have

$$(x_{n+1}, y_{n+1}, x_n, y_n) \in M$$

and

$$d(x_n, x_{n+1}) + d(y_n, y_{n+1}) \leq r^n[d(x_0, x_1) + d(y_0, y_1)]. \tag{2.5}$$

Now, if $d(x_0, x_1) + d(y_0, y_1) = 0$, then we get immediately that

$$x_0 = x_1 = F(x_0, y_0), \;\; y_0 = y_1 = F(y_0, x_0)$$

and (x_0, y_0) is a coupled fixed point of F. Then, we assume that

$$d(x_0, x_1) + d(y_0, y_1) > 0.$$

In this case, from (2.5) and since $r < 1$, $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in the complete metric space (X, d). Hence, there exists $(x^*, y^*) \in X \times X$ such that

$$\lim_{n\to+\infty} x_n = x^* \text{ and } \lim_{n\to+\infty} y_n = y^*. \tag{2.6}$$

Now, we will prove that (x^*, y^*) is a coupled fixed point of F. First note that the continuity of F implies that

$$\lim_{n\to+\infty} F(x_n, y_n) = F(x^*, y^*), \;\; \lim_{n\to+\infty} F(y_n, x_n) = F(y^*, x^*). \tag{2.7}$$

By the triangular inequality, we have

$$d(F(x^*, y^*), x^*) \leq d(F(x^*, y^*), F(x_n, y_n)) + d(x_{n+1}, x^*).$$

Taking $n \to +\infty$ and using again (2.6) and (2.7), we get

$$d(F(x^*, y^*), x^*) \leq 0,$$

i.e., $F(x^*, y^*) = x^*$. Similarly,

$$d(F(y^*, x^*), y^*) \leq d(F(y^*, x^*), F(y_n, x_n)) + d(y_{n+1}, y^*).$$

Taking $n \to +\infty$ and using (2.6) and (2.7), we get

$$d(F(y^*, x^*), y^*) \leq 0,$$

i.e., $F(y^*, x^*) = y^*$. Then (x^*, y^*) is a coupled fixed point of F. This makes end to the proof. □

Remark 2.4. From Theorem 2.3 we can derive corollaries involving specific contractive conditions, by a suitable choice of the nonnegative constants $\alpha, \beta, \theta, \gamma$ and δ.

Now, reasoning on Theorem 2.3, some questions arise naturally. To be precise, one can ask himself

(Q1) is possible to guarantee the uniqueness of the coupled fixed point of F?
(Q2) is possible to have the equality between the components of the couple?

Motivated by the interest in this research, we give positive answers to these questions adding to Theorem 2.3 some hypotheses. We proceed with order. Then, to have the uniqueness, we state and prove the following theorem.

Theorem 2.5. *Adding to the hypotheses of Theorem 2.3 the following condition:*

$$(H)\quad \forall\,(x,y),(x',y')\in X^2,\ \exists\,(u,v)\in X^2\,|\,(x,y,u,v)\in M\ \textit{and}\ (x',y',u,v)\in M,$$

we obtain the uniqueness of the coupled fixed point of F.

Proof. Assume that (a,b) is another coupled fixed point of F, i.e., $a=F(a,b)$ and $b=F(b,a)$. By (H), there exists $(u,v)\in X^2$ such that

$$(x^*,y^*,u,v)\in M,\ \ (a,b,u,v)\in M. \tag{2.8}$$

Now, for $(x,y)\in X^2$, we denote

$$F^{n+1}(x,y)=F(F^n(x,y),F^n(y,x))\ \text{ for all } n\in\mathbb{N},$$

where $F^0:X\times X\to X$ is given by $F^0(x,y)=x$ for all $x,y\in X$ and $F^1=F$. Since M is F-invariant, from (2.8) and for all $n\in\mathbb{N}$, we get

