Annals of Functional Analysis
Ann. Funct. Anal. 2 (2011), no. 2, 92–100
ANNALS OF FUNCTIONAL ANALYSIS
ISSN: 2008-8752 (electronic)
URL: www.emis.de/journals/AFA/

CENTRAL ELEMENTS IN TOPOLOGICAL ALGEBRAS WITH THE EXPONENTIAL MAP

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Communicated by Z. Lykova

ABSTRACT. Topological algebras with the exponential map are examined and several conditions are obtained for an element in a Gelfand–Mazur algebra with the exponential map to be central modulo the left topological radical.

1. INTRODUCTION

Characterizations of commutativity in different classes of topological algebras have been considered in many papers (see, for example, [9, 12, 14, 16, 17, 22, 26]) and many of the results obtained in this area use the extended Jacobson density theorem for Banach algebras due to A. M. Sinclair [24, Theorem 6.7]. It is the object of the present paper to examine commutativity criteria (modulo either the left topological radical or the Jacobson radical) in topological algebras. We show, among others, that one can use a result analogous to Sinclair's theorem also within the class of Gelfand–Mazur algebras with the exponential map (Theorem 3.3). Moreover, we obtain several conditions for an element in a topological algebra to be central modulo the left topological radical and we also describe several classes of topological algebras possessing the exponential map.

2. PRELIMINARIES

Throughout this paper, all algebras are assumed to be complex, associative and unital.

Let A be an algebra with the identity e_A. We denote by $InvA$ the set of all invertible elements of A, $\sigma_A(a)$ is the *spectrum* of an element a in A and $r_A(a)$

Date: Received: 20 March 2011; Accepted: 20 August 2011.
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2010 *Mathematics Subject Classification.* Primary 46H05; Secondary 46H20.
Key words and phrases. Gelfand–Mazur algebra, exponential map, commutativity, central element, left topological radical.

is the *spectral radius* of $a \in A$, i.e. $r_A(a) = sup\{|\alpha| : \alpha \in \sigma_A(a)\}$. In case $\sigma_A(a)$ is empty we put $r_A(a) = 0$ and if $\sigma_A(a)$ is unbounded then $r_A(a)$ is defined to be ∞. Further, in what follows, an ideal of A is always a proper ideal, $\mathcal{L}$ is the set of all maximal left ideals of A, $\mathcal{R}$ is the set of all maximal right ideals of A, $[a, b] = ab - ba$ is the commutator of $a, b \in A$ and

$$exp(a)_n = \sum_{i=0}^{n} a^i/i!$$

for any a in an algebra A and any $n = 1, 2, \ldots$.

Now, if L is in $\mathcal{L}$ and $a \in A \setminus L$ then $L_a = \{x \in A : xa \in L\}$ is a maximal left ideal of A. Indeed, L_a is a left ideal of A and if $ba \notin L$ for some b in $L_1 \in \mathcal{L}$ containing L_a then, because of $Aba + L = A$, there is $c \in A$ such that $cba - a \in L$. So, $cb - e_A \in L_a \subset L_1$ and, consequently, $L_1 = A$.

Further, let $A_L = \{x \in A : Lx \subseteq L\}$ stand for the *normalizer* of the ideal $L \in \mathcal{L}$. Clearly A_L is a subalgebra of A containing the center $Z(A)$ of A. Besides, L is a two-sided ideal of A_L and an element $a \in A \setminus L$ is in A_L if and only if $L \subseteq L_a$, or equivalently, if and only if $L_a \subseteq L$. Moreover, the quotient algebra A_L/L is a division algebra ([19, Theorem 2.1.2] or [21, Lemma 2.1]). Analogously, $A_R = \{x \in A : xR \subseteq R\}$ is the normalizer of $R \in \mathcal{R}$.

We shall need the following lemmas.

