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SOME EXISTENCE RESULTS ON A CLASS OF INCLUSIONS

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ABSTRACT. In this paper, we introduce the generalized system nonlinear variational inclusions and prove the existence of its solution in normed spaces. We provide examples of applications related to a system nonlinear variational inclusions in the sense of Verma, a coupled fixed point problem, considered by Bhaskar and Lakshmikantham, a coupled coincidence point considered by Lakshmikantham and Ćirić. Also, we generalized coupled best approximations theorem.

1. Introduction and preliminaries

In the sequel, if not otherwise stated, let I be any finite index set. For each $i \in I$, let K_i be a nonempty subset of a real topological vector space X_i, $s_i : K \to X_i$ be a mapping and $M_i : K_i \multimap X_i$ be a multivalued mapping with nonempty values, where $K = \prod_{i\in I} K_i$ and $X = \prod_{i\in I} X_i$. For each $x \in X$ denoted by $x = (x_i)_{i\in I}$ where x_i the ith coordinate.

In this paper, we study the following system of general nonlinear variational inclusion problem:

(SGNVI) Find $\overline{x} = (\overline{x}_i)_{i\in I} \in K$ such that for each $i \in I$,

$$0 \in s_i(\overline{x}) + M_i(\overline{x}_i). \tag{1.1}$$

Below are some special cases of problem (1.1).

(1) If $X_i = \mathbb{R}$ and $M_i(x_i) = (-\infty, -m_i(x_i)]$, where $m_i(\cdot)$ is a mapping $m_i : K_i \to \mathbb{R}$ then problem SGNVI reduces to finding $\overline{x} \in K$ such that for each

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$i \in I$,

$$s_i(\overline{x}) \geq m_i(x_i).$$

(2) If $X_i = \mathbb{R}$ and $M_i(x_i) = \{-m_i(x_i)\}$, then problem SGNVI reduces to finding $\overline{x} \in K$ such that for each $i \in I$,

$$s_i(\overline{x}) = m_i(x_i).$$

(3) If

$$I = \{1, 2\}, X = X_1 = X_2, K = K_1 = K_2,$$

$$s_1(x_1, x_2) = -F(x_1, x_2), s_2(x_1, x_2) = -F(x_2, x_1),$$

$M_1(x_1) = G(x_1), M_2(x_2) = G(x_2)$ for all $x_1, x_2 \in K$ then (1.1) reduces to finding $(x_1, x_2) \in K \times K$, such that

$$F(x_1, x_2) \in G(x_1), F(x_2, x_1) \in G(x_2), \tag{1.2}$$

which is a multivalued coupled coincidence point problem.

(4) If G is a single-valued mapping and $G(x) = \{g(x)\}$ then (1.2) reduces to finding $(x_1, x_2) \in K \times K$, such that

$$F(x, y) = g(x), F(y, x) = g(y).$$

which is a coupled coincidence point problem considered by Lakshmikantham and Ćirić [9].

(5) If $G(x) = \{x\}$ is an identity mapping, then (1.2) is equivalent to finding $(x_1, x_1) \in X \times X$, such that

$$F(x_1, x_2) = x_1, F(x_2, x_1) = x_1,$$

which is known as a coupled fixed point problem, considered by Bhaskar and Lakshmikantham [3].

(6) In the paper [15] Verma introduced the system of nonlinear variational inclusion (SNVI) problem: finding $(x_0, y_0) \in H_1 \times H_2$ such that

$$0 \in S(x_0, y_0) + M(x_0),\ 0 \in T(x_0, y_0) + N(y_0), \tag{1.3}$$

where H_1 and H_2 are real Hilbert spaces,

$$S : H_1 \times H_2 \to H_1, T : H_1 \times H_2 \to H_2$$

any mappings and $M : H_1 \multimap H_1, N : H_2 \multimap H_2$ any multivalued mappings. If $I = \{1, 2\}$ then (1.1) reduces to (1.3).

(i) If $M(\cdot) = \partial f(\cdot)$ and $N(\cdot) = \partial g(\cdot)$ where $\partial f(\cdot)$ is the subdifferential of a proper, convex and lower semicontinuous functions,

$$f, g : X \to \mathbb{R} \cup \{+\infty\}$$

then problem SNVI reduces to finding $(x_0, y_0) \in K_1 \times K_2$ such that

$$\langle S(x_0, y_0), x - x_0\rangle + f(x) - f(x_0) \geq 0 \text{ for all } x \in K_1,$$

$$\langle T(x_0, y_0), y - y_0\rangle + g(x) - g(x_0) \geq 0 \text{ for all } y \in K_2,$$

where K_1 and K_2, respectively, are nonempty closed convex subsets of H_1 and H_2.