$$(x^*,y^*,F^n(u,v),F^n(v,u))\in M,\ \ (a,b,F^n(u,v),F^n(v,u))\in M.$$

Now, by (iii) of Theorem 2.3, we have

$$\begin{aligned}
&d(F(x^*,y^*),F(F^n(u,v),F^n(v,u)))\le\frac{\alpha}{2}[d(x^*,F(x^*,y^*))+d(y^*,F(y^*,x^*))]\\
&+\frac{\beta}{2}[d(F^n(u,v),F(F^n(u,v),F^n(v,u)))+d(F^n(v,u),F(F^n(v,u),F^n(u,v)))]\\
&+\frac{\theta}{2}[d(x^*,F(F^n(u,v),F^n(v,u)))+d(y^*,F(F^n(v,u),F^n(u,v)))]\\
&+\frac{\gamma}{2}[d(F^n(u,v),F(x^*,y^*))+d(F^n(v,u),F(y^*,x^*))]\\
&+\frac{\delta}{2}[d(x^*,F^n(u,v))+d(y^*,F^n(v,u))],
\end{aligned}$$

that is

$$\begin{aligned}
&d(x^*,F^{n+1}(u,v))\le\frac{\alpha}{2}[d(x^*,x^*)+d(y^*,y^*)]+\frac{\beta}{2}[d(F^n(u,v),x^*)+d(x^*,F^{n+1}(u,v))\\
&+d(F^n(v,u),y^*)+d(y^*,F^{n+1}(v,u))]+\frac{\theta}{2}[d(x^*,F^{n+1}(u,v))+d(y^*,F^{n+1}(v,u))]\\
&+\frac{\gamma}{2}[d(F^n(u,v),x^*)+d(F^n(v,u),y^*)]+\frac{\delta}{2}[d(x^*,F^n(u,v))+d(y^*,F^n(v,u))].
\end{aligned}\tag{2.9}$$

Analogously, as $(F^n(v,u),F^n(u,v),y^*,x^*)\in M$, we obtain

$$\begin{aligned}
&d(F^{n+1}(v,u),y^*)\le\frac{\alpha}{2}[d(F^n(v,u),y^*)+d(y^*,F^{n+1}(v,u))+d(F^n(u,v),x^*)\\
&+d(x^*,F^{n+1}(u,v))]+\frac{\beta}{2}[d(y^*,y^*)+d(x^*,x^*)]+\frac{\theta}{2}[d(F^n(v,u),y^*)+d(F^n(u,v),x^*)]\\
&+\frac{\gamma}{2}[d(y^*,F^{n+1}(v,u))+d(x^*,F^{n+1}(u,v))]+\frac{\delta}{2}[d(F^n(v,u),y^*)+d(F^n(u,v),x^*)].
\end{aligned}\tag{2.10}$$

Now, combining (2.9) and (2.10), we get

$$d(x^*,F^{n+1}(u,v))+d(F^{n+1}(v,u),y^*)\le r[d(F^n(u,v),x^*)+d(F^n(v,u),y^*)],$$

where r is given by (2.4).

Proceeding thus, we obtain

$$d(x^*, F^{n+1}(u,v)) + d(F^{n+1}(v,u), y^*) \leq r^n[d(F(u,v), x^*) + d(F(v,u), y^*)].$$

Now as $r < 1$, taking $n \to +\infty$, we get

$$d(x^*, F^{n+1}(u,v)) + d(F^{n+1}(v,u), y^*) \to 0.$$

Therefore, it follows

$$\lim_{n\to+\infty} F^{n+1}(u,v) = x^* = F(x^*, y^*), \lim_{n\to+\infty} F^{n+1}(v,u) = y^* = F(y^*, x^*).$$

Now, as (a,b) is another coupled fixed point of F, reasoning as above, we obtain

$$\lim_{n\to+\infty} F^{n+1}(u,v) = a = F(a,b) \text{ and } \lim_{n\to+\infty} F^{n+1}(v,u) = b = F(b,a).$$

By the uniqueness of the limit, we conclude that $(x^*, y^*) = (a,b)$ is the unique coupled fixed point of F. □

Now, to have the equality between the components of the coupled fixed point, we state and prove the following theorem.

