Ann. Funct. Anal. 4 (2013), no. 2, 97–109
ANNALS OF FUNCTIONAL ANALYSIS
ISSN: 2008-8752 (electronic)
URL:www.emis.de/journals/AFA/

ISHIKAWA TYPE ALGORITHM OF TWO MULTI-VALUED QUASI-NONEXPANSIVE MAPS ON NONLINEAR DOMAINS

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Communicated by M. A. Japon Pineda

ABSTRACT. We study an Ishikawa type algorithm for two multi-valued quasi-nonexpansive maps on a special class of nonlinear spaces namely hyperbolic metric spaces; in particular, strong and $\triangle-$convergence theorems for the proposed algorithms are established in a uniformly convex hyperbolic space which improve and extend the corresponding known results in uniformly convex Banach spaces. Our new results are also valid in geodesic spaces.

1. Introduction and preliminaries

A nonempty subset D of a metric space X is called proximinal if for each $x \in X$, there exists an element $y \in D$ such that $d(x,y) = d(x,D)$, where $d(x,D) = \inf\{d(x,z) : z \in D\}$. Let $CB(D)$, $K(D)$ and $P(D)$ denote the family of nonempty, closed and bounded subsets; nonempty, compact subsets and nonempty, proximinal and bounded subsets of D, respectively. Hausdorff metric on $CB(D)$ is defined by:

$$H(A,B) = \max\left\{\sup_{x\varepsilon A} d(x,B),\ \sup_{y\varepsilon B} d(y,A)\right\}$$

for all $A, B \in CB(D)$.
Let $T : D \to CB(D\ )$ be a multi-valued map. An element $p \in D$ is a fixed point of T if $p \in Tp$. The set of all fixed points of T is denoted by $F(T)$.We say that T is:

Date: Received: 12 September 2012; Revised: 11 December 2012; Accepted: 10 January 2013.

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2010 *Mathematics Subject Classification.* Primary 47H04; Secondary 47H09.

Key words and phrases. Hyperbolic space, multi-valued map, common fixed point, asymptotic centre, convergence.

(i) nonexpansive if $H(Tx,Ty)\leq d(x,y)$ for all $x,y\in D$
(ii) quasi-nonexpansive if $F(T)\neq\emptyset$ and $H(Tx,Tp)\leq d(x,p)$ for all $x\in D$ and all $p\in F(T)$
(iii) Lipschitzian if there exists a constant $L>0$ such that $H(Tx,Ty)\leq L\, d(x,y)$ for all $x,y\in D$
(iv) Lipschitzian quasi-nonexpansive if both (ii) and (iii) hold.

If $F(T)\neq\emptyset$, then the class of multi-valued quasi-nonexpansive maps properly contains the class of multi-valued nonexpansive maps.

In 1968, Markin [15] established convergence results for multi-valued nonexpansive maps in a Hilbert space. Later, some classical fixed point theorems for single-valued maps were extended to multi-valued maps; for example, Banach Contraction Principle was extended for multi-valued contractive maps in complete metric spaces by Nadler [16]. Shimizu and Takahashi [20] established existence of fixed points of multi-valued nonexpansive maps in certain convex metric spaces. The study of multi-valued maps is a rapidly growing area of research (see, for instance [1, 18, 19, 22]).

The algorithms with error term for single-valued maps in Banach spaces have been studied by many authors, see, e.g., [8, 21] and references therein.

Recently, Cholamjiak and Suntai [4] proposed and analyzed algorithms with bounded error term for multi-valued maps in Banach spaces as follows:

Let T_1 and T_2 be two quasi-nonexpansive multi-valued maps from D into $CB(D)$ where D is a convex subset of a Banach space. Then for $x_1\in D$, generate $\{x_n\}$ as

$$\begin{aligned} y_n &= \alpha_n' z_n' + \beta_n' x_n + \left(1-\alpha_n' - \beta_n'\right) u_n,\ \ n\geq 1 \\ x_{n+1} &= \alpha_n z_n + \beta_n x_n + (1-\alpha_n-\beta_n) v_n,\ \ n\geq 1 \end{aligned} \tag{1.1}$$

where $z_n'\in T_1x_n, z_n\in T_2y_n, 0\leq \alpha_n,\beta_n,\alpha_n+\beta_n,\alpha_n',\beta_n',\alpha_n'+\beta_n'\leq 1$ and $\{u_n\},\{v_n\}$ are bounded sequences in D.

Let T_1,T_2 be two multi-valued maps from D into $P(D)$ and $P_{T_i}x=\{y\in T_ix : d\left(x,y\right)=d(x,T_ix)\}, i=1,2$. Then for $x_1\in D$, generate $\{x_n\}$ as

$$\begin{aligned} y_n &= \alpha_n' z_n' + \beta_n' x_n + \left(1-\alpha_n' - \beta_n'\right) u_n,\ \ n\geq 1 \\ x_{n+1} &= \alpha_n z_n + \beta_n x_n + (1-\alpha_n-\beta_n) v_n,\ \ n\geq 1 \end{aligned} \tag{1.2}$$

where $z_n'\in P_{T_1}x_n$ and $z_n\in P_{T_2}y_2, 0\leq \alpha_n,\beta_n,\alpha_n+\beta_n,\alpha_n',\beta_n',\alpha_n'+\beta_n'\leq 1$ and $\{u_n\},\{v_n\}$ are bounded sequences in D.

Inspired and motivated by the work of Cholamjiak and Suntai [4], we translate algorithms (1.1- 1.2) in the general setup of $W-$hyperbolic spaces and approximate a common fixed point of two multi-valued quasi-nonexpansive maps.

Kohlenbach [11] introduced a general setup known as $W-$hyperbolic spaces which contains as a special case Banach spaces as well as $CAT(0)$ spaces.

