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OPTIMAL BUNDLES IN NORMED SPACES

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ABSTRACT. In this paper, we prove existence of optimal bundles for a countable set of data in a broad class of normed spaces, which extend previous known results for a finite data set in a Hilbert space. In addition, we study the behavior of deviations and diameters for an increasing sequence of data sets.

1. INTRODUCTION

In [1] it was introduced the following:

Given a Hilbert space F, a finite subset $Y \subset F$ and a closed subspace V of F, let $E(Y,V)$ be the total distance of the data set Y to the subspace V, i.e. $E(Y,V) = \sum_{f\in Y} d^2(f,V)$, where d is the Euclidean metric. If $\Pi_n(F) = \{V \text{ subspace of } F : \dim V \le n\}$ and

$$E_F(Y) = \inf_{V\in\Pi_n(F)} E(Y,V),$$

a subspace $V_0 \subset F$ is called optimal if $E(Y,V_0) = E_F(Y)$.

In [1] the authors gave a constructive proof of existence of optimal subspaces and applications to problem of finding a model space that describes a given class of signals or images.

In [2, 3] the authors introduced and solved a new problem for $F = \mathbb{R}^d$ and $F = L^2(\mathbb{R}^d)$: To prove the existence of l subspaces in $\Pi_n(F)$ minimizing

$$\sum_{f\in Y} \min_{1\le j\le k} d^2(f,V_j), \quad V_j \in \Pi_n(F).$$

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A solution to the last problem is called an optimal bundle. The problem of finding an optimal bundle appears in several situations of practical interest, for examples the reader can see [3, 8].

Our main objective in this paper is to establish existence results of optimal bundle in normed spaces, as general as possible, for a data set not necessarily finite. A natural extension to the above problems is consider a countable data set Y and a norm defined on the set of real number sequences.

Let $m \in \mathbb{N} \cup \{\infty\}$ and let $\{u_k\}_{k=1}^m$ be the canonical basis in $\mathbb{R}^m$, where $\mathbb{R}^\infty$ means $\mathbb{R}^{\mathbb{N}}$. Let $(F, \| \ \|)$ be a normed space. For $n, l \in \mathbb{N}$ we consider the set

$$\mathcal{B}_{l,n}(F) := \{\mathbf{V} = (V_1, \cdots, V_l) : V_i \in \Pi_n(F), 1 \leq i \leq l\}.$$

If $\mathbf{V} \in \mathcal{B}_{l,n}(F)$, it will be called a *bundle*.

Let $Y = \{f_k\}_{k=1}^m \subset F$ and let $\rho : \mathbb{R}^m \to [0, \infty]$ be a monotone norm, i.e., ρ is a norm such that if $|x_k| \leq |y_k|$ for all k, $\rho\left(\sum_{k=1}^m x_k u_k\right) \leq \rho\left(\sum_{k=1}^m y_k u_k\right)$. For $\emptyset \neq Y_1 \subset Y$ and $\mathbf{V} = (V_1, \cdots, V_l) \in \mathcal{B}_{l,n}(F)$, we define the *deviation* of the set Y_1 from the bundle $\mathbf{V}$ as

$$E(Y_1, \mathbf{V}) = \rho\left(\sum_{f_k \in Y_1} \min_{1 \leq j \leq l} d(f_k, V_j) u_k\right), \tag{1.1}$$

where $d(f, U) = \inf_{u \in U} \|f - u\|$ is the distance of f to the set U. Note that

$$E(Y_1, \mathbf{V}) = \rho\left(\sum_{f_k \in Y_1} d\left(f_k, \cup_{j=1}^l V_j\right) u_k\right).$$

The *diameter* of the set Y_1 is defined by

$$E_F(Y_1) = \inf_{\mathbf{V} \in \mathcal{B}_{l,n}(F)} E(Y_1, \mathbf{V}).$$

A bundle $\mathbf{V}_0 \in \mathcal{B}_{l,n}(F)$ is called an *optimal bundle* for Y_1 if

$$E(Y_1, \mathbf{V}_0) = E_F(Y_1) < \infty.$$

If $l = 1$, an optimal bundle is known as optimal subspace (see [5]).

Given $Z \subset F$, we denote *spanZ* the linear space generated by the elements of Z. Throughout this paper we write $X = spanY$.

