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UNIFORM BOUNDEDNESS PRINCIPLES FOR ORDERED TOPOLOGICAL VECTOR SPACES

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ABSTRACT. We obtain uniform boundedness principles for a new class of families of mappings from topological vector spaces to ordered topological vector spaces. The new class of families of mappings includes the family of linear mappings and many other families which consist of nonlinear mappings.

1. Introduction and preliminaries

The uniform boundedness principle is well known as one of the foundation stones of functional analysis. Its basic form [3] asserts that a pointwise boundedness family of continuous linear operators from a Banach space X to a normed space Y is equicontinuous at each $x \in X$, and uniformly bounded on each bounded subset of X. For its importance there has been a lot of work (see books [1, 16]) on uniform boundedness principles since Banach and Steinhaus gave the above version of uniform boundedness principles in 1927. Especially, today we can find various versions [2], [4]-[11], [13]-[15] of uniform boundedness principles in many different mathematical fields.

In this paper, we will give two uniform boundedness principles for a new class of families of mappings from topological vector spaces to ordered topological vector spaces, and point out that the new class of families of mappings includes the family of linear mappings and many other families which consist of nonlinear mappings. Our results have strongly generalized the above version of uniform

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boundedness principle, but are very different from any of the results in relevant literature.

In the whole paper, all topological vector spaces are Hausdorff and over the real scalar field $\mathbb{R}$. For a topological vector space X, denote by $\mathcal{N}(X)$ the family of neighborhoods of $0 \in X$.

Definition 1.1. Let X, Y be topological vector spaces. A family Γ of mappings from X to Y is said to be UB if there exist $U \in \mathcal{N}(X),\ \delta,\ M > 0$ and $\varphi : (0, \delta) \to (0, +\infty)$ with $\lim_{t\to 0} \varphi(t) = 0$ such that for all $f \in \Gamma$:

(1) $f(0) = 0$;
(2) For all $u \in U$ and $x \in X$, there exist $r, s \in [-M, M]$ such that $f(x-u) = rf(x) + sf(u)$;
(3) For all $u \in U$ and $0 < t < \delta$, there exists a real s such that $|s| \leq \varphi(t)$ and $f(tu) = sf(u)$.

Remark 1.2. The family Γ which consists of linear mappings from X to Y is UB with $U = X$, $M = 1$ and $\varphi(t) = t$.

Example 1.3. Let $(X, \|\cdot\|)$ be a normed space. Then the mapping $\|\cdot\| : X \to [0, +\infty)$ belongs to the UB family of mappings from X to $\mathbb{R}$ with $U = X$, $M = 1$ and $\varphi(t) = t$.

In fact, since $\|x\| - \|u\| \leq \|x - u\| \leq \|x\| + \|u\|$ for all $x, u \in X$ we have $\|x - u\| = \|x\| + s\|u\|$ for some $|s| \leq 1$.

Example 1.4. Let $(X, \|\cdot\|)$ be a normed space and Y a vector space. For a linear mapping T from X to Y, define $f_T : X \to Y$ by

$$f_T(x) = \frac{1}{1 + \|x\|}T(x),\ \forall\, x \in X.$$

Then the family $\Gamma = \{f_T : X \to Y | T : X \to Y \text{ is linear}\}$ is UB with $U = \{x \in X : \|x\| < 1\}$, $M = 2$ and $\varphi(t) = 2t$.

For each $f_T \in \Gamma$, $f_T(0) = 0$ is obvious. For $x \in X$ and $u \in U$,

$$\begin{aligned} f_T(x-u) &= \frac{1}{1 + \|x-u\|}T(x-u) = \frac{1}{1 + \|x-u\|}T(x) - \frac{1}{1 + \|x-u\|}T(u) \\ &= \frac{1 + \|x\|}{1 + \|x-u\|}f_T(x) - \frac{1 + \|u\|}{1 + \|x-u\|}f_T(u) \end{aligned}$$

and

$$\frac{1 + \|x\|}{1 + \|x-u\|} < 2 = M, \quad \frac{1 + \|u\|}{1 + \|x-u\|} < 2 = M$$

since $2(1 + \|x-u\|) \geq 2 + \|x-u\| + \|x\| - \|u\| \geq 1 + \|x\| + 1 - \|u\| > 1 + \|x\|$ and $2(1 + \|x-u\|) \geq 2 > 1 + \|u\|$.