A $W-$hyperbolic space (X,d,W) is a metric space (X,d) together with a map $W: X^2\times[0,1]\to X$ satisfying

(i) $d(u,W(x,y,\alpha))\leq(1-\alpha)d(u,x)+\alpha d(u,y)$
(ii) $d(W(x,y,\alpha),W(x,y,\beta))=|\alpha-\beta|\, d(x,y)$
(iii) $W(x,y,\alpha)=W(y,x,1-\alpha)$
(iv)$d(W(x,z,\alpha),W(y,w,\alpha))\leq(1-\alpha)d(x,y)+\alpha d(z,w)$

for all $x, y, z, w \in X$ and $\alpha, \beta \in [0,1]$. The triplet (X, d, W) satisfying only (i) is the convex metric space due to Takahashi [23]. A subset K of a $W-$hyperbolic space X is convex if $W(x, y, \alpha) \in K$ for all $x, y \in K$ and $\alpha \in [0,1]$.
The class of $W-$hyperbolic spaces contains normed spaces and their convex subsets as subclasses and $CAT(0)$ spaces form a very special subclass of the class of $W-$hyperbolic spaces with unique geodesic paths.

A $W-$hyperbolic space X is uniformly convex [20] if for all $u, x, y \in X$, $r > 0$ and $\varepsilon \in (0,2]$, there exists a $\delta \in (0,1]$ such that $d\left(W\left(x, y, \frac{1}{2}\right), u\right) \leq (1-\delta)r$, whenever $d(x,u) \leq r, d(y,u) \leq r$ and $d(x,y) \geq \varepsilon r$.

A map $\eta : (0,\infty) \times (0,2] \to (0,1]$ which provides such a $\delta = \eta(r,\varepsilon)$ for given $r > 0$ and $\varepsilon \in (0,2]$, is called a modulus of uniform convexity of X . We call η monotone if it decreases with r (for a fixed ε).

It has been shown in [13] that $CAT(0)$ spaces are uniformly convex $W-$hyperbolic spaces with modulus of uniform convexity $\eta(r,\varepsilon) = \frac{\varepsilon^2}{8}$. Thus, uniformly convex $W-$hyperbolic spaces are a natural generalization of both uniformly convex Banach spaces and $CAT(0)$ spaces. For details about $CAT(0)$ spaces, see [2] and [9].

Now we transform (1.1) and (1.2) in a $W-$hyperbolic space.

Let T_1 and T_2 be two quasi-nonexpansive multi-valued maps from D into $CB(D)$ where D is a convex subset of a hyperbolic space. Then for $x_1 \in D$, generate $\{x_n\}$ as

$$\begin{aligned} y_n &= W\left(z_n^{'}, W\left(x_n, u_n, \tfrac{\beta_n^{'}}{1-\alpha_n^{'}}\right), \alpha_n^{'}\right),\ n \geq 1, \\ x_{n+1} &= W\left(z_n, W\left(y_n, v_n, \tfrac{\beta_n}{1-\alpha_n}\right), \alpha_n\right),\ n \geq 1, \end{aligned} \tag{1.3}$$

where $z_n^{'} \in T_1 x_n, z_n \in T_2 y_n, 0 \leq \alpha_n, \beta_n, \alpha_n + \beta_n, \alpha_n^{'}, \beta_n^{'}, \alpha_n^{'} + \beta_n^{'} \leq 1, \{u_n\}$ and $\{v_n\}$ are bounded in D.

Let T_1 and T_2 be two multi-valued maps from D into $P(D)$ and $P_{T_i}x = \{y \in T_i x : d(x,y) = d(x, T_i x)\}, i = 1,2$. Then for $x_1 \in D$, generate $\{x_n\}$ as

$$\begin{aligned} y_n &= W\left(z_n^{'}, W\left(x_n, u_n, \tfrac{\beta_n^{'}}{1-\alpha_n^{'}}\right), \alpha_n^{'}\right),\ n \geq 1, \\ x_{n+1} &= W\left(z_n, W\left(y_n, v_n, \tfrac{\beta_n}{1-\alpha_n}\right), \alpha_n\right),\ n \geq 1, \end{aligned} \tag{1.4}$$

where $z_n^{'} \in P_{T_1} x_n$ and $z_n \in P_{T_2} y_2, 0 \leq \alpha_n, \beta_n, \alpha_n + \beta_n, \alpha_n^{'}, \beta_n^{'}, \alpha_n^{'} + \beta_n^{'} \leq 1, \{u_n\}$ and $\{v_n\}$ are bounded in D.

It is worth mentioning that the algorithms (1.3-1.4) coincide with the algorithms (1.1-1.2) when $W(x,y,\alpha) = \alpha x + (1-\alpha)y$ and X is a Banach space. Moreover, they provide algorithms in a $CAT(0)$ space if $W(x,y,\alpha) = \alpha x \oplus (1-\alpha)y$.

Let $\{x_n\}$ be a bounded sequence in a metric space X. For $x \in X$, define a continuous functional

$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n).$$

Then
(i) $r_K(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in K\}$ is called the asymptotic radius of $\{x_n\}$ with respect to $K \subset X$,

(ii) for any $y \in K$, the set $A_K(\{x_n\}) = \{x \in K : r(x, \{x_n\} \leq r(y, \{x_n\})\}$ is called the asymptotic center of $\{x_n\}$ with respect to $K \subset X$.

If the asymptotic radius and the asymptotic center is taken with respect to X, then these are simply denoted by $r(\{x_n\})$ and $A(\{x_n\})$, respectively. In general, $A(\{x_n\})$ may be empty or may contain infinitely many points. Through asymptotic center technique of Edelstein [5] in Banach fixed point theory, one can conclude that bounded sequences in general $W-$hyperbolic and normed spaces do not have unique asymptotic center with respect to closed convex subsets. However, it is remarkable that a complete uniformly convex $W-$hyperbolic space with monotone modulus of uniform convexity enjoys this property [13].