The concepts of diameter and optimal subspace were introduced by Kolmogorov in [9], when ρ is the supremum norm. With this norm, Garkavi in [6] proved existence of optimal subspaces for a compact data set. Other work about these concepts can be seen in [12]. Recently, in [5] it was proved existence of optimal subspace in reflexive Banach spaces when ρ is a monotone norm and Y a finite set. Also the authors studied properties of deviations and diameters.

If ρ is not the supreme norm, it is unknown to us results of existence for an infinite set of data, even in a Hilbert space. In Section 2 we prove existence of optimal bundles for a countable set of data in a broad class of normed spaces. Also, we give an algorithm to construct optimal bundles when the data set is finite. In Section 3 we study the behavior of deviations and diameters for an increasing sequence of data sets. Finally, we give some properties of deviations and diameters in Section 4.

2. Existence of optimal bundles

Observe that if $l \geq m$, then $\mathbf{V}_0 = (span\{f_1\}, \cdots, span\{f_m\}, \{0\}, \cdots, \{0\}) \in \mathcal{B}_{l,n}(F)$ is an optimal bundle for Y. If X has dimension at most n, then $\mathbf{V}_0 = (X, \cdots, X) \in \mathcal{B}_{l,n}(F)$ is an optimal bundle for Y. So, from now on we assume $n < dim X$ and $1 \leq l < m$.

We will need the following lemma which was proved in ([11] p. 273).

Lemma 2.1. *Let G be a Banach space of dimension n and let G^* be its conjugate space. Then there exist n linearly independent elements $e_1, \cdots, e_n \in G$ and n functionals $g_1, \cdots, g_n \in G^*$ such that $\|e_k\| = \|g_k\| = 1$, $g_i(e_k) = 1$ if $i = k$, and $g_i(e_k) = 0$ if $i \neq k$, $1 \leq i, k \leq n$.*
Consequently, for every $e = \sum_{i=1}^{n} \alpha_i e_i \in G$ we have then $|\alpha_i| \leq \|e\|$, $1 \leq i \leq n$.

For a linear space W with norm $\|\cdot\|$, we consider the following sets

$$\Lambda(W) = \left\{ (w_k)_{k=1}^m \in W^m : \rho\left(\sum_{k=1}^{m} \|w_k\| u_k \right) < \infty \right\},$$

and

$$\Lambda_0(W) = \left\{ (w_k)_{k=1}^m \in \Lambda(W) : \lim_{N\to\infty} \rho\left(\sum_{k=N+1}^{\infty} \|w_k\| u_k \right) = 0 \right\}.$$

If $m < \infty$, we observe that $\Lambda_0(W) = \Lambda(W)$.

The following example shows that some condition about the data set must be required to assure existence of optimal bundles.

Example 2.2. Let $F = \mathbb{R}^2$ be with the Euclidean norm, $m = \infty$, $n = l = 1$, and $\rho\left(\sum_{k=1}^m x_k u_k\right) = \left(\sum_{k=1}^m x_k^2\right)^{\frac{1}{2}}$. We consider $f_k = \left(\frac{1}{\sqrt{k}}, 1\right)$. It is easy to see that $Y = \{f_k\}_{k=1}^\infty \notin \Lambda_0(F)$. Suppose that for some $V \in \mathcal{B}_{1,1}(F)$, $E(Y, V) < \infty$. If $V = span\{(v_1, v_2)\}$, $v_1^2 + v_2^2 = 1$, we have

$$E(Y, V)^2 = \sum_{k=1}^{\infty} \left(1 + \frac{1}{k} - \left(\frac{v_1}{\sqrt{k}} + v_2 \right)^2 \right).$$

Then $|v_2| = 1$ and $v_1 = 0$. So $E(Y, V)^2 = \sum_{k=1}^\infty \frac{1}{k} = \infty$, a contradiction. Therefore there is not an optimal bundle for Y.

Next, we establish the main result of this section.