Theorem 2.6. *Adding to the hypotheses of Theorem 2.3 the following condition:*

$$(E) \quad \forall\,(x,y,x',y') \in X^4,\ \exists\,(u,v) \in X^2 \,|\, (x,y,u,v) \in M \text{ and } (x',y',u,v) \in M,$$

we obtain the equality between the components of the coupled fixed point of F.

Proof. The proof is straightforward, following the same lines of the proof of Theorem 2.5. Then, in order to avoid repetition, the details are omitted. □

Now, to stimulate the interest in this study, we are ready to extend naturally the concepts of coupled fixed point and F-invariant subset with the following definitions.

Definition 2.7. Let X be a non-empty set and $F : X^N \to X$ be a given mapping ($N \geq 2$). An element $(x_1, x_2, \cdots, x_N) \in X^N$ is said to be a fixed point of N-order of the mapping F if

$$\begin{cases} F(x_1, x_2, \cdots, x_N) = x_1, \\ F(x_2, x_3, \cdots, x_N, x_1) = x_2, \\ \vdots \\ F(x_N, x_1, \cdots, x_{N-1}) = x_N. \end{cases}$$

Definition 2.8. Let (X,d) be a metric space and $F : X^N \to X$ be a given mapping. Let M be a non-empty subset of X^{2N}. We say that M is F-invariant subset of X^{2N} if and only if for all $x_1, x_2, \cdots, x_{2N} \in X$, we have

(A)

$$(x_1, x_2, \cdots, x_{2N}) \in M \Leftrightarrow \begin{cases} (x_2, x_3, \cdots, x_{2N}, x_1) \in M, \\ (x_3, x_4, \cdots, x_{2N}, x_1, x_2) \in M, \\ \vdots \\ (x_{2N}, x_1, \cdots, x_{2N-1}) \in M; \end{cases}$$

(B) $(F(x_1, x_2, \cdots, x_N), \ldots, F(x_N, x_1, \cdots, x_{N-1}), F(x_{N+1}, x_{N+2}, \cdots, x_{2N}), \ldots,$
$F(x_{2N}, x_{N+1}, \cdots, x_{2N-1})) \in M$ whenever $(x_1, x_2, \cdots, x_{2N}) \in M$.

Finally, in virtue of the results obtained in the previous section, we can state analogous results for fixed point of N-order in complete metric spaces and in partially ordered sets. Here, we give the equivalent of Theorem 2.3, Theorem 2.5 and Theorem 2.6.

Theorem 2.9. *Let (X,d) be a complete metric space, $F : X^N \to X$ be a continuous mapping and M be a non-empty subset of X^{2N}. We assume that*
(i) *M is F-invariant;*
(ii) *there exists $(x_1, x_2, \cdots, x_N) \in X^N$ such that*

$$(F(x_1, x_2, \cdots, x_N), \cdots, F(x_N, x_1, \cdots, x_{N-1}), x_1, x_2, \cdots, x_N) \in M;$$

(iii) *for all $(x_1, x_2, \cdots, x_N, x_{i1}, x_{i2}, \cdots, x_{iN}) \in M$, we have*

$$\begin{aligned} d(F(x_1, x_2, \cdots, x_N), F(x_{i1}, x_{i2}, \cdots, x_{iN})) &\leq \frac{\alpha}{N}[d(x_1, F(x_1, x_2, \cdots, x_N)) + \cdots \\ + d(x_N, F(x_N, x_1, x_2, \cdots, x_{N-1}))] &+ \frac{\beta}{N}[d(x_{i1}, F(x_{i1}, x_{i2}, \cdots, x_{iN})) + \cdots \\ + d(x_{iN}, F(x_{iN}, x_{i1}, \cdots, x_{i(N-1)}))] &+ \frac{\theta}{N}[d(x_1, F(x_{i1}, x_{i2}, \cdots, x_{iN})) + \cdots \\ + d(x_N, F(x_{iN}, x_{i1}, x_{i2}, \cdots, x_{i(N-1)}))] &+ \frac{\gamma}{N}[d(x_{i1}, F(x_1, x_2, \cdots, x_N)) + \cdots \\ + d(x_{iN}, F(x_N, x_1, x_2, \cdots, x_{N-1}))] &+ \frac{\delta}{N}[d(x_1, x_{i1}) + \cdots + d(x_N, x_{iN})], \end{aligned}$$