In 2008, Kirk and Panyanak [10] proposed a new type of convergence in geodesic spaces, namely $\triangle-$convergence, which was originally introduced by Lim [14]. They showed that $\triangle-$convergence coincides with weak convergence in Banach spaces satisfying the Opial condition and both concepts share many common properties. For a general iteration scheme in $CAT(0)$ spaces, we refer the reader to [6].

A sequence $\{x_n\}$ in X is said to $\triangle-$converge to $x \in X$ if x is the unique asymptotic center for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case, we write x as $\triangle-$limit of $\{x_n\}$, *i.e.*, $\triangle - \lim_n x_n = x$.

For two multi-valued maps T_1 and T_2, we set $F = F(T_1) \cap F(T_2) \neq \emptyset$.

Lemma 1.1. [3]*If $\{a_n\}$ and $\{b_n\}$ are sequences of non-negative real numbers satisfying $a_{n+1} \leq a_n + b_n$, $n \geq 1$ and $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.*

Lemma 1.2. [7]*Let (X, d, W) be a uniformly convex hyperbolic space with monotone modulus of uniform convexity η. Let $x \in X$ and $\{\alpha_n\}$ be a sequence in $[b, c]$ for some $b, c \in (0, 1)$. If $\{x_n\}$ and $\{y_n\}$ are sequences in X with*

$$\limsup_{n\longrightarrow\infty} d(x_n, x) \leq r, \limsup_{n\longrightarrow\infty} d(y_n, x) \leq r, \lim_{n\longrightarrow\infty} d(W(x_n, y_n, \alpha_n), x) = r$$

for some $r \geq 0$, then $\lim_{n\to\infty} d(x_n, y_n) = 0$.

Lemma 1.3. [7]*Let K be a nonempty, closed convex subset of a uniformly convex hyperbolic space and $\{x_n\}$ a bounded sequence in K such that $A(\{x_n\}) = \{y\}$. If $\{y_m\}$ is another sequence in K such that $\lim_{m\to\infty} r(y_m, \{x_n\}) = \rho$, then $\lim_{m\to\infty} y_m = y$.*

2. Main results

The following lemma collects some inequalities which are needed in the sequel.

Lemma 2.1. *Let D be a nonempty, closed and convex subset of a $W-$hyperbolic space X. Let T_1 and T_2 be two multi-valued quasi-nonexpansive maps from D into $CB(D)$ such that $T_1 p = \{p\} = T_2 p$ for all $p \in F \neq \emptyset$. Then for the algorithm $\{x_n\}$ defined by (1.3) with $0 < l \leq \alpha_n, \alpha_n^{'} \leq k < 1, p \in F$, we have*

(i) $d(y_n,p) \le d(x_n,p) + \left(1-\alpha_n^{'}-\beta_n^{'}\right) h$ *for some* $h>0$
(ii) $d(x_{n+1},p) \le d(x_n,p) + \left\{(\alpha_n+\beta_n)\left(1-\alpha_n^{'}-\beta_n^{'}\right) + (1-\alpha_n-\beta_n)\right\} h$ *for some* $h>0$
(iii) $d\left(W\left(y_n,v_n,\frac{\beta_n}{1-\alpha_n}\right),p\right) \le d(y_n,p) + \left(\frac{1-\alpha_n-\beta_n}{1-k}\right) d\left(y_n,v_n\right)$
(iv) $d(y_n,z_n) \le \left(\frac{1-\alpha_n-\beta_n}{1-k}\right) d(y_n,v_n) + d\left(z_n, W\left(y_n,v_n,\frac{\beta_n}{1-\alpha_n}\right)\right)$
(v) $d\left(W\left(x_n,u_n,\frac{\beta_n^{'}}{1-\alpha_n^{'}}\right),p\right) \le d(x_n,p) + \left(\frac{1-\alpha_n^{'}-\beta_n^{'}}{1-k}\right) d\left(u_n,x_n\right)$
(vi) $d(z_n^{'},x_n) \le d\left(z_n^{'}, W\left(x_n,u_n,\frac{\beta_n^{'}}{1-\alpha_n^{'}}\right)\right) + \left(\frac{1-\alpha_n^{'}-\beta_n^{'}}{1-k}\right) d(u_n,x_n).$

Proof. (i) Set $\max\{\sup_{n\in N} d(u_n,p), \sup_{n\in N} d(v_n,p)\} < h$ for some $h>0$ because $\{u_n\}$ and $\{v_n\}$ are bounded sequences.

We observe that

$$\begin{aligned}
d(y_n,p) &= d\left(W\left(z_n^{'}, W\left(x_n,u_n,\frac{\beta_n^{'}}{1-\alpha_n^{'}}\right),\alpha_n^{'}\right),p\right)\\
&\le \alpha_n^{'} d\left(z_n^{'},p\right) + \left(1-\alpha_n^{'}\right) d\left(W\left(x_n,u_n,\frac{\beta_n^{'}}{1-\alpha_n^{'}}\right),p\right)\\
&\le \alpha_n^{'} d\left(z_n^{'},p\right) + \beta_n^{'} d(x_n,p) + \left(1-\alpha_n^{'}-\beta_n^{'}\right) d(u_n,p)\\
&\le \alpha_n^{'} d\left(z_n^{'},T_1p\right) + \beta_n^{'} d(x_n,p) + \left(1-\alpha_n^{'}-\beta_n^{'}\right) h\\
&\le \alpha_n^{'} H\left(T_1x_n,T_1p\right) + \beta_n^{'} d(x_n,p) + \left(1-\alpha_n^{'}-\beta_n^{'}\right) h\\
&\le \alpha_n^{'} d(x_n,p) + \beta_n^{'} d(x_n,p) + \left(1-\alpha_n^{'}-\beta_n^{'}\right) h\\
&= (\alpha_n^{'}+\beta_n^{'}) d(x_n,p) + \left(1-\alpha_n^{'}-\beta_n^{'}\right) h\\
&\le d(x_n,p) + \left(1-\alpha_n^{'}-\beta_n^{'}\right) h.
\end{aligned}$$