Theorem 2.3. *Let G be a Banach space and let G^* be its conjugate space. If $Z = \{h_k\}_{k=1}^m \in \Lambda_0(G^*)$, then there exists $\mathbf{V}_0 \in \mathcal{B}_{l,n}(G^*)$ such that $\mathbf{V}_0$ is an optimal bundle for Z.*

Proof. We only consider $m = \infty$. The case $m < \infty$ follows with the obvious modifications. If $h_k = 0$ for all k, $\mathbf{V}_0 = (0, \cdots, 0)$ is the optimal bundle. Assume $h_k \neq 0$ for some k. Let $\{\mathbf{V}_s\}_{s\in\mathbb{N}} \subset \mathcal{B}_{l,n}(G^*)$ be such that

$$E(Y, \mathbf{V}_s) \leq E_{G^*}(Y) + \frac{1}{s}. \tag{2.1}$$

We write $\mathbf{V}_s = (V_{s1}, \cdots, V_{sl})$. Given two bundles $\mathbf{V} = (V_1, \cdots, V_l)$ and $\mathbf{U} = (U_1, \cdots, U_l)$ with $V_j \subset U_j$ we have $E(Y, \mathbf{U}) \leq E(Y, \mathbf{V})$. So, w.l.o.g. we can assume $dim V_{sj} = n$, $1 \leq j \leq l$. By Lemma 2.1, for each $s \in \mathbb{N}$ and $1 \leq j \leq l$, there exists a basis $\left\{e_{sj}^i\right\}_{i=1}^n$ of V_{sj} such that $\|e_{sj}^i\| = 1$, and $|c^i| \leq \|g\|$ for $g = \sum_{i=1}^n c^i e_{sj}^i$.

Let $g_{sj}^k = \sum_{i=1}^n c_{sj}^{ki} e_{sj}^i \in V_{sj}$ be such that $d(h_k, V_{sj}) = \|h_k - g_{sj}^k\|$. It is easy to see that $\|g_{sj}^k\| \leq 2\|h_k\|$. So,

$$|c_{sj}^{ki}| \leq \|g_{sj}^k\| \leq 2\|h_k\| =: \beta_k,$$

for all $s, k \in \mathbb{N}$, $1 \leq i \leq n$ and $1 \leq j \leq l$.

Let B_{G^*} be the closed unit ball of G^* and let P be the product topological space

$$P = \prod_{i=1}^{nl} B_{G^*} \times \prod_{k=1}^{\infty} \prod_{i=1}^{nl} [-\beta_k, \beta_k],$$

where B_{G^*} has the weak-star topology and $[-\beta_k, \beta_k] \subset \mathbb{R}$ has its natural topology. By Banach Alaoglu's Theorem B_{G^*} is weakly-star compact, so Tychonoff's Theorem implies that P is compact. Hence, the sequence $\{p_s\}_{s\in\mathbb{N}} \subset P$ given by

$$p_s = (e_{s1}^1, .., e_{s1}^n, .., e_{sl}^1, .., e_{sl}^n, c_{s1}^{11}, .., c_{s1}^{1n}, .., c_{sl}^{11}, .., c_{sl}^{1n}, c_{s1}^{21}, .., c_{s1}^{2n}, .., c_{sl}^{21}, .., c_{sl}^{2n}, ..)$$

has one limit point, say

$$p = (e_{01}^1, .., e_{01}^n, .., e_{0l}^1, .., e_{0l}^n, c_{01}^{11}, .., c_{01}^{1n}, .., c_{0l}^{11}, .., c_{0l}^{1n}, c_{01}^{21}, .., c_{01}^{2n}, .., c_{0l}^{21}, .., c_{0l}^{2n}, ..).$$

Since $Z \in \Lambda_0(G^*)$ and $h_k \neq 0$ for some k, for $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\rho\left(\sum_{k=N+1}^{\infty} \|h_k\| u_k\right) < \epsilon \quad \text{and} \quad \min_{h_k \neq 0, 1 \leq k \leq N} \|h_k\| =: M > 0. \tag{2.2}$$

By the definition of the norm in G^*, for each $1 \leq k \leq N$, $1 \leq j \leq l$, there is $x_{kj} \in G$ such that

$$\|x_{kj}\| \leq 1 \quad \text{and} \quad \left\|h_k - \sum_{i=1}^n c_{0j}^{ki} e_{0j}^i\right\| - M\epsilon \leq \left|h_k(x_{kj}) - \sum_{i=1}^n c_{0j}^{ki} e_{0j}^i(x_{kj})\right|.$$