Lemma 2.1. *Let A be an algebra, let $L \in \mathcal{L}$ and $a, x, y \in A$. If $a \notin A_L$, then there is $b \in A$ such that $ba - x, b - y \in L$.*

Proof. Since $a \notin A_L$, there are $c_1, c_2 \in A$ such that $c_1 \in L$, $c_1 \notin L_a$, $c_2 \notin L$, $c_2 \in L_a$. Now, $c_1 a \notin L$ and, because L is a maximal left ideal, $Ac_1 a + L = A$. Analogously, $Ac_2 + L = A$. So, there are $d_1, d_2 \in A$ such that $d_1 c_1 a - x \in L$ and $d_2 c_2 - y \in L$. Finally, put $b = d_1 c_1 + d_2 c_2$. □

Lemma 2.2. *Let A be an algebra. If $\Im \subset \mathcal{L}$ is such that $L_a \in \Im$ for any $L \in \Im$ and $a \in A \setminus L$, then*

$$\Re(\Im) = \cap\{L : L \in \Im\}$$

is a two-sided ideal of A.

Moreover, if $b \in A$ is such that for any $L \in \Im$ there is $\alpha \in \mathbb{C}$ satisfying $b - \alpha e_A \in L$, then $[b, x] \in \Re(\Im)$ for any $x \in A$.

Proof. See [21, Lemmas 2.3 and 2.4]. □

3. Central elements in Gelfand–Mazur algebras

Recall that a *topological algebra* is an algebra which is also a Hausdorff topological vector space in such a way that the ring multiplication is separately continuous; and we say that a topological algebra A is a *topological algebra with the exponential map* if every $a \in A$ has an exponent $exp(a) \in InvA$.

If A is a topological algebra, then by rad$A$ we denote the *left topological radical* of A, i.e. rad$A = \cap\{L : L \in \Im_c\}$, where $\Im_c$ is the set of all closed maximal left ideals of A. By Lemma 2.2, rad$A$ is a two-sided ideal of A and an element a in A is said to be *central modulo* rad$A$ if $[a, b] \in$ rad$A$ for every $b \in A$.

Furthermore, A is called a *Gelfand–Mazur algebra* if the quotient algebra A/L is topologically isomorphic to $\mathbb{C}$ for every closed two-sided ideal L of A which is maximal in A as a left or as a right ideal. Also, we say that a Gelfand–Mazur algebra A is *hereditarily Gelfand–Mazur algebra* if every unital closed subalgebra of A is a Gelfand–Mazur algebra in the subalgebra topology. For different classes of Gelfand–Mazur algebras we refer to [1, 5, 23].

Proposition 3.1. *Let A be a hereditarily Gelfand–Mazur algebra. The following statements are equivalent for an element $a \in A$:*

(1) *$a \in A_L$ for every $L \in \Im_c$,*
(2) *for every $L \in \Im_c$ there is $\alpha \in \mathbb{C}$ satisfying $a - \alpha e_A \in L$,*
(3) *a is central modulo rad$A$.*

Proof. (1)⇒(2). Take $L \in \Im_c$. Then L is a closed two-sided ideal of the subalgebra A_L and, as we mentioned above, the quotient algebra A_L/L is a division algebra. Hence L is maximal in A_L as a left or as a right ideal. Indeed, if there is a left ideal I in A_L such that $L \subset I$ and $i \in I \setminus L$, then $i + L$ is invertible in A_L/L. Therefore there exists $x \in A_L$ such that $xi - e_A \in L$, so that $e_A = xi - (xi - e_A) \in I$ which is impossible. The proof for right ideals is similar.

Now, since A_L is, by assertion, a Gelfand–Mazur algebra, A_L/L is topologically isomorphic to $\mathbb{C}$. So, there is a multiplicative linear functional Λ on A_L with the kernel ker$\Lambda = L$ and since $e_A, a \in A_L$, we have $a - \Lambda(a)e_A \in L$.

(2)⇒(3). Note first that $L_x \in \Im_c$ for any $L \in \Im_c$ and $x \in A \setminus L$. Then use Lemma 2.2.

(3)⇒(1). Take any $L \in \Im_c$ and any $l \in L$. Then $[a, l] \in \text{rad}A \subset L$. Thus, $la \in L$ for any $l \in L$, that is $a \in A_L$. □

Remark 3.2. An analogous result is also valid when considering the closed maximal right ideals and the right topological radical of A.