(ii) Utilizing (i), we have

$$\begin{aligned}
d(x_{n+1},p) &= d\left(W\left(z_n, W\left(y_n,v_n,\frac{\beta_n}{1-\alpha_n}\right),\alpha_n\right),p\right)\\
&\le \alpha_n d(z_n,p) + (1-\alpha_n) d\left(W\left(y_n,v_n,\frac{\beta_n}{1-\alpha_n}\right),p\right)\\
&\le \alpha_n d(z_n,p) + \beta_n d(y_n,p) + (1-\alpha_n-\beta_n) d(v_n,p)\\
&\le \alpha_n H(T_2y_n,T_2p) + \beta_n d(y_n,p) + (1-\alpha_n-\beta_n) h\\
&\le (\alpha_n+\beta_n) d(y_n,p) + (1-\alpha_n-\beta_n) h\\
&\le (\alpha_n+\beta_n)\left\{d(x_n,p) + \left(1-\alpha_n^{'}-\beta_n^{'}\right) h\right\} + (1-\alpha_n-\beta_n) h\\
&\le d(x_n,p) + \left\{(\alpha_n+\beta_n)\left(1-\alpha_n^{'}-\beta_n^{'}\right) + (1-\alpha_n-\beta_n)\right\} h.
\end{aligned}$$

(iii) Since

$$\begin{aligned} d\Big(W\left(y_n, v_n, \frac{\beta_n}{1-\alpha_n}\right), p\Big) &\leq \frac{\beta_n}{1-\alpha_n} d(y_n, p) + \left(1 - \frac{\beta_n}{1-\alpha_n}\right) d(v_n, p) \\ &\leq \frac{\beta_n}{1-\alpha_n} d(y_n, p) + \left(1 - \frac{\beta_n}{1-\alpha_n}\right) \{d(v_n, y_n) + d(y_n, p)\} \\ &\leq d(y_n, p) + \left(\frac{1-\alpha_n-\beta_n}{1-\alpha_n}\right) d\left(v_n, y_n\right) \end{aligned}$$

and $0 < l \leq \alpha_n \leq k < 1$, therefore we have

$$d\left(W\left(y_n, v_n, \frac{\beta_n}{1-\alpha_n}\right), p\right) \leq d\left(y_n, p\right) + \left(\frac{1-\alpha_n-\beta_n}{1-k}\right) d\left(v_n, y_n\right).$$

(iv) From

$$\begin{aligned} d(y_n, x_{n+1}) &= d\left(y_n, W\left(z_n, W\left(y_n, v_n, \frac{\beta_n}{1-\alpha_n}\right), \alpha_n\right)\right) \\ &\leq \alpha_n d(y_n, z_n) + (1-\alpha_n) d\left(y_n, W\left(y_n, v_n, \frac{\beta_n}{1-\alpha_n}\right)\right) \\ &\leq \alpha_n d(y_n, z_n) + (1-\alpha_n-\beta_n) d(y_n, v_n) \end{aligned}$$

and

$$\begin{aligned} d(z_n, x_{n+1}) &= d\left(z_n, W\left(z_n, W\left(y_n, v_n, \frac{\beta_n}{1-\alpha_n}\right), \alpha_n\right)\right) \\ &\leq (1-\alpha_n) d\left(z_n, W\left(y_n, v_n, \frac{\beta_n}{1-\alpha_n}\right)\right), \end{aligned}$$

we have

$$\begin{aligned} d(y_n, z_n) &\leq d(y_n, x_{n+1}) + d(x_{n+1}, z_n) \\ &\leq \alpha_n d(y_n, z_n) + (1-\alpha_n-\beta_n) d(y_n, v_n) \\ &\quad + (1-\alpha_n) d\left(z_n, W\left(y_n, v_n, \frac{\beta_n}{1-\alpha_n}\right)\right). \end{aligned}$$

Rearranging the terms in the above inequality and using $0 < l \leq \alpha_n \leq k < 1$, we get

$$d\left(y_n, z_n\right) \leq \left(\frac{1-\alpha_n-\beta_n}{1-k}\right) d\left(y_n, v_n\right) + d\left(z_n, W\left(y_n, v_n, \frac{\beta_n}{1-\alpha_n}\right)\right).$$

(v) Since

$$\begin{aligned} d\left(W\left(x_n, u_n, \frac{\beta_n'}{1-\alpha_n'}\right), p\right) &\leq \frac{\beta_n'}{1-\alpha_n'} d(x_n, p) + \left(1 - \frac{\beta_n'}{1-\alpha_n'}\right) d(u_n, p) \\ &\leq \left(1 - \frac{\beta_n'}{1-\alpha_n'}\right)\{d(u_n, x_n) + d(x_n, p)\} \\ &\quad + \frac{\beta_n'}{1-\alpha_n'} d(x_n, p) \\ &\leq d(x_n, p) + \left(\frac{1-\alpha_n' - \beta_n'}{1-k}\right) d(u_n, x_n). \end{aligned}$$

and $0 < l \leq \alpha_n' \leq k < 1$, therefore we have

$$d\left(W\left(x_n, u_n, \frac{\beta_n'}{1-\alpha_n'}\right), p\right) \leq d(x_n, p) + \left(\frac{1-\alpha_n' - \beta_n'}{1-k}\right) d(u_n, x_n).$$