For $u \in U$ and $t > 0$,

$$f_T(tu) = \frac{1}{1 + \|tu\|}T(tu) = \frac{1}{1 + \|tu\|}tT(u) = \frac{1 + \|u\|}{1 + \|tu\|}tf_T(u)$$

where

$$\Big|\frac{1+\|u\|}{1+\|tu\|}t\Big| < 2|t| = \varphi(t) \text{ with } t > 0.$$

Remark 1.5. Each $f_T \in \Gamma$ in Example 1.4 is nonlinear.

Remark 1.6. The family of demi-linear mappings [9] from X to Y related to γ and U is UB with U, $\delta = 1$, $M = 1 + |\gamma(-1)|$, $\varphi = |\gamma|\big|_{(0,\delta)} : (0, \delta) \to (0, +\infty)$.

Definition 1.7. [12] Let Y be a topological vector space and an ordered vector space. Y is called an ordered topological vector space if the positive cone $C = \{y \in Y : y \geq 0\}$ is closed in Y.

For topological vector spaces X and Y, a family $\Gamma \subset Y^X$ is said to be pointwise bounded if for each $x \in X$ the set $\{f(x) : f \in \Gamma\}$ is bounded in Y. If for each bounded subset $B \subset X$ the set $\{f(x) : f \in \Gamma, x \in B\}$ is bounded in Y, then Γ is said to be uniformly bounded on each bounded subset of X.

And for Y an ordered topological vector space, a family $\Gamma \subset Y^X$ is said to be pointwise order bounded if for each $x \in X$ the set $\{f(x) : f \in \Gamma\}$ is order bounded in Y. If for each bounded subset $B \subset X$ the set $\{f(x) : f \in \Gamma, x \in B\}$ is order bounded in Y, then Γ is said to be uniformly order bounded on each bounded subset of X.

For more notations and terminologies in topological vector spaces or ordered topological vector spaces, we can refer to [16] or [12].

2. Main results

Theorem 2.1. *Let X be a topological vector space of second category and Y an ordered topological vector space with an order unit e. If the UB family Γ of continuous mappings from X to Y with $U \in \mathcal{N}(X)$, δ, $M > 0$ and $\varphi : (0, \delta) \to (0, +\infty)$ satisfying $\lim_{t\to 0} \varphi(t) = 0$ is pointwise order bounded, then Γ is uniformly order bounded on each bounded subset of X.*

Proof. Let U be balanced and closed, and $x \in X$. Then there exists $\theta \in (0, 1)$ such that $\theta x \in U$ and $\{f(\theta x) : f \in \Gamma\}$ is order bounded in Y. Pick a, $b \in Y$ with

$$\{f(\theta x) : f \in \Gamma\} \subset [a, b] := \{y \in Y : a \leq y \leq b\}.$$

Since e is an order unit in Y, there exists $n \in \mathbb{N}$ for which

$$[a, b] \subset [-ne, ne] = n[-e, e].$$

Thus

$$\{f(\theta x) : f \in \Gamma\} \subset n[-e, e].$$

Since $\lim_{t\to 0} \varphi(t) = 0$ there exists $m \in \mathbb{N}$ such that $1/m < \delta$ and $\varphi(1/m) \leq (1/n)$. But $f((1/m)\theta x) = s_f f(\theta x)$, where $|s_f| \leq \varphi(1/m) \leq (1/n)$ and $s_f n[-e, e] \subset [-e, e]$ for all $f \in \Gamma$. Then

$$\{f((1/m)\theta x) : f \in \Gamma\} = \{s_f f(\theta x) : f \in \Gamma\} \subset \cup_{f\in\Gamma}(s_f n[-e, e]) \subset [-e, e]$$

and $(1/m)\theta x \in \cap_{f\in\Gamma} f^{-1}([-e, e])$.