The next theorem could be looked as a version of the above mentioned Sinclair's theorem for the class of topological algebras possessing the exponential map.

Theorem 3.3. *Let A be a topological algebra with the exponential map, $L \in \Im_c$ and let $a \in A \setminus A_L$. Moreover, let Z be the linear subspace in A, generated by the elements a and e_A and let $\{x, y\}$ be a linearly independent system of elements of Z. Then, there is $b \in A$ such that $exp(b)a - x, exp(b) - y \in L$.*

Proof. Note that a and e_A are linearly independent elements. Consider Z as the 2-dimensional coordinate space with respect to the base $\{a, e_A\}$, i.e. for any element $z = \alpha a + \beta e_A \in Z$ $(\alpha, \beta \in \mathbb{C})$ put $\|z\| = \sqrt{\alpha\bar{\alpha} + \beta\bar{\beta}}$. Moreover, let $LB(Z)$ be the Banach algebra of all bounded linear operators on Z. Define an operator $P \in LB(Z)$ by $P(\alpha a + \beta e_A) = \alpha x + \beta y$ for any $\alpha, \beta \in \mathbb{C}$. Since $\{x, y\}$ is a linearly independent system, P is invertible, so that $0 \notin \sigma_{LB(Z)}(P)$. Furthermore, $\sigma_{LB(Z)}(P)$ is a finite set (see, for example, [19, Proposition 1.10.1]). Put $\mathbb{R}_- = \{\alpha \in \mathbb{R} : \alpha \leq 0\}$. Now, there is $\lambda \in \mathbb{C} \setminus \mathbb{R}_-$ such that $\sigma_{LB(Z)}(\lambda P) \subset \mathbb{C} \setminus \mathbb{R}_-$ and it readily follows that there is $R \in LB(Z)$ such that $exp_n(R)(z) \longrightarrow P(z)$ $(z \in Z)$ in the coordinate space topology (see, for example, [11, Proposition 1.8.3]). But then

$exp(R)_n(z) \longrightarrow P(z)$ in the topology of A for every $z \in Z$. Further, by Lemma 2.1, there is $b \in A$ such that $ba - R(a), b - R(e_A) \in L$. So, $bz - R(z) \in L$ for every $z \in Z$ and also, $b^n z - R^n(z) = b(b^{n-1}z - R^{n-1}(z)) + b(R^{n-1}(z)) - R(R^{n-1}(z)) \in L$ ($z \in Z$, $n = 2, 3, \ldots$). Hence $exp_n(b)z - exp_n(R)(z) \in L$ for any $z \in Z$ ($n = 1, 2, \ldots$) and since L is closed, we conclude that $exp(b)z - P(z) \in L$ for any $z \in Z$. □

Corollary 3.4. *Let A be a topological algebra with the exponential map and let L be an element in $\Im_c$. If $a \in A \setminus A_L$ and $\alpha \in \mathbb{C}$ is nonzero, then there are elements $b_1, b_2 \in InvA$ such that*

(1) $\alpha \in \sigma_A(a(b_1^{-1}ab_1))$,
(2) $\alpha \in \sigma_A(a - b_2^{-1}ab_2)$.

Proof. Since by assertion $a \notin A_L$, elements a and e_A are linearly independent and therefore $\{e_A, a/\alpha\}$ and $\{a + \alpha e_A, e_A\}$ are two linearly independent systems of elements of the linear subspace of A generated by a and e_A. Hence, by Theorem 3.3, there are $b_1, b_2 \in InvA$ such that $b_1 a - e_A, \alpha b_1 - a, b_2 a - a - \alpha e_A, b_2 - e_A \in L$. Now $ab_1 a - a, ab_2 - a \in L$, so that $ab_1 a - \alpha b_1, [b_2, a] - \alpha b_2 \in L$. Consequently, $b_1^{-1}ab_1 a - \alpha e_A, b_2^{-1}[b_2, a] - \alpha e_A \in L$. But this yields $\alpha \in \sigma_A(a(b_1^{-1}ab_1))$ and $\alpha \in \sigma_A(a - b_2^{-1}ab_2)$. □

Proposition 3.1 and Corollary 3.4 give us

Corollary 3.5. *Let A be a hereditarily Gelfand–Mazur algebra with the exponential map and let a be an element of A. If one of the following two conditions*

(1) $sup\{r_A(a(b^{-1}ab)) : b \in InvA\} < \infty$,
(2) $sup\{r_A(a - b^{-1}ab) : b \in InvA\} < \infty$

is satisfied, then a is central modulo rad$A$.