(vi) From

$$\begin{aligned} d\left(z_n', y_n\right) &= d\left(z_n', W\left(z_n', W\left(x_n, u_n, \frac{\beta_n'}{1-\alpha_n'}\right), \alpha_n'\right)\right) \\ &\leq \left(1-\alpha_n'\right) d\left(z_n', W\left(x_n, u_n, \frac{\beta_n'}{1-\alpha_n'}\right)\right) \end{aligned}$$

and

$$\begin{aligned} d(y_n, x_n) &\leq d\left(W\left(z_n', W\left(x_n, u_n, \frac{\beta_n'}{1-\alpha_n'}\right), \alpha_n'\right), x_n\right) \\ &\leq \alpha_n' d\left(x_n, z_n'\right) + \left(1-\alpha_n' - \beta_n'\right) d\left(x_n, u_n\right), \end{aligned}$$

we obtain

$$\begin{aligned} d(z_n', x_n) &\leq d\left(z_n', y_n\right) + d(y_n, x_n) \\ &\leq \left(1-\alpha_n'\right) d\left(z_n', W\left(x_n, u_n, \frac{\beta_n'}{1-\alpha_n'}\right)\right) \\ &\quad + \alpha_n' d\left(x_n, z_n'\right) + \left(1-\alpha_n' - \beta_n'\right) d\left(x_n, u_n\right). \end{aligned}$$

Rearranging the terms in the above inequality and using $0 < l \leq \alpha_n' \leq k < 1$, we get $d(z_n', x_n) \leq d\left(z_n', W\left(x_n, u_n, \frac{\beta_n'}{1-\alpha_n'}\right)\right) + \left(\frac{1-\alpha_n'-\beta_n'}{1-k}\right) d\left(x_n, u_n\right).$ □

Lemma 2.2. *Let D be a nonempty, closed and convex subset of a uniformly convex $W-$hyperbolic space X. Let T_1 and T_2 be two multi-valued Lipschitzian quasi-nonexpansive maps from D into $CB(D)$ such that $T_1 p = \{p\} = T_2 p$ for all $p \in F \neq \emptyset$. Then for the algorithm $\{x_n\}$ defined by (1.3) with $0 < l \leq \alpha_n, \alpha_n' \leq k < 1$, $\sum_{n=1}^{\infty}(1-\alpha_n-\beta_n) < \infty$ and $\sum_{n=1}^{\infty}\left(1-\alpha_n'-\beta_n'\right) < \infty$, we have*

$$\lim_{n\to\infty} d(x_n, T_1 x_n) = 0 = \lim_{n\to\infty} d(x_n, T_2 x_n).$$

Proof. Since $\sum_{n=1}^{\infty}(1-\alpha_n-\beta_n)<\infty$ and $\sum_{n=1}^{\infty}(1-\alpha_n^{'}-\beta_n^{'})<\infty$, therefore Lemma 2.1 (ii) and Lemma 1.1 give that $\lim_{n\to\infty} d(x_n,p)$ exists. Assume that $\lim_{n\to\infty} d(x_n,p)=c\geq 0$. Then it follows from Lemma 2.1 (i) that $\limsup_{n\to\infty} d(y_n,p)\leq c$. As $\{x_n\}$, $\{y_n\}$, $\{u_n\}$ and $\{v_n\}$ are bounded sequences, so $\max\{\sup_{n\in N} d(v_n,y_n),\sup_{n\in N} d(u_n,x_n)\}<\infty$. Also observe that

$$\lim_{n\to\infty} d\left(W\left(z_n, W\left(y_n, v_n, \frac{\beta_n}{1-\alpha_n}\right),\alpha_n\right),p\right)=\lim_{n\to\infty} d(x_{n+1},p)=c.$$

Moreover, the inequality $d(z_n,p)\leq H(T_2y_n,T_2p)\leq d(y_n,p)$ and Lemma 2.1 (iii) imply that $\limsup_{n\to\infty} d(z_n,p)\leq c$ and $\limsup_{n\to\infty} d\left(W\left(y_n,v_n,\frac{\beta_n}{1-\alpha_n}\right),p\right)\leq c$, respectively. By Lemma 1.2, we have

$$\lim_{n\to\infty} d\left(W\left(y_n, v_n, \frac{\beta_n}{1-\alpha_n}\right),z_n\right)=0. \tag{2.1}$$

Taking lim sup on both sides in Lemma 2.1 (iv) and using (2.1), we have

$$\lim_{n\to\infty} d(y_n,z_n)=0. \tag{2.2}$$

Further,

$$\begin{aligned} d(x_{n+1},p) &= d\left(W\left(z_n, W\left(y_n, v_n, \frac{\beta_n}{1-\alpha_n}\right),\alpha_n\right),p\right) \\ &\leq \alpha_n d(z_n,p)+\beta_n d(y_n,p)+(1-\alpha_n-\beta_n)d(v_n,p) \\ &\leq \alpha_n d(z_n,y_n)+(\alpha_n+\beta_n)d(y_n,p)+(1-\alpha_n-\beta_n)h \end{aligned}$$

implies that $c\leq \liminf_{n\to\infty} d(y_n,p)$. This, in conjunction with $\limsup_{n\to\infty} d(y_n,p)\leq c$, implies that

$$\lim_{n\to\infty} d\left(W\left(z_n^{'}, W\left(x_n, u_n, \frac{\beta_n^{'}}{1-\alpha_n^{'}}\right),\alpha_n^{'}\right),p\right)=\lim_{n\to\infty} d(y_n,p)=c.$$