We consider the neighborhood U of p defined by

$$\begin{aligned} U = \{ & (w_1^1, .., w_1^n, .., w_l^l, .., w_l^n, a_1^{11}, .., a_1^{1n}, .., a_l^{11}, .., a_l^{1n}, a_1^{21}, .., a_1^{2n}, .., a_l^{21}, .., a_l^{2n}, ..) \\ & \in P : \left|w_j^i(x_{uv}) - e_{0j}^i(x_{uv})\right| < \epsilon \text{ and } \left|a_j^{ki} - c_{0j}^{ki}\right| < M\epsilon, \ 1 \leq j, v \leq l, \\ & 1 \leq i \leq n, \ 1 \leq u, k \leq N \}. \end{aligned}$$

Since p is a limit point of $\{p_s\}_{s\in\mathbb{N}}$, there exists $r \in \mathbb{N}$, $r \geq \frac{1}{\epsilon}$ such that

$$\left|e^i_{rj}(x_{uv}) - e^i_{0j}(x_{uv})\right| < \epsilon \quad \text{and} \quad \left|c^{ki}_{rj} - c^{ki}_{0j}\right| < M\epsilon,$$

for all $1 \leq j, v \leq l$, $1 \leq i \leq n$ and $1 \leq u, k \leq N$. Consequently, if $1 \leq k \leq N$, $h_k \neq 0$, and $1 \leq j \leq l$, we have

$$\begin{aligned}
\left\|h_k - \sum_{i=1}^n c^{ki}_{0j} e^i_{0j}\right\| - M\epsilon &\leq \left|h_k(x_{kj}) - \sum_{i=1}^n c^{ki}_{0j} e^i_{0j}(x_{kj})\right| \\
&\leq \left|h_k(x_{kj}) - g^k_{rj}(x_{kj})\right| + \left|\sum_{i=1}^n c^{ki}_{0j} e^i_{0j}(x_{kj}) - \sum_{i=1}^n c^{ki}_{rj} e^i_{rj}(x_{kj})\right| \\
&\leq \|h_k - g^k_{rj}\| + \sum_{i=1}^n \left|c^{ki}_{0j} e^i_{0j}(x_{kj}) - c^{ki}_{rj} e^i_{rj}(x_{kj})\right| \\
&\leq d(h_k, V_{rj}) + \sum_{i=1}^n \left|c^{ki}_{0j} - c^{ki}_{rj}\right| \left|e^i_{0j}(x_{kj})\right| \\
&\quad + \sum_{i=1}^n \left|c^{ki}_{rj}\right| \left|e^i_{0j}(x_{kj}) - e^i_{rj}(x_{kj})\right| \\
&\leq d(h_k, V_{rj}) + nM\epsilon + n2\|h_k\|\epsilon \leq d(h_k, V_{rj}) + 3n\epsilon\|h_k\|.
\end{aligned}$$

From definition of $g^k_{0,j}$, if $h_k = 0$ then $c^{ki}_{0j} = 0$ for all j, i. So

$$\left\|h_k - \sum_{i=1}^n c^{ki}_{0j} e^i_{0j}\right\| \leq d(h_k, V_{rj}) + (3n+1)\epsilon\|h_k\|, \quad 1 \leq k \leq N,\ 1 \leq j \leq l.$$

Let $V_j = span\{e^1_{0j}, \cdots, e^n_{0j}\}$, $1 \leq j \leq l$, and $\mathbf{V}_0 = (V_1, \cdots, V_l) \in \mathcal{B}_{l,n}(G^*)$. Then

$$\min_{1\leq j\leq l} d(h_k, V_j) \leq \min_{1\leq j\leq l} d(h_k, V_{rj}) + (3n+1)\epsilon\|h_k\|, \quad 1 \leq k \leq N.$$

Finally from (2.1) and (2.2) we have,

$$\begin{aligned}
E(Y, \mathbf{V}_0) &= \rho\left(\sum_{k=1}^N \min_{1\leq j\leq l} d(h_k, V_j) u_k + \sum_{k=N+1}^\infty \min_{1\leq j\leq l} d(h_k, V_j) u_k\right) \\
&\leq \rho\left(\sum_{k=1}^N \min_{1\leq j\leq l} d(h_k, V_j) u_k\right) + \rho\left(\sum_{k=N+1}^\infty \min_{1\leq j\leq l} d(h_k, V_j) u_k\right) \\
&\leq \rho\left(\sum_{k=1}^N \min_{1\leq j\leq l} d(h_k, V_{rj}) u_k\right) + (3n+1)\epsilon\rho\left(\sum_{k=1}^N \|h_k\| u_k\right) + \rho\left(\sum_{k=N+1}^\infty \|h_k\| u_k\right) \\
&\leq E(Y, \mathbf{V}_r) + \epsilon\left(1 + (3n+1)\rho\left(\sum_{k=1}^\infty \|h_k\| u_k\right)\right) \\
&\leq E_{G^*}(Y) + \epsilon\left(2 + (3n+1)\rho\left(\sum_{k=1}^\infty \|h_k\| u_k\right)\right).
\end{aligned}$$