(ii) When $M(x) = \partial_{K_1}(x)$ and ∂_{K_2} denote indicator functions of K_1 and

K_2, respectively, the SNVI problem (1.3) reduces to system of nonlinear variational inequalities problem: finding $(x_0, y_0) \in K_1 \times K_2$ such that

$$\langle S(x_0, y_0), x - x_0 \rangle \geq 0 \text{ for all } x \in K_1,$$

$$\langle T(x_0, y_0), y - y_0 \rangle \geq 0 \text{ for all } y \in K_2.$$

The aim of this paper is to obtain the results of existence a solution of SGNVI problem (1.1) using the KKM technique.

We need the following definitions and results.

Let X and Y be real vector spaces, $F : X \multimap Y$ is a multivalued mapping from a set X into the power set of a set Y. For $A \subseteq X$, let

$$F(A) = \cup \{F(x) : x \in A\}.$$

For any $B \subseteq Y$, the lower inverse and upper inverse of B under F are defined by

$$F^-(B) = \{x \in X : F(x) \cap B \neq \emptyset\} \text{ and } F^+(B) = \{x \in X : F(x) \subseteq B\},$$

respectively.

A mapping F is upper (lower) semicontinuous on X if and only if for every open $V \subseteq Y$, the set $F^+(V)$ ($F^-(V)$) is open. A mapping F is continuous if and only if it is upper and lower semicontinuous. A mapping F with compact values is continuous if and only if F is a continuous mapping in the Hausdorff distance, see for example [4].

Let X be a normed space. If A and B are nonempty subsets of X, we define

$$A + B = \{a + b : a \in A, b \in B\} \text{ and } ||A|| = \inf\{||a|| : a \in A\}.$$

We using the notion a C-convex map for multivalued maps.

Definition 1.1. (Borwein, [5]) Let X and Y be real vector spaces, K a nonempty convex subset of X and C is a cone in Y. A multivalued mapping $F : K \multimap Y$ is said to be C-convex if,

$$\lambda F(x_1) + (1 - \lambda)F(x_2) \subset F(\lambda x_1 + (1 - \lambda)x_2) + C \tag{1.4}$$

for all $x_1, x_2 \in K$ and all $\lambda \in [0, 1]$.

A mapping F is convex if it satisfies condition (1.4) with $C = \{0\}$ (see for example, Nikodem [11], Nikodem and Popa [12]). If F is a single-valued mapping, $Y = \mathbb{R}$ and $C = [0, +\infty)$, we obtain the standard definition of convex functions. The convex multivalued mappings play an important role in convex analysis, economic theory and convex optimization problems see for example [1, 2, 5, 14].

Lemma 1.2. *(Nikodem, [11]) If a multivalued mapping $F : K \multimap Y$ is C-convex, then*

$$\lambda_1 F(x_1) + \ldots + \lambda_n F(x_n) \subset F(\lambda_1 x_1 + \ldots + \lambda_n x_n) + C,$$

for all $n \in \mathbb{N}, x_1, \ldots, x_n \in K$ and $\lambda_1, \ldots, \lambda_n \in [0, 1]$ such that $\lambda_1 + \ldots + \lambda_n = 1$.

Lemma 1.3. *Let K be a convex subset of normed space X and a multivalued mapping $F : K \multimap X$ is convex, then*

$$||F(\sum_{i=1}^{n} \lambda_i x_i) + u|| \leq \sum_{i=1}^{n} \lambda_i ||F(x_i) + u|| \tag{1.5}$$

for all $n \in \mathbb{N}, x_1, \ldots, x_n \in K, u \in X$ and $\lambda_1, \ldots, \lambda_n \in [0,1]$ such that $\lambda_1 + \ldots + \lambda_n = 1$.

Remark 1.4. If $F : K \to K$ is single valued and almost-affine mapping (see for example Prolla [13]) then the condition (1.5) is hold.

Definition 1.5. (Dugundji and Granas [6, Definition 1.1]) Let K be a nonempty subset of topological vector space a X. A multivalued mapping $H : K \multimap X$ is called a KKM mapping if, for every finite subset $\{x_1, x_2, \ldots, x_n\}$ of K,

$$co\{x_1, x_2, \ldots, x_n\} \subset \bigcup_{i=1}^{n} H(x_i),$$

where *co* denotes the convex hull.