Also, the inequality $d\left(z_n^{'},p\right)\leq H(T_1x_n,T_1p)\leq d(x_n,p)$ and Lemma 2.1 (v) imply that $\limsup_{n\to\infty} d\left(z_n^{'},p\right)\leq c$ and $\limsup_{n\to\infty} d\left(W\left(x_n,u_n,\frac{\beta_n^{'}}{1-\alpha_n^{'}}\right),p\right)\leq c$, respectively. Again by Lemma 1.2, we have

$$\lim_{n\to\infty} \ d\left(z_n^{'}, W\left(x_n, u_n, \frac{\beta_n^{'}}{1-\alpha_n^{'}}\right)\right)=0. \tag{2.3}$$

Taking lim sup on both sides in Lemma 2.1 (vi) and using (2.3), we get

$$\lim_{n\to\infty} d(z_n^{'},x_n)=0. \tag{2.4}$$

As $z_n^{'}\in T_1x_n$, so $d(x_n,T_1x_n)\leq d(z_n^{'},x_n)$ which implies, on letting $n\to\infty$,

$$\lim_{n\to\infty} d(x_n,T_1x_n)=0.$$

As $\{x_n\}$ and$\{u_n\}$ are bounded, so is $\left\{d\left(u_n,z_n^{'}\right)\right\}$. Let $K=sup_{n\in N}d\left(u_n,z_n^{'}\right)$.

Then it follows from an inequality in the proof of Lemma 2.1 (vi) and (2.4) that

$$\begin{aligned} d\left(y_n, z_n^{'}\right) &\leq \beta_n^{'} d\left(z_n^{'}, x_n\right) + \left(1-\alpha_n^{'}-\beta_n^{'}\right) d\left(u_n, z_n^{'}\right) \\ &\leq \beta_n^{'} d\left(z_n^{'}, x_n\right) + \left(1-\alpha_n^{'}-\beta_n^{'}\right) K \to 0 \text{ as } n \to \infty. \end{aligned} \tag{2.5}$$

It follows from (2.4) and (2.5) that

$$d\left(y_n, x_n\right) \leq d\left(y_n, z_n^{'}\right) + d\left(z_n^{'}, x_n\right) \to 0 \text{ as } n \to \infty. \tag{2.6}$$

Using (2.2), (2.6) and the fact that $z_n \in T_2 y_n$, we get

$$\begin{aligned} d(x_n, T_2 x_n) &\leq d(x_n, y_n) + d(y_n, z_n) + d(z_n, T_2 x_n) \\ &\leq d(x_n, y_n) + d(y_n, z_n) + H\left(T_2 y_n, T_2 x_n\right) \\ &\leq d(x_n, y_n) + d(y_n, z_n) + L d\left(y_n, x_n\right) \to 0 \text{ as } n \to \infty. \end{aligned}$$

That is, $\lim_{n\to\infty} d(x_n, T_1 x_n) = 0 = \lim_{n\to\infty} d(x_n, T_2 x_n)$. □

Our next result deals with $\triangle-$convergence of the algorithm (1.3).

Theorem 2.3. *Let D be a nonempty, closed and convex subset of a complete uniformly convex $W-$hyperbolic space X with monotone modulus of uniform convexity η and let T_1 and T_2 be two multi-valued Lipschitzian quasi-nonexpansive maps from D into $CB(D)$ with $T_1 p = \{p\} = T_2 p$ for all $p \in F \neq \emptyset$.Then the algorithm $\{x_n\}$ in (1.3) with $0 < l \leq \alpha_n, \alpha_n^{'} \leq k < 1$, $\sum_{n=1}^{\infty}(1-\alpha_n-\beta_n) < \infty$ and $\sum_{n=1}^{\infty}\left(1-\alpha_n^{'}-\beta_n^{'}\right) < \infty$, $\triangle-$ converges to a point in F.*

Proof. As $\{d(x_n, p)\}$ converges, therefore $\{x_n\}$ is bounded. Hence $\{x_n\}$ has a unique asymptotic centre, that is, $A(\{x_n\}) = \{x\}$. Let $\{u_n\}$ be any subsequence of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. Then by Lemma 2.2, we have $\lim_{n\to\infty} d(u_n, T_1 u_n) = 0 = \lim_{n\to\infty} d(u_n, T_2 u_n)$. Denote $w_w(x_n) = \cup A(\{u_n\})$, where union is taken over all subsequences $\{u_n\}$ of $\{x_n\}$. Let $u \in w_w(x_n)$. Now we show that $u \in T_1 u$. For this, we consider a sequence $z_{n_k} \in T_1 u$ such that

$$\begin{aligned} d(z_{n_k}, u_n) &\leq d(z_{n_k}, T_1 u_n) + d(T_1 u_n, u_n) \\ &\leq H(T_1 u, T_1 u_n) + d(T_1 u_n, u_n) \\ &\leq d(u, u_n) + d(T_1 u_n, u_n). \end{aligned}$$

Therefore, we have

$$r(z_{n_k}, \{u_n\}) = \limsup_{n\to\infty} d(z_{n_k}, u_n) \leq \limsup_{n\to\infty} d(u, u_n) = r(u, \{u_n\}).$$

This implies that $|r(z_{n_k}, \{u_n\}) - r(u, \{u_n\})| \to 0$ as $k \to \infty$. It follows from Lemma 1.3 that $\lim_{k\to\infty} z_{n_k} = u$. Since $T_1 u$ is closed, therefore $u \in T_1 u$. That is, $u \in F(T_1)$. Similarly, we can show that $u \in F(T_2)$. Hence $u \in F$. Next, we show that every subsequence $\{u_n\}$ of $\{x_n\}$ has the the same center. That is, $w_w(x_n)$ is singleton.We have already assumed that $A(\{x_n\}) = \{x\}$ and $A(\{u_n\}) = \{u\}$.