As $\epsilon > 0$ is arbitrary, the proof is complete. □

Corollary 2.4. *Let F be a Banach space which is isometrically isomorphic to some conjugate space. If $Y = \{f_k\}_{k=1}^m \in \Lambda_0(F)$, then there exists $\mathbf{V}_0 \in \mathcal{B}_{l,n}(F)$ such that $\mathbf{V}_0$ is an optimal bundle for Y. In particular, if F is a reflexive space, the assertion remains.*

Example 2.5. The classical Lebesgue spaces $L_p(\Omega)$, $1 < p \leq \infty$, where Ω is an open set in $\mathbb{R}^d$, satisfy the hypothesis of Corollary 2.4.

The following simple property is an immediate consequence of the definition.

Lemma 2.6. *Let G and H be normed spaces, $G \subset H$, and $Y = \{f_k\}_{k=1}^m \in \Lambda_0(G)$. Then*

$$E_H(Y) \leq E_G(Y).$$

Let F^{**} be the second conjugate space of F and let $J_F : F \to F^{**}$ be the canonical mapping. We write $\overline{f} = J_F(f)$, $f \in F$.

As another consequence of Theorem 2.3, we also obtain the following result.

Theorem 2.7. *Let F be a Banach space. Assume that there exists a linear projector $p : F^{**} \to F$ with unit norm. If $Y = \{f_k\}_{k=1}^m \in \Lambda_0(F)$, then there is $\mathbf{V}_0 \in \mathcal{B}_{l,n}(F)$ such that $\mathbf{V}_0$ is an optimal bundle for Y.*

Proof. By Theorem 2.3, there exists an optimal bundle $\mathbf{W}_0 = (W_1, \cdots, W_l) \in \mathcal{B}_{l,n}(F^{**})$ for $J_F(Y)$. Set $\mathbf{V}_0 = (p(W_1), \cdots, p(W_l)) \in \mathcal{B}_{l,n}(F)$, and let g_{kj} be a best approximation to $\overline{f_k}$ from W_j, for all k, j. Since $p\left(\overline{f_k}\right) = f_k$ and $\|p\| = 1$, then

$$d(f_k, p(W_j)) \leq \|f_k - p(g_{kj})\| = \|p\left(\overline{f_k} - g_{kj}\right)\| \leq \left\|\overline{f_k} - g_{kj}\right\| = d\left(\overline{f_k}, W_j\right). \quad (2.3)$$

By (2.3) we have

$$E(Y, \mathbf{V}_0) \leq E(J_F(Y), \mathbf{W}_0) = E_{F^{**}}(J_F(Y)).$$

Since J_F is an isometry, from Lemma 2.6 we get $E_{F^{**}}(J_F(Y)) \leq E_F(Y)$. Thus, $E(Y, \mathbf{V}_0) \leq E_F(Y)$ and the theorem is proved. □

Example 2.8. The space $L_1([0, 1])$, satisfies the hypothesis of Theorem 2.7 (see [10]).

Remark 2.9. (a) In a conjugate space, the Krein Milman Theorem implies that the unit ball has extreme points. On the other hand, it is well known that the unit ball of $L_1([0, 1])$ has not extreme points, so $L_1([0, 1])$ is not isometrically isomorphic to any conjugate space and Corollary 2.4 cannot be applied in this space.

(b) Theorem 2.7 implies Corollary 2.4. In fact, if F is isometrically isomorphic to some conjugate space, there exists a linear operator with unit norm projecting F^{**} onto F (see [4], p. 55).

(c) It is interesting to note that if F is a Banach space not containing a subspace isometrically isomorphic to l_1, then F *is isometrically isomorphic to some conjugate space* if only if *there exists a linear projector $p : F^{**} \to F$ with unit norm* (see [7], p. 221).