Lemma 1.6. *(Ky Fan [7], Lemma 1.) Let X be a topological vector space, K be a nonempty subset of X and $H : K \multimap X$ a mapping with closed values and KKM mapping. If $H(x)$ is compact for at least one $x \in K$ then $\bigcap_{x \in K} H(x) \neq \emptyset$.*

2. Main results

Theorem 2.1. *For each $i \in I$, suppose that*

(1) *K_i is a nonempty convex compact subset of a normed space X_i,*
(2) *$s_i : K \to X_i$ continuous mapping,*
(3) *$M_i : K_i \multimap X_i$ continuous convex multivalued mapping with compact values.*

Then there exists $\overline{x} \in K$ such that

$$\sum_{i \in I} ||M_i(\overline{x}_i) + s_i(\overline{x})|| = \inf_{x \in K} \sum_{i \in I} ||M_i(x_i) + s_i(\overline{x})||.$$

Proof. Define a multivalued mapping $H : K \multimap K$ by

$$H(y) = \{x \in K : \sum_{i \in I} ||M_i(x_i) + s_i(x)|| \leq \sum_{i \in I} ||M_i(y_i) + s_i(x)||\}$$

for each $y = (y_i)_{i \in I} \in K$.

We have that $y \in H(y)$, hence $H(y)$ is nonempty for all $y \in K$.

The mappings s_i and M_i are continuous and we have that $H(y)$ is closed for each $y \in K$.

Since K is a compact set we have that $H(y)$ is compact for each $y \in K$.

Mapping H is a KKM map. Namely, suppose for any $y^j \in K, j \in J$, where J finite subset of $\mathbb{N}$, there exists

$$y^0 \in co\{y^j : j \in J\}, \tag{2.1}$$

such that

$$y^0 \notin \bigcup_{j \in J} H(y^j). \tag{2.2}$$

From (2.1) we obtain that there exist $\lambda_j \geq 0, j \in J$, such that

$$y^0 = \sum_{j \in J} \lambda_j y^j \text{ and } \sum_{j \in J} \lambda_j = 1.$$

From condition (2.2) we obtain that

$$\sum_{i \in I} ||M_i(y_i^0) + s_i(y^0)|| > \sum_{i \in I} ||M_i(y_i^j) + s_i(y^0)|| \text{ for each } j \in J. \tag{2.3}$$

From (2.3) we obtain,

$$\sum_{j \in J} \lambda_j \sum_{i \in I} ||M_i(y_i^0) + s_i(y^0)|| > \sum_{j \in J} \lambda_j \sum_{i \in I} ||M_i(y_i^j) + s_i(y^0)||,$$

so, we have

$$\sum_{i \in I} ||M_i(y_i^0) + s_i(y^0)|| > \sum_{i \in I} \sum_{j \in J} \lambda_j ||M_i(y_i^j) + s_i(y^0)||.$$

Since M_i is convex mapping for each $i \in I$ from Lemma 1.3, we obtain

$$||M_i(\sum_{j \in J} \lambda_j y_i^j) + s_i(y^0)|| \leq \sum_{j \in J} \lambda_j ||M_i(y_i^j) + s_i(y^0)|| \text{ for each } i \in I,$$

and

$$\sum_{i \in I} ||M_i(\sum_{j \in J} \lambda_j y_i^j) + s_i(y^0)|| \leq \sum_{i \in I} \sum_{j \in J} \lambda_j ||M_i(y_i^j) + s_i(y^0)||$$

This is a contradiction with (2.3) and H is KKM mapping. From Lemma 1.6 it follows that there exists $\overline{x} \in K$ such that

$$\overline{x} \in H(x) \text{ for all } x \in K.$$

So,

$$\sum_{i \in I} ||M_i(\overline{x}_i) + s_i(\overline{x})|| \leq \sum_{i \in I} ||M_i(x_i) + s_i(\overline{x})|| \text{ for all } x \in K.$$

□

3. Some Applications

3.1. **Existence solutions the SNVI problem.** Applying Theorem 2.1, we have the following theorem on existence solutions the SNVI problem (1.3).

Theorem 3.1. *Let X be a normed space, K a nonempty convex compact subset of X, $S, T : K \times K \to X$ continuous mappings and $M, N : K \multimap X$ continuous convex mappings with compact values such that for every $(x, y) \in K \times K$*

$$0 \in M(K) + S(x, y) \text{ and } 0 \in N(K) + T(x, y). \tag{3.1}$$

Then there exists $(x_0, y_0) \in K \times K$ such that

$$0 \in S(x_0, y_0) + M(x_0) \text{ and } 0 \in T(x_0, y_0) + N(y_0).$$

Proof. From Theorem 2.1, we have that there exists $(x_0, y_0) \in K \times K$ such that

$$||M(x_0) + S(x_0, y_0)|| + ||N(y_0) + T(x_0, y_0)|| =$$

$$\inf_{(x,y)\in K\times K}\{||M(x) + S(x_0, y_0)|| + ||N(y) + T(x_0, y_0)||\}.$$