As $u \in F$, so $\lim_{n\to\infty} d(x_n, u)$ exists by applying Lemma 1.1 to (ii) in Lemma 2.1. Suppose $x \neq u$. Then by the uniqueness of asymptotic centre, we have

$$\begin{aligned}\limsup_{n\to\infty} d(u_n, u) &< \limsup_{n\to\infty} d(u_n, x) \\ &\leq \limsup_{n\to\infty} d(x_n, x) \\ &< \limsup_{n\to\infty} d(x_n, u) \\ &= \limsup_{n\to\infty} d(u_n, u),\end{aligned}$$

a contradiction. This proves that $\{x_n\}$, $\triangle-$ converges to a point in F. □

Remark 2.4. *Theorem 2.3 extends Theorem 4.6 in [12] to the case of two multi-valued quasi-nonexpansive maps in a uniformly convex $W-$hyperbolic space. Moreover, the algorithm (1.3) is independent of compactness of the domain of maps.*

Recall that a multi-valued map $T : D \to CB(D)$ is *hemi-compact* if any bounded sequence $\{x_n\}$ in D satisfying $d(x_n, Tx_n) \to 0$ as $n \to \infty$, has a convergent subsequence.

A multi-valued map $T : D \to CB(D)$ is said to satisfy *condition (I)* if there is a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$, $f(t) > 0$ for $t \in (0, \infty)$ such that $d(x, Tx) \geq f(d(x, F))$ for all $x \in D$.
Two multi-valued maps $T_1, T_2 : D \to CB(D)$ are said to satisfy *condition(II)* if there is a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$, $f(r) > 0$ for $r \in (0, \infty)$
such that either $d(x, T_1x) \geq f(d(x, F))$ or $d(x, T_2x) \geq f(d(x, F))$ holds for all $x \in D$.

The following result gives a necessary and sufficient condition for strong convergence of the algorithm (1.3) in a complete $W-$hyperbolic space.

Theorem 2.5. *Let D be a nonempty, closed and convex subset of a complete uniformly convex $W-$hyperbolic space X and let T_1, T_2 be two multi-valued Lipschitzian quasi-nonexpansive maps from D into $CB(D)$ with $F \neq \emptyset$. Then the algorithm $\{x_n\}$ in (1.3) with $\sum_{n=1}^{\infty}(1-\alpha_n-\beta_n) < \infty$ and $\sum_{n=1}^{\infty}\left(1-\alpha_n^{'}-\beta_n^{'}\right) < \infty$, converges strongly to a point in F if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$.*

Proof. If $\{x_n\}$ converges to $p \in F$, then $\lim_{n\to\infty} d(x_n, p) = 0$. Since $0 \leq d(x_n, F) \leq d(x_n, p)$, we have $\liminf_{n\to\infty} d(x_n, F) = 0$.
Conversely, suppose $\liminf_{n\to\infty} d(x_n, F) = 0$. Since $\liminf_{n\to\infty} d(x_n, F) = 0$ and $\lim_{n\to\infty} d(x_n, F)$ exists through Lemma 2.1 (ii), therefore $\lim_{n\to\infty} d(x_n, F) = 0$.
Next, we show that $\{x_n\}$ is a Cauchy sequence. Let $\varepsilon > 0$. Since $\lim_{n\to\infty} d(x_n, F) = 0$ and $\sum_{n=1}^{\infty} h_n < \infty$ where $h_n = \left\{(\alpha_n + \beta_n)\left(1-\alpha_n^{'}-\beta_n^{'}\right) + (1-\alpha_n-\beta_n)\right\} h$ for some $h > 0$ as in Lemma 2.1 (ii), therefore there exists $n_0 \geq 1$ such that for all $n \geq n_0$, we have that $d(x_n, F) < \frac{\varepsilon}{5}$ and $\sum_{j=n_0}^{\infty} h_j < \frac{\varepsilon}{4}$. In particular, $d(x_{n_0}, F) < \frac{\varepsilon}{5}$. That is, $\inf\{d(x_{n_0}, p) : p \in F\} < \frac{\varepsilon}{5}$.There must exist $p^* \in F$ such that $d(x_{n_0}, p^*) < \frac{\varepsilon}{4}$.

Note that, for any $n > m \geq n_0$, we have

$$\begin{aligned} d(x_{n+m}, x_n) &\leq d(x_{n+m}, p^*) + d(x_n, p^*) \\ &\leq d(x_{n+m-1}, p^*) + h_{n+m-1} + d(x_{n-1}, p^*) + h_{n-1} \\ &\leq 2d(x_{n_0}, p^*) + \sum_{j=n_0}^{n+m-1} h_j + \sum_{j=n_0}^{n-1} h_j \\ &\leq 2\left(d(x_{n_0}, p^*) + \sum_{j=n_0}^{n+m-1} h_j\right) \\ &\leq 2\left(d(x_{n_0}, p^*) + \sum_{j=n_0}^{\infty} h_j\right) \\ &\leq 2\left(\frac{\varepsilon}{4} + \frac{\varepsilon}{4}\right) = \varepsilon. \end{aligned}$$

This proves that $\{x_n\}$ is a Cauchy sequence in X and so $\lim_{n\to\infty} x_n = q$ (say). We claim that $q \in F$. Indeed, let $\varepsilon > 0$, then there exists an integer $n_1 \geq 1$ such that $d(x_n, q) < \frac{\varepsilon}{4}$ for all $n \geq n_1$. Also $\lim_{n\to\infty} d(x_n, F) = 0$ implies that there exists an integer $n_2 \geq 1$ such that $d(x_n, F) < \frac{\varepsilon}{5}$ for all $n \geq n_2$. Choose $n_3 = \max(n_1, n_2)$. Hence there exists $q_0 \in F$ such that $d(x_{n_3}, q_0) < \frac{\varepsilon}{4}$. Therefore, we have

$$\begin{aligned} d(T_1 q, q) &\leq d(T_1 q, q_0) + d(q, q_0) \leq 2d\,(q, q_0) \leq 2\,(d(x_{n_3}, q) + d(x_{n_3}, q_0)) \\ &< 2\left(\frac{\varepsilon}{4} + \frac{\varepsilon}{4}\right) = \varepsilon. \end{aligned}$$

Therefore, we have $d(T_1 q, q) = 0$. Similarly, we can show that $d(T_2 q, q) = 0$. Hence $q \in F$. □

As an application of Theorem 2.5, the following strong convergence result can be easily proved by using Lemma 2.2.