Next, we give another theorem on the existence of optimal bundles.

Theorem 2.10. *Let F be a Banach space and let $Y = \{f_k\}_{k=1}^m \in \Lambda_0(F)$. If X is isometrically isomorphic to some conjugate space, and there exists a lineal metric selection $P_X : F \to X$, with unit norm, then there is $\mathbf{V}_0 \in \mathcal{B}_{l,n}(F)$ such that $\mathbf{V}_0$ is an optimal bundle for Y.*

Proof. By Corollary 2.4, there is a linear optimal bundle $\mathbf{V_0} = (V_1, \cdots, V_l) \in \mathcal{B}_{l,n}(X)$ for Y. Let $\mathbf{W} = (W_1, \cdots, W_l) \in \mathcal{B}_{l,n}(F)$ and $\mathbf{W}' = (P_X(W_1), \cdots, P_X(W_l))$. Since P_X is a lineal metric selection, we have $P_X(f_k) = f_k$ and $\mathbf{W}' \in \mathcal{B}_{l,n}(X)$. We choose $g_{kj} \in W_j$ such that $d(f_k, W_j) = \|f_k - g_{kj}\|$. Then $P_X(g_{kj}) \in P_X(W_j)$ and

$$\begin{aligned} d(f_k, P_X(W_j)) &\le \|f_k - P_X(g_{kj})\| = \|P_X(f_k - g_{kj})\| \le \|f_k - g_{kj}\| \\ &= d(f_k, W_j). \end{aligned} \tag{2.4}$$

So, $E(Y, \mathbf{V_0}) \le E(Y, \mathbf{W}') \le E(Y, \mathbf{W})$. □

Remark 2.11. In ([5], p. 196) the authors show an example such that Theorem 2.10 can be applied, but Corollary 2.4 cannot.

Next, we give an algorithm to construct optimal bundles using optimal subspaces when Y is a finite set. It extends a similar algorithm proved in ([2], Theorem 2.2) for a Hilbert space.

Theorem 2.12. *Let F be a normed space, $Y = \{f_k\}_{k=1}^m \subset F$, $m < \infty$, and let ρ be the p-norm in $\mathbb{R}^m$, $1 \le p \le \infty$. Assume that there is an optimal subspace for Y_1, for all $Y_1 \subset Y$, $Y_1 \ne \emptyset$. Then there exists $\mathbf{V}_0 \in \mathcal{B}_{l,n}(F)$ such that $\mathbf{V}_0$ is an optimal bundle for Y.*

Proof. Suppose $1 \le p < \infty$. We denote by π_l the set of all l-tuples $P = (Y_1, \cdots, Y_l)$ of subsets of Y such that

$$Y = \bigcup_{i=1}^{l} Y_i \quad \text{and} \quad Y_i \cap Y_j = \emptyset \quad \text{for} \quad i \ne j.$$

Note that we allow some of the elements of $P \in \pi_l$ to be the empty set.
For $P = (Y_1, \cdots, Y_l) \in \pi_l$, and $Y_s \ne \emptyset$, we choose $W_s(P) \in \mathcal{B}_{1,n}(F)$ an optimal subspace for Y_s. Since the cardinal of π_l is finite, there exists $P^* = (Y_1^*, \cdots, Y_l^*) \in \pi_l$ such that

$$\sum_{s=1}^{l} \rho^p \left(\sum_{f_k \in Y_s^*} d(f_k, W_s(P^*)) u_k \right) \le \sum_{s=1}^{l} \rho^p \left(\sum_{f_k \in Y_s} d(f_k, W_s(P)) u_k \right), \tag{2.5}$$

for all $P = (Y_1, \cdots, Y_l) \in \pi_l$. Since ρ is a p-norm, it follows that

$$\sum_{s=1}^{l} \rho^p \left(\sum_{f_k \in Y_s^*} d(f_k, W_s(P^*)) u_k \right) = \rho^p \left(\sum_{s=1}^{l} \sum_{f_k \in Y_s^*} d(f_k, W_s(P^*)) u_k \right). \tag{2.6}$$