From condition (3.1) we obtain that

$$\inf_{(x,y)\in K\times K}\{||M(x) + S(x_0, y_0)|| + ||N(y) + T(x_0, y_0)||\} = 0,$$

so, we have

$$||M(x_0) + S(x_0, y_0)|| + ||N(y_0) + T(x_0, y_0)|| = 0,$$

hence,

$$0 \in M(x_0) + S(x_0, y_0) \text{ and } 0 \in N(y_0) + T(x_0, y_0).$$

□

3.2. **A Coupled Coincidence Point.**

Theorem 3.2. *Let X be a normed space, K a nonempty convex compact subset of X, $F : K \times K \to X$ continuous mapping and $G : K \multimap X$ continuous convex mapping with compact values such that $F(K \times K) \subseteq G(K)$. Then F and G have a multivalued coupled coincidence point.*

Proof. Put

$$S(x, y) = -F(x, y),\ T(x, y) = -F(y, x) \text{ for } x, y \in K,$$

$$M(x) = G(x),\ N(y) = G(y) \text{ for } x, y \in K.$$

Then S, T, M and N satisfies all of the requirements of Theorem 3.1. Therefore, there exists $(x_0, y_0) \in K$ such that

$$0 \in -F(x_0, y_0) + G(x_0) \text{ and } 0 \in -F(y_0, x_0) + G(y_0)$$

i. e.

$$F(x_0, y_0) \in G(x_0) \text{ and } F(y_0, x_0) \in G(y_0).$$

□

Corollary 3.3. *Let X be a normed space, K a nonempty convex compact subset of X, $F : K \times K \to X$ continuous mapping and $g : K \to X$ continuous convex mapping such that $F(K \times K) \subseteq g(K)$. Then F and g have a coupled coincidence point.*

Proof. Let $G(x) = \{g(x)\}$ and apply Theorem 3.2. □

Corollary 3.4. *(*[10, Theorem 3.2]*) Let X be a normed space, K a nonempty convex compact subset of X, $F : K \times K \to K$ continuous mapping. Then F has a coupled fixed point.*

Proof. Let $G(x) = \{x\}$ and apply Theorem 3.2. □

3.3. **A Coupled Best Approximations.**

Theorem 3.5. *Let X be a normed space, K a nonempty convex compact subset of X, $F : K \times K \to X$ continuous mapping and $G : K \multimap X$ continuous convex mapping with compact values. Then there exists $(x_0, y_0) \in K \times K$ such that*

$$||G(x_0) - F(x_0, y_0)|| + ||G(y_0) - F(y_0, x_0)|| = \qquad (3.2)$$

$$\inf_{(x,y)\in K\times K} \{||G(x) - F(x_0, y_0)|| + ||G(y) - F(y_0, x_0)||\}.$$

Proof. Put

$$S(x, y) = -F(x, y),\ T(x, y) = -F(y, x) \text{ for } x, y \in K,$$

$$M(x) = G(x),\ N(y) = G(y) \text{ for } x, y \in K.$$

Then S, T, M and N satisfies all of the requirements of Theorem 2.1. Therefore, there exists $(x_0, y_0) \in K \times K$ such that (3.2) holds. □

Corollary 3.6. *Let X be a normed space, K a nonempty convex compact subset of X, $F : K \times K \to X$ continuous mapping and $g : K \to X$ continuous almost-affine mapping. Then there exists $(x_0, y_0) \in K \times K$ such that*

$$||g(x_0) - F(x_0, y_0)|| + ||g(y_0) - F(y_0, x_0)|| =$$

$$\inf_{(x,y)\in K\times K} \{||g(x) - F(x_0, y_0)|| + ||g(y) - F(y_0, x_0)||\}.$$

Corollary 3.7. *Let X be a normed space, K a nonempty convex compact subset of X, $F : K \times K \to X$ continuous mapping. Then there exists $(x_0, y_0) \in K \times K$ such that*

$$||x_0 - F(x_0, y_0)|| + ||y_0 - F(y_0, x_0)|| = \inf_{(x,y)\in K\times K} \{||x - F(x_0, y_0)|| + ||y - F(y_0, x_0)||\}.$$

3.4. **Applications on best approximations.**

(1) (Ky Fan [8], Best approximation theorem.) Let K be a nonempty compact, convex subset of a normed linear space X and $f : K \to X$ a continuous function. Then there is an $x_0 \in K$ such that

$$||x_0 - f(x_0)|| = \inf_{x\in K} ||x - f(x_0)||.$$

(2) (Prolla [13], Best approximation theorem.) Let K be a nonempty compact, convex subset of a normed linear space X and $f : K \to X$ a continuous function and $g : K \to X$ a continuous, almost-affine, onto map. Then there is an $x_0 \in K$ such that

$$||g(x_0) - f(x_0)|| = \inf_{x\in K} ||x - f(x_0)||.$$

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