Proof. By Corollary 3.4 $a \in A_L$ for every $L \in \Im_c$. So, by Proposition 3.1, a is central modulo rad$A$. □

A topological algebra A is called a *Q-algebra* if the set $InvA$ is open in the topology of A. It is a well-known fact that in a Q-algebra A every maximal ideal is closed and $\sigma_A(a)$ is bounded for any $a \in A$ ([23], p. 72). Thus, for topological Q-algebras rad$A$ = Rad$A = \cap\{R : R \in \mathcal{R}\}$ and we can easily deduce the following theorem (cf. [13, Theorems 4.1 and 4.2], [15, Theorem 2.1], [18, Theorem 5.6], [20, Theorem 2], [21, Theorem 3.3], [26, Theorems 3.1 and 5.1]).

Theorem 3.6. *Let A be a hereditarily Gelfand–Mazur Q-algebra with the exponential map. The following statements are equivalent for an element $a \in A$:*

(1) *a is central modulo the Jacobson radical* Rad$A$ *of A,*
(2) $\sigma_A(a + b) \subseteq \sigma_A(a) + \sigma_A(b)$ *for each* $b \in A$,
(3) $\sigma_A(ab) \setminus \{0\} \subseteq \sigma_A(a)\sigma_A(b)$ *for each* $b \in A$,
(4) *there is* $K > 0$ *such that* $r_A(a + b) \leq K(r_A(a) + r_A(b))$ *for each* $b \in A$,
(5) *there is* $K > 0$ *such that* $r_A(ab) \leq K r_A(a) r_A(b)$ *for each* $b \in A$,
(6) *there is* $M > 0$ *such that* $sup\{r_A(a - b^{-1}ab) : b \in InvA\} < M$,
(7) *there is* $M > 0$ *such that* $sup\{r_A(a(b^{-1}ab)) : b \in InvA\} < M$.

Proof. (1)⇒(2). Take any $b \in A$ and $\lambda \in \sigma_A(a+b)$. Then $a+b-\lambda e_A \in M$ for some $M \in \mathcal{L}$ or $M \in \mathcal{R}$. Suppose that $M \in \mathcal{L}$. Then, by Proposition 3.1, there is $\alpha \in \sigma_A(a)$ such that $a - \alpha e_A \in M$. Now $a - \alpha e_A + b - (\lambda - \alpha)e_A \in M$ and so $(\lambda - \alpha) \in \sigma_A(b)$. The proof in case $M \in \mathcal{R}$ is analogous (see Remark 3.2).

(1)⇒(3). Suppose $ab - \lambda e_A \in M$ for some $b \in A$, nonzero $\lambda \in \mathbb{C}$ and $M \in \mathcal{L}$ or $M \in \mathcal{R}$. If $M \in \mathcal{L}$ then, again by Proposition 3.1, there is $\alpha \in \sigma_A(a)$ with $a - \alpha e_A \in M$. Also, $\alpha b - ba \in M$ and, consequently, $\alpha b - \lambda e_A \in M$ since $[a, b] \in M$. Hence, α is nonzero and $\lambda/\alpha \in \sigma_A(b)$. In view of Remark 3.2 the proof for $M \in \mathcal{R}$ is analogous.

(2)⇒(4) and (3)⇒(5). Take any $K \geq 1$.

(4)⇒(6). Take any $M > 2Kr_A(a)$.

(5)⇒(7). Take any $M > Kr_A(a)^2$.