It should be noted that $\{W_s(P^*) : 1 \le s \le l, Y_s^* \ne \emptyset\} =: \{W_1^*, \cdots, W_d^*\}$ with $d \le l$, $W_i^* \ne W_j^*$. Let $\mathbf{V}_0 = (W_1^*, \cdots, W_d^*, \{0\}, \cdots, \{0\}) \in \mathcal{B}_{l,n}(F)$. Now, (2.5) and (2.6) imply,

$$E^p(Y,\mathbf{V}_0) \le \rho^p\left(\sum_{s=1}^{l}\sum_{f_k\in Y_s^*} d(f_k, W_s(P^*))u_k\right) \le \sum_{s=1}^{l}\rho^p\left(\sum_{f_k\in Y_s} d(f_k, W_s(P))u_k\right), \tag{2.7}$$

for all $P=(Y_1,\cdots,Y_l)\in\pi_l$.
On the other hand, given $\mathbf{V}=(V_1,\cdots,V_l)\in\mathcal{B}_{l,n}(F)$ we can find a l-tuple $P'=(Y_1',\cdots,Y_l')\in\pi_l$ such that

$$Y_s' = \left\{ f_k \in Y' : d(f_k, V_s) = \min_{1\le j\le l} d(f_k, V_j)\right\}.$$

Since $W_s(P')$ is an optimal subspace for Y_s', we have

$$\sum_{s=1}^{l}\rho^p\left(\sum_{f_k\in Y_s'} d(f_k, W_s(P'))u_k\right) \le \sum_{s=1}^{l}\rho^p\left(\sum_{f_k\in Y_s'} d(f_k, V_s)u_k\right) = E^p(Y,\mathbf{V}).$$

In consequence, from (2.7) with $P=P'$ we obtain $E(Y,\mathbf{V}_0)\le E(Y,\mathbf{V})$, $\mathbf{V}\in\mathcal{B}_{l,n}(F)$.
The case $p=\infty$ follows with the obvious modifications. □

Now, we give an example where only Theorem 2.12 can be applied because the normed space is not Banach space. Let $(S_f(\mathbb{R}),\|\cdot\|_2)$ be the space of sequences of real number with finite support. Let ρ be the Euclidian norm in $\mathbb{R}^3$, $n=2$, and let $Y=\{f_k\}_{k=1}^3$ be such that $f_i(j)=\delta_{ij}$, δ_{ij} the Kronecker Delta function. A straightforward computation shows that for all $Y_1\subset Y$, $Y_1\neq\emptyset$, there exists an optimal subspace.

3. Convergence of deviations and diameters for an increasing sequence of data

The following theorem immediately follows from the monotony property of the norm.

Proposition 3.1. *Let $Y=\{f_k\}_{k=1}^m\subset F$, $\mathbf{V}\in\mathcal{B}_{l,n}(F)$ and $Y_N=\{f_k\}_{k=1}^N$, $1\le N<m$. Then*
(a) $E(Y_N,\mathbf{V})\le E(Y_{N+1},\mathbf{V})$;
(b) $E(Y_N,\mathbf{V})\le E(Y,\mathbf{V})$;
(c) $E_F(Y_N)\le E_F(Y_{N+1})$;
(d) $E_F(Y_N)\le E_F(Y)$.

Theorem 3.2. *Let $Y=\{f_k\}_{k=1}^\infty\in\Lambda_0(F)$ and $Y_N=\{f_k\}_{k=1}^N$, $N\in\mathbb{N}$. Then*
(a) $\lim_{N\to\infty} E_F(Y_N)=E_F(Y)$;
(b) If $\mathbf{V}_N$ is an optimal bundle for Y_N then $\lim_{N\to\infty} E(Y,\mathbf{V}_N)=E_F(Y)$.

Proof. (a) According to Proposition 3.1 (c)-(d), we have $E_F(Y_N)\uparrow\alpha$, as $N\to\infty$, with $\alpha\le E_F(Y)<\infty$. Let $\epsilon=E_F(Y)-\alpha>0$. By hypothesis there exists $M\in\mathbb{N}$

such that

$$\rho\left(\sum_{k=M+1}^{\infty}\|f_k\|u_k\right) < \frac{\epsilon}{2}.$$

Let $\mathbf{V} = (V_1, \cdots, V_l) \in \mathcal{B}_{l,n}(F)$ be such that $E(Y_M, \mathbf{V}) < E_F(Y_M) + \frac{\epsilon}{2}$. Therefore,