Obviously, $(1/m)\theta x \in U$. Let

$$W = U \cap (\cap_{f\in\Gamma} f^{-1}([-e, e])).$$

From the above deducing process, we know $X = \cup_{n=1}^{\infty} nW$. By Definition 1.7, the positive cone $C = \{y \in Y : y \geq 0\}$ is closed in Y. So $[-e, e]$ is also closed in Y. Then $f^{-1}([-e, e])$ is closed in X since each $f \in \Gamma$ is continuous. Thus

$$W = U \cap (\cap_{f\in\Gamma} f^{-1}([-e, e]))$$

is closed in X. But X is a topological vector space of second category. There exists $n_0 \in \mathbb{N}$ with $n_0 W$ having nonempty interior. So W has nonempty interior and $W - W = \{x - z : x, z \in W\} \in \mathcal{N}(X)$.

Let x, $z \in W$ and $f \in \Gamma$. Then $f(x)$, $f(z) \in [-e, e]$ and

$$f(x - z) = r_f f(x) + s_f f(z)$$

where $|r_f| \leq M$ and $|s_f| \leq M$. Pick $p \in \mathbb{N}$ with $p > M$. It is easy to know $r_f f(x)$, $s_f f(z) \in p[-e, e]$. Thus

$$f(x - z) = r_f f(x) + s_f f(z) \in 2p[-e, e].$$

So $f(W - W) \subset 2p[-e, e]$ for all $f \in \Gamma$.

Let $B \subset X$ be bounded, and without loss of generality suppose that $W - W$ is balanced. Since $(W - W) \cap U \in \mathcal{N}(X)$, pick $q \in \mathbb{N}$ for which

$$(1/q)B \subset (W - W) \cap U.$$

Observe that q is independent of every individual $x \in B$. For $x \in B$ and $f \in \Gamma$, $x = q(x/q)$ where $(x/q) \in (W - W) \cap U$ and $f(-(x/q)) \in 2p[-e, e]$.

$$\begin{aligned}
f(x) &= f(q\frac{x}{q}) = f[(q-1)\frac{x}{q} + \frac{x}{q}] \\
&= r_{1f} f[(q-1)\frac{x}{q}] + s_{1f} f(-\frac{x}{q}) \\
&= r_{1f} r_{2f} f[(q-2)\frac{x}{q}] + r_{1f} s_{2f} f(-\frac{x}{q}) + s_{1f} f(-\frac{x}{q}) \\
&\qquad \cdots \quad \cdots \\
&= r_{1f} \cdots r_{q-1f} f(\frac{x}{q}) + (r_{1f} \cdots r_{q-2f} s_{q-1f} + \cdots + r_{1f} s_{2f} + s_{1f}) f(-\frac{x}{q}) \\
&= r_{1f} \cdots r_{q-1f} r_{qf} f(0) + (r_{1f} \cdots r_{q-1f} s_{qf} + \cdots + r_{1f} s_{2f} + s_{1f}) f(-\frac{x}{q}) \\
&= (r_{1f} \cdots r_{q-1f} s_{qf} + \cdots + r_{1f} s_{2f} + s_{1f}) f(-\frac{x}{q}),
\end{aligned}$$

where

$$|r_{if}| \leq M, \; i = 1, 2, \cdots, q-1$$

and

$$|s_{if}| \leq M, \; i = 1, 2, \cdots, q.$$

Obviously,

$$\begin{aligned}|r_{1f}r_{2f}\cdots r_{q-1f}s_{qf}+\cdots+r_{1f}s_{2f}+s_{1f}| &\le M^q+M^{q-1}+\cdots+M^2+M\\ &< p^q+p^{q-1}+\cdots+p^2+p\\ &= \frac{p(1-p^q)}{1-p}.\end{aligned}$$