(6)⇒(1) and (7)⇒(1). Use Corollary 3.5. □

4. Classes of topological algebras with the exponential map

4.1. **Fundamental topological algebras with bounded elements.** A topological algebra A is a *fundamental topological algebra* if there exists $b > 1$ such that for every sequence (a_n) of A the convergence of $b^n(a_n - a_{n-1})$ to zero in A implies that (a_n) is a Cauchy sequence. This class of topological algebras was introduced in [6]. It is known (see [3, Proposition 2.3]) that every locally pseudoconvex algebra (in particular, every locally convex algebra and every locally bounded algebra) is a fundamental topological algebra.

An element a of a topological algebra A is called to be *bounded* if there exists $\lambda > 0$ such that the set $\{\left(\frac{a}{\lambda}\right)^n : n \in \mathbb{N}\}$ is bounded in A. If every element of A is bounded, then A is called a *topological algebra with bounded elements.*

Proposition 4.1. *Every unital sequentially complete fundamental topological algebra with bounded elements is a topological algebra with the exponential map.*

Proof. Let a be an element in A, O, O'' neighbourhoods of zero in A and O' a neighbourhood of zero in $\mathbb{C}$ such that $O'O'' \subset O$. Let $\lambda > 0$ be a number such that the set $\{\left(\frac{a}{\lambda}\right)^n : n \in \mathbb{N}\}$ is bounded. Then there is $\mu > 0$ such that $\left(\frac{a}{\lambda}\right)^n \in \mu O''$ for each n. Moreover, put

$$S_n = \sum_{i=0}^{n} \frac{a^i}{i!}$$

for each $n \in \mathbb{N}$, and let $b > 1$ be a fixed number. Since

$$\sum_{n=0}^{\infty} \frac{b^n \lambda^n}{n!}$$

converges, then there is an $n_0 \in \mathbb{N}$ such that $\frac{b^n\lambda^n}{n!} \in \mu^{-1}O'$ whenever $n > n_0$ and

$$b^n(S_n - S_{n-1}) = \frac{b^n\lambda^n}{n!}\left(\frac{a}{\lambda}\right)^n \in O'O'' \subset O$$

whenever $n > n_0$. Hence, $b^n(S_n - S_{n-1})$ converges to zero in A. Therefore, (S_n) is a Cauchy sequence which converges in A, because A is a sequentially complete fundamental topological algebra. Consequently,

$$exp(a) = \lim_{n\to\infty} S_n = \sum_{i=0}^{\infty} \frac{a^i}{i!} \in A.$$

Similarily, $exp(-a) \in A$. Since $exp(a)\, exp(-a) = exp(-a)\, exp(a) = e_A$, then A is a topological algebra with the exponential map. ☐

Corollary 4.2. *Every unital sequentially complete locally pseudoconvex algebra with bounded elements is a topological algebra with the exponential map.*

4.2. **Fundamental locally convex algebras.** A fundamental topological algebra A is called to be locally multiplicative (see [7] or [8]), if in A there exists a neighbourhood U_0 of zero such that for every neighbourhood V of zero of A, the sufficiently large powers of U_0 lie in V. It is shown in [8, p. 1765] or [7, Theorem 5.4], that every unital fundamental locally convex Fréchet algebra is a topological algebra with the exponential map.

4.3. **Locally idempotent galbed algebras.** Let l^0 be the set of number sequences (α_n), where $\alpha_n \neq 0$ for only a finite number of elements, $k > 0$, l^k the set of number sequences (α_n) for which the series

$$\sum_{v=0}^{\infty} |\alpha_v|^k$$

converges and let $l = l^1 \setminus l^0$. A topological algebra A is called a *galbed algebra* if there exsists a sequence $(\alpha_n) \in l$ such that for each neighbourhood O of zero in A there is another neighbourhood U of A such that

$$\Big\{ \sum_{k=0}^{n} \alpha_k\, a_k : a_0, \dots, a_n \in U \Big\} \subset O$$

for each $n \in \mathbb{N}$. In case when we have already specified the sequence $(\alpha_n) \in l$, we will talk about an (α_n)-galbed algebra. In particular, when $\alpha_n = \frac{1}{2^n}$ for each $n \in \mathbb{N}$, a galbed algebra is called an *exponentially galbed algebra*. The class of exponentially galbed algebras has been introduced in [25] and the class of galbed algebras in [2]. It is easy to see that every locally pseudoconvex algebra is an exponentally galbed algebra.