where $\alpha, \beta, \theta, \gamma, \delta$ are nonnegative constants such that $\alpha+\beta+\theta+\gamma+\delta < 1$. Then F has a fixed point of N-order.

Theorem 2.10. *Adding to the hypotheses of Theorem 2.9 the following condition:*

$$(HN) \quad \forall\, (x_1, x_2, \cdots, x_N), (x_{i1}, x_{i2}, \cdots, x_{iN}) \in X^N,\ \exists\, (x_{j1}, x_{j2}, \cdots, x_{jN}) \in X^N \mid$$
$$(x_1, \cdots, x_N, x_{j1}, x_{j2}, \cdots, x_{jN}) \in M \text{ and } (x_{i1}, \cdots, x_{iN}, x_{j1}, x_{j2}, \cdots, x_{jN}) \in M,$$

we obtain the uniqueness of the fixed point of N-order of the mapping F.

Theorem 2.11. *Adding to the hypotheses of Theorem 2.9 the following condition:*

$$(EN) \quad \forall\, (x_1, x_2, \cdots, x_N, x_{i1}, x_{i2}, \cdots, x_{iN}) \in X^{2N},\ \exists\, (x_{j1}, x_{j2}, \cdots, x_{jN}) \in X^N \mid$$
$$(x_1, \cdots, x_N, x_{j1}, x_{j2}, \cdots, x_{jN}) \in M \text{ and } (x_{i1}, \cdots, x_{iN}, x_{j1}, x_{j2}, \cdots, x_{jN}) \in M,$$

we obtain the equality between the components of the fixed point of N-order of the mapping F.

Example 2.12. Let $X = \mathbb{R}$ endowed with the usual metric $d(u,v) = |u - v|$ for all $u, v \in X$. Let $N \geq 2$ and $F : X^N \to X$ be the mapping defined by

$$F(x_1, x_2, \cdots, x_N) = \frac{x_1 + x_2 + \cdots + x_N + 5N}{N+2},\ \forall\, x = (x_1, x_2, \cdots, x_N) \in X^N.$$

For all $x = (x_1, x_2, \cdots, x_N), y = (y_1, y_2, \cdots, y_N) \in X^N$, we have

$$d(F(x), F(y)) \leq \frac{\delta}{N}[d(x_1, y_1) + d(x_2, y_2) + \cdots + d(x_N, y_N)],$$

where $\delta = \dfrac{N}{N+2}$. Applying Theorem 2.9 and Theorem 2.10 with $M = X^{2N}$, we obtain that F has a unique fixed point of N-order, that is

$$x^* = (5N/2, 5N/2, \cdots, 5N/2).$$

3. Applications

We start this section giving a simple application of our results. To be precise, we deduce theorems in partially ordered cone metric spaces. At first, we recall some definitions and preliminaries.

3.1. **Preliminaries.** We always assume that (E, P) is a real ordered Banach space, P is a cone in E and the partial order $\leq$ in E is defined by P. We denote by θ_E the zero vector of E. Concerning the definitions and properties of cone metric spaces, the reader can refer to Huang and Zhang [12]. We need also the following concepts and theorem of [10].

Definition 3.1. (see Du [10].) Let X be a nonempty set. A vector-valued function $\rho : X \times X \to E$ is said to be a topological vector cone metric, if the following conditions hold:

(C1) $\theta_E \leq \rho(x, y)$ for all $x, y \in X$ and $\rho(x, y) = \theta_E$ if and only if $x = y$;
(C2) $\rho(x, y) = \rho(y, x)$ for all $x, y \in X$;
(C3) $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$ for all $x, y, z \in X$.