So

$$f(x)=(r_{1f}r_{2f}\cdots r_{q-1f}s_{qf}+\cdots+r_{1f}s_{2f}+s_{1f})f(-x/q)\in\frac{p(1-p^q)}{1-p}2p[-e,e]$$

for all $f\in\Gamma$ and $x\in B$. Thus, Γ is uniformly order bounded on $B\subset X$ and is uniformly order bounded on each bounded subset of X. □

Theorem 2.2. *Let X be a topological vector space of second category and Y an ordered topological vector space with the positive cone having nonempty interior. If the UB family Γ of continuous mappings from X to Y with $U\in\mathcal{N}(X)$, δ, $M>0$ and $\varphi:(0,\delta)\to(0,+\infty)$ satisfying $\lim_{t\to 0}\varphi(t)=0$ is pointwise bounded or pointwise order bounded, then Γ is uniformly order bounded on each bounded subset of X.*

Proof. Let U be balanced and closed, and $C=\{y\in Y: y\ge 0\}$ the positive cone in Y. By the hypothesis, C is closed in Y and there exists y_0 in the interior of C. Obviously, $C-y_0$ and y_0-C are both neighborhoods of $0\in Y$. Then

$$V=(C-y_0)\cap(y_0-C)$$

is also a neighborhood of $0\in Y$, and a closed and balanced convex set. Let $x\in X$. There exists $\theta\in(0,1)$ such that $\theta x\in U$. And since $\{f(\theta x): f\in\Gamma\}$ is bounded in Y there exists $n\in\mathbb{N}$ such that $\{f(\theta x): f\in\Gamma\}\subset nV$. In another case, we know $\{f(\theta x): f\in\Gamma\}$ is order bounded in Y. Then there exist a, $b\in Y$ such that

$$\{f(\theta x): f\in\Gamma\}\subset[a,b]:=\{y\in Y: a\le y\le b\}.$$

But $C-y_0$ and y_0-C absorbs $[a,b]$. Pick $m\in\mathbb{N}$ with $m>n$ and

$$[a,b]\subset[-my_0,my_0]=m[-y_0,y_0].$$

Hence $\{f(\theta x): f\in\Gamma\}\subset m[-y_0,y_0]=mV$. Observing $nV\subset mV$ we obtain $\{f(\theta x): f\in\Gamma\}\subset mV$ in two cases.

Since $\lim_{t\to 0}\varphi(t)=0$ there exists $p\in\mathbb{N}$ such that $1/p<\delta$ and $\varphi(1/p)\le(1/m)$. But

$$f((1/p)\theta x)=s_f f(\theta x),\quad\text{where}\quad |s_f|\le\varphi(1/p)\le(1/m).$$

Then $|ms_f|\le 1$ for all $f\in\Gamma$ and

$$\{f((1/m)\theta x): f\in\Gamma\}=\{s_f f(\theta x): f\in\Gamma\}\subset\cup_{f\in\Gamma}s_f mV\subset V$$

Thus $(1/m)\theta x\in\cap_{f\in\Gamma}f^{-1}(V)$. Let

$$W=U\cap(\cap_{f\in\Gamma}f^{-1}(V))$$

where

$$V=(C-y_0)\cap(y_0-C)=[-y_0,y_0].$$

Obviously, $(1/m)\theta x \in U$. Hence, $X = \cup_{n=1}^{\infty} nW$. Since V is closed in Y, each $f \in \Gamma$ is continuous and U is closed in X, we know W is closed in X. But X is of second category. So $W - W = \{x - z : x, z \in W\} \in \mathcal{N}(X)$.

As the same in the proof of Theorem 2.1, we can obtain the result. □

Remark 2.3. In Theorem 2.2, if the positive cone C in Y is normal in the weak topology, then Γ is uniformly bounded on each bounded subset of X. In fact, $Y' = C' - C'$ where $C' = \{f \in Y' : f(y) \geq 0 \text{ for all } y \in C\}$. Then an order bounded subset in Y must be bounded in Y. See details for [12].

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