Theorem 2.6. *Let D be a nonempty, closed and convex subset of a complete uniformly convex $W-$hyperbolic space X. Let T_1, T_2 be two multi-valued Lipschitzian quasi-nonexpansive maps from D into $CB(D)$ with $F \neq \emptyset$ and either of the two maps is hemi-compact or satisfies Condition (II). Then the algorithm $\{x_n\}$ in (1.3) with $0 < l \leq \alpha_n, \alpha_n' \leq k < 1$, $\sum_{n=1}^{\infty}(1 - \alpha_n - \beta_n) < \infty$ and $\sum_{n=1}^{\infty}\left(1 - \alpha_n' - \beta_n'\right) < \infty$, strongly converges to a point in F.*

Remark 2.7. (i) The algorithm (1.3) generalizes algorithm (2.1) of [4] and extends algorithm (1.2) of [17] for multi-valued maps in $W-$hyperbolic spaces (ii) Theorem 2.5 extends ([1], Theorem 4) to the case of two multi-valued quasi-nonexpansive maps for the algorithm (1.3) which is different from the algorithm defined by Abbas et al .[1] (iii) Theorem 2.5 generalizes ([4], Theorem 2.5) from Banach spaces to $W-$hyperbolic spaces (iv) Our results also hold in $CAT(0)$ spaces and generalizes the corresponding results in [12, 18].

We can also obtain approximation results for the algorithm (1.4). As the calculations in these results are similar to those in the above results, so we omit their proofs.

Theorem 2.8. *Let D be a nonempty, closed and convex subset of a complete uniformly convex $W-$hyperbolic space X with monotone modulus of uniform convexity η and let T_1 and T_2 be two multi-valued maps from D into $P(D)$ with $F \neq \emptyset$ such that P_{T_1} and P_{T_2} are nonexpansive. Then the algorithm $\{x_n\}$ in (1.4) with $0 < l \leq \alpha_n, \alpha_n^{'} \leq k < 1$, $\sum_{n=1}^{\infty}(1-\alpha_n-\beta_n) < \infty$ and $\sum_{n=1}^{\infty}\left(1-\alpha_n^{'}-\beta_n^{'}\right) < \infty$, $\triangle-$ converges to a point in F.*

Theorem 2.9. *Let D be a nonempty, closed and convex subset of a complete uniformly convex $W-$hyperbolic space X and let T_1 and T_2 be two multi-valued maps from D into $P(D)$ with $F \neq \emptyset$ such that P_{T_1} and P_{T_2} are nonexpansive. Then the algorithm $\{x_n\}$ in (1.4) with $0 < l \leq \alpha_n, \alpha_n^{'} \leq k < 1$, $\sum_{n=1}^{\infty}(1-\alpha_n-\beta_n) < \infty$ and $\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\left(1-\alpha_n^{'}-\beta_n^{'}\right) < \infty$, converges strongly to a point in F if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$.*

Theorem 2.10. *Let D be a nonempty, closed and convex subset of a complete uniformly convex $W-$hyperbolic space X. Let T_1 and T_2 be two multi-valued maps from D into $P(D)$ with $F \neq \emptyset$ such that P_{T_1} and P_{T_2} are nonexpansive. If one of the maps is hemi-compact or satisfies Condition (II), then the algorithm $\{x_n\}$ in (1.4) with $0 < l \leq \alpha_n, \alpha_n^{'} \leq k < 1, \sum_{n=1}^{\infty}(1-\alpha_n-\beta_n) < \infty$ and $\sum_{n=1}^{\infty}\left(1-\alpha_n^{'}-\beta_n^{'}\right) < \infty$, strongly converges to a point in F.*

Remark 2.11. *The essentials of hypotheses in our results are natural in view of the following observations: $X = [0,1] \times [0,1]$ under the Euclidean distance. Define maps $S, T : X \to CB(X)$ by $S(x,y) = \left\{\frac{1}{4}(2x+1, 2y+1)\right\}$ and $T(x,y) = \left\{\frac{1}{6}(4x+1, 4y+1)\right\}$ and the parameters as $\alpha_n = \alpha_n^{'} = \frac{1}{2}$ and $\beta_n = \beta_n^{'} = \frac{n^2+2n-1}{2(n+1)^2}$. Now the computations:$S\left(\frac{1}{2},\frac{1}{2}\right) = \left\{\left(\frac{1}{2},\frac{1}{2}\right)\right\} = T\left(\frac{1}{2},\frac{1}{2}\right)$and $\sum_{n=1}^{\infty}(1-\alpha_n-\beta_n) = \sum_{n=1}^{\infty}\left(1-\frac{1}{2}-\frac{n^2+2n-1}{2(n+1)^2}\right) = \sum_{n=1}^{\infty}\left(\frac{1}{2}-\frac{(n+1)^2-2}{2(n+1)^2}\right) = \sum_{n=1}^{\infty}\frac{1}{(n+1)^2} < \infty$ guarantee the conclusions.*

Acknowledgement. The second author (the first author) is grateful to King Fahd University of Petroleum & Minerals for supporting project IN101037 (this research).

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