$$\begin{aligned} E_F(Y) &\leq E(Y, \mathbf{V}) = \rho\left(\sum_{k=1}^{M}\min_{1\leq j\leq l} d(f_k, V_j)u_k + \sum_{k=M+1}^{\infty}\min_{1\leq j\leq l} d(f_k, V_j)u_k\right) \\ &\leq \rho\left(\sum_{k=1}^{M}\min_{1\leq j\leq l} d(f_k, V_j)u_k\right) + \rho\left(\sum_{k=M+1}^{\infty}\min_{1\leq j\leq l} d(f_k, V_j)u_k\right) \\ &\leq E(Y_M, \mathbf{V}) + \rho\left(\sum_{k=M+1}^{\infty}\|f_k\|u_k\right) \\ &< E_F(Y_M) + \epsilon \leq \alpha + \epsilon = E_F(Y), \end{aligned} \tag{3.1}$$

a contradiction. So, $\lim_{N\to\infty} E_F(Y_N) = E_F(Y)$.
(b) From (3.1) and Proposition 3.1 (d) we have

$$\begin{aligned} E_F(Y) &\leq E(Y, \mathbf{V}_N) \leq E(Y_N, \mathbf{V}_N) + \rho\left(\sum_{k=N+1}^{\infty}\min_{1\leq j\leq l} d(f_k, V_j)u_k\right) \\ &\leq E_F(Y_N) + \rho\left(\sum_{k=N+1}^{\infty}\|f_k\|u_k\right) \leq E_F(Y) + \rho\left(\sum_{k=N+1}^{\infty}\|f_k\|u_k\right), \end{aligned}$$

for all $N \in \mathbb{N}$. Taking the limit as $N \to \infty$, we get $\lim_{N\to\infty} E(Y, \mathbf{V}_N) = E_F(Y)$. □

4. Properties of deviations and diameters

The properties established in the next theorem immediately follow.

Proposition 4.1. *Let $Y = \{f_k\}_{k=1}^m, Z = \{h_k\}_{k=1}^m \in \Lambda(F)$ and let $\mathbf{V} = (V_1, \cdots, V_l)$, $\mathbf{W} = (W_1, \cdots, W_l) \in \mathcal{B}_{l,n}(F)$. Then the following statements hold.*

(a) $|E(Y, \mathbf{V}) - E(Z, \mathbf{V})| \leq \rho\left(\sum_{k=1}^{m}\|f_k - h_k\|u_k\right)$;
(b) If $W_j \subset V_j$, $1 \leq j \leq l$, *then* $E(Y, \mathbf{V}) \leq E(Y, \mathbf{W})$;
(c) $|E_F(Y) - E_F(Z)| \leq \rho\left(\sum_{k=1}^{m}\|f_k - h_k\|u_k\right)$;
(d) $|E(Y, \mathbf{V}) - E(Y, \mathbf{W})| \leq \rho\left(\sum_{k=1}^{m}\left(d(f_k, T) - d(f_k, U)\right)u_k\right)$, *where* $T = \bigcup_{j=1}^{l} V_j$ *and* $U = \bigcup_{j=1}^{l} W_j$.

We recall that the metric projection on a set $D \subset F$, when it is well defined, is given by $P_D(f) = \{g \in D : \|g - f\| = d(f, D)\}$.

Next, our propose is to study the continuity of the function $E(Y,.)$. So, we consider a notion of distance between two bundles $\mathbf{V} = (V_1, \cdots, V_l)$ and $\mathbf{W} = (W_1, \cdots, W_l)$, which was introduced in [5],

$$d_*(\mathbf{V}, \mathbf{W}) = \sup\left\{\frac{\|g-h\|}{\|f\|} : \ f \neq 0, g \in P_T(f), h \in P_U(f)\right\},$$

where P_T and P_U are the metric projections on the sets $T = \bigcup_{j=1}^{l} V_j$ and $U = \bigcup_{j=1}^{l} W_j$, respectively.

The following theorem is a direct consequence of Proposition 4.1, (d).

Theorem 4.2. *Assume the same hypothesis of Proposition 4.1. Then*

$$|E(Y, \boldsymbol{V}) - E(Y, \boldsymbol{W})| \leq \rho \left(\sum_{k=1}^{m} \|f_k\| u_k\right) d_*(\boldsymbol{V}, \boldsymbol{W}).$$

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