Moreover, a topological algebra A is called a *locally idempotent algebra* if A has a base of idempotent neibourghoods of zero.

Proposition 4.3. *Every unital sequentially complete locally idempotent (α_n)-galbed algebra A is a topological algebra with the exponential map if*

$$\lim_{n\to\infty} \frac{\alpha_n}{(n+1)\, \alpha_{n+1}} = 0. \tag{4.1}$$

Proof. Let a be an element in A and let O be a neighbourhood of zero in A. Then there exist an idempotent neigbourhood U of zero such that $U \subset O$ and $\lambda > 0$ such that $a \in \lambda U$. Put

$$S_n = \sum_{i=0}^{n} \frac{a^i}{i!}$$

for each $n \in \mathbb{N}$. Then

$$S_{n+p} - S_n = \sum_{i=n+1}^{n+p} \frac{a^i}{i!} = \sum_{i=n+1}^{n+p} \alpha_i u_i,$$

where $p \geq 1$ and

$$u_{n+k} = \frac{a^{n+k}}{(n+k)!\,\alpha_{n+k}} \in \frac{\lambda^{n+k}}{(n+k)!\,\alpha_{n+k}} U$$

for each $k \geq 1$. Since

$$\sum_{n=0}^{\infty} \frac{\lambda^{n+k}}{(n+k)!\,\alpha_{n+k}}$$

converges by (4.1) for each $k \geq 1$, there exists $n_0 \in \mathbb{N}$ such that

$$\left| \frac{\lambda^{n+k}}{(n+k)!\,\alpha_{n+k}} \right| \leq 1$$

whenever $n > n_0$ and $k \geq 1$. Hence $u_{n+k} \in U$ for each $k \geq 1$ whenever $n > n_0$. Therefore,

$$S_{n+p} - S_n = \sum_{i=0}^{n+p} \alpha_i u_i \in \Big\{ \sum_{i=0}^{n+p} \alpha_i u_i : u_0, \ldots u_{n+p} \in U \Big\} \subset O$$

whenever $n > n_0$ and $p \geq 1$ (here $u_i = \theta_A$ for each $i \in \{1, \ldots, n\}$). Thus (S_n) is a Cauchy sequence which converges in A because A is sequentially complete. Similary as in the proof of Proposition 4.1, A is a topological algebra with the exponential map. □

Corollary 4.4. *Every unital sequentially complete locally idempotent exponentially galbed (in particular, every unital sequentially complete locally m-pseudoconvex*[1]*) algebra A is a topological algebra with the exponential map.*

Theorem 4.5. *Let A be*

a) *a unital sequentially complete locally pseudoconvex Q-algebra with bounded elements or*

b) *a unital sequentially complete locally idempotent exponentially galbed Q-algebra with bounded elements.*

If $a \in A$, then the statements (1)–(7) *of Theorem 3.6 are equivalent.*

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[1]Corollary 4.2 for a complete locally m-pseudoconvex algebra was proved in [10], Proposition 5.2.2.

Proof. It is known (see [4, Theorem 1] or [1, Corollary 2]) that every exponentially galbed (in particular, locally pseudoconvex) algebra with bounded elements is a Gelfand–Mazur algebra. In addition, every subalgebra of such an algebra is also an exponentially galbed (respectively, locally pseudoconvex) algebra with bounded elements. Hence, in both cases, A is a hereditarily Gelfand–Mazur Q-algebra. Since A is a topological algebra with the exponential map (Corollaries 4.2 and 4.4), the statements (1)–(7) are, by Theorem 3.6, equivalent. □

Acknowledgement. This research is in part supported by Estonian Science Foundation grant 7320 and by Estonian Targeted Financing Project SF0180039s08.

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