The pair (X, ρ) is then called a topological vector cone metric space.

In the following, the notation $x \ll y$ will stand for $y - x \in \operatorname{int}(P)$, where $\operatorname{int}(P)$ denotes the interior of P.

Definition 3.2. (see Du [10].) Let (X, ρ) be a topological vector cone metric space, $x \in X$ and $\{x_n\}_{n\in\mathbb{N}}$ a sequence in X.
(i) $\{x_n\}$ topological vector cone converges to x whenever for every $c \in E$ with $\theta_E \ll c$ there is a natural number N_0 such that $\rho(x_n, x) \ll c$ for all $n \geq N_0$;
(ii) $\{x_n\}$ is a topological vector cone Cauchy sequence whenever for every $c \in E$ with $\theta_E \ll c$ there is a natural number N_0 such that $\rho(x_n, x_m) \ll c$ for all $n, m \geq N_0$;
(iii) (X, ρ) is topological vector complete cone space if every topological vector cone Cauchy sequence in X is topological vector cone convergent.

Theorem 3.3. (see Du [10].) *Let (X, ρ) be a topological vector cone metric space, $x \in X$ and $\{x_n\}_{n\in\mathbb{N}}$ a sequence in X. Let $d_\rho : X \times X \to [0, +\infty)$ be defined as $d_\rho = \xi_e \circ \rho$, where $\xi_e(y) = \inf\{r \in \mathbb{R} : y \in re - P\}$ for all $y \in E$ and $e \in \operatorname{int}(P)$. Then the following statements hold:*
(i) *if $\{x_n\}$ topological vector cone converges to x, then $d_\rho(x_n, x) \to 0$ as $n \to +\infty$;*
(ii) *if $\{x_n\}$ is a topological vector cone Cauchy sequence in (X, ρ), then $\{x_n\}$ is a Cauchy sequence (in usual sense) in (X, d_ρ);*
(iii) *if (X, ρ) is topological vector complete cone space, then (X, d_ρ) is a complete metric space.*

Remark 3.4. Note that the conditions (i), (ii) and (iii) of Theorem 3.3, should be correctly presented in the form of necessary and sufficient conditions as follows:

(i) $\{x_n\}$ topological vector cone converges to x if and only if $d_\rho(x_n, x) \to 0$ as $n \to +\infty$;

(ii) $\{x_n\}$ is a topological vector cone Cauchy sequence in (X, ρ) if and only if $\{x_n\}$ is a Cauchy sequence (in usual sense) in (X, d_ρ);

(iii) (X, ρ) is topological vector complete cone space if and only if (X, d_ρ) is a complete metric space.

In fact in [10], the proof of the above Theorem 3.3 deals with necessary and sufficient conditions.

3.2. **Coupled fixed point results in a partially ordered cone metric space.** As a consequence of Theorem 2.3, we obtain the following result.

Corollary 3.5. *Let (X, ρ) be a topological vector complete cone metric space endowed with a partial order $\leq$. Let $F : X \times X \to X$ be a continuous mapping having the mixed monotone property on X. We assume that the mapping F satisfies, for all $(x, y, u, v) \in X^4$ with $u \leq x$ and $y \leq v$, the following contractive condition:*

$$\begin{aligned}\rho(F(x,y),F(u,v)) &\leq \frac{\alpha}{2}[\rho(x,F(x,y))+\rho(y,F(y,x))] \\ &+\frac{\beta}{2}[\rho(u,F(u,v))+\rho(v,F(v,u))]+\frac{\theta}{2}[\rho(x,F(u,v))+\rho(y,F(v,u))] \\ &+\frac{\gamma}{2}[\rho(u,F(x,y))+\rho(v,F(y,x))]+\frac{\delta}{2}[\rho(x,u)+\rho(y,v)],\end{aligned} \tag{3.1}$$

where $\alpha, \beta, \theta, \gamma, \delta$ are nonnegative constants such that $\alpha+\beta+\theta+\gamma+\delta < 1$. If there exist $x_0, y_0 \in X$ such that $x_0 \leq F(x_0, y_0)$ and $F(y_0, x_0) \leq y_0$, then F has a coupled fixed point, i.e., there exist $x^*, y^* \in X$ such that $F(x^*, y^*) = x^*$ and $F(y^*, x^*) = y^*$.

Proof. Define $d_\rho := \xi_e \circ \rho$. By Theorem 3.3, it follows that (X, d_ρ) is a complete metric space. Now condition (3.1) implies that

$$\begin{aligned}d_\rho(F(x,y),F(u,v)) &\leq \frac{\alpha}{2}[d_\rho(x,F(x,y))+d_\rho(y,F(y,x))] \\ &+\frac{\beta}{2}[d_\rho(u,F(u,v))+d_\rho(v,F(v,u))]+\frac{\theta}{2}[d_\rho(x,F(u,v))+d_\rho(y,F(v,u))] \\ &+\frac{\gamma}{2}[d_\rho(u,F(x,y))+d_\rho(v,F(y,x))]+\frac{\delta}{2}[d_\rho(x,u)+d_\rho(y,v)],\end{aligned}$$

for all $(x, y, u, v) \in X^4$ with $u \leq x$ and $y \leq v$. Now, defining $M \subseteq X^4$ as

$$M = \{(a,b,c,d) \in X^4 \,|\, c \leq a,\ b \leq d\},$$

we check easily that all hypotheses of Theorem 2.3 are satisfied. Then, we get the desired result. □

Remark 3.6. Corollary 3.5 generalizes Theorem 2.1 of Bhaskar and Lakshmikantham in [4].

As a consequence of Theorem 2.5, we have the following result.

Corollary 3.7. *Adding to the hypotheses of Corollary 3.5 the following condition:*

$(H1)$ $\forall\,(x,y),(x',y')\in X^2, \exists (u,v)\in X^2$ *that is comparable to* (x,y) *and* (x',y'),

we obtain the uniqueness of the coupled fixed point of F*. Here,* X^2 *is endowed with the partial order*

$$(x,y)\le(x',y')\Leftrightarrow x\le x',\, y'\le y.$$

Proof. The proof follows easily by Theorem 2.5 defining again $M\subseteq X^4$ as

$$M=\{(a,b,c,d)\in X^4\,|\,c\le a,\, b\le d\}.$$

□

Remark 3.8. Corollary 3.7 is a generalization of Theorem 2.4 of Bhaskar and Lakshmikantham in [4].

As a consequence of Theorem 2.6, we obtain the following result.

Corollary 3.9. *Adding to the hypotheses of Corollary 3.5 the following condition:*

$(E1)$ *every pair of elements of* X *has an upper bound or a lower bound in* X*.*

Then $x=y$*, i.e.,* $x=F(x,x)$*.*

Proof. The proof follows easily by Theorem 2.6 defining, once again, $M\subseteq X^4$ as

$$M=\{(a,b,c,d)\in X^4\,|\,c\le a,\, b\le d\}.$$

□

Remark 3.10. Corollary 3.9 is a generalization of Theorem 2.5 of Bhaskar and Lakshmikantham in [4].

Remark 3.11. Denoting

$$f(x)=F(x,x),\ \forall\, x\in X,$$

Corollary 3.9 can be considered also as a generalization of results in [2].

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[1] Universié de Tunis, Ecole Supérieure des Sciences et Techniques de Tunis, Département de Mathématiques, 5, avenue Taha Hussein-Tunis, B.P.:56, bab menara-1008, tunisie.

E-mail address: bessem.samet@gmail.com

[2] Department of Mathematics and Informatics, University of Palermo, Via Archirafi 34, 90123 Palermo, Italy.

E-mail address: cvetro@math.unipa.it