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ON CLASS $\mathcal{A}(k^*)$ OPERATORS

NAIM L. BRAHA[1]*, ILMI HOXHA[1] AND SALAH MECHERI[2]

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Abstract. This paper deals with some classes of bounded linear operators on Hilbert spaces. The main emphasis is put onto the classes $\mathcal{A}(k^*)$ and $\mathcal{A}_{(k^*)}P$, $k>0$. Some additional results are given for other classes, like $P\mathcal{A}(k^*)$, $M-\mathcal{A}(k^*)$ and spectral properties of operators belonging to $\mathcal{A}(k^*)$ are considered. We also describe under what conditions a matrix-operator $T_{A,B}$ belongs to $\mathcal{A}(k^*)$, $\mathcal{A}_{(k^*)}P$ or $P\mathcal{A}(k^*)$.

1. Introduction

Let us denote by H a complex Hilbert space and with $B(H)$ the space of all bounded linear operators defined in H. In the following we will mention some known classes of operators defined in H. An operator $T\in B(H)$ is said to be positive (denoted $T\geq 0$) if $\langle Tx,x\rangle\geq 0$ for all $x\in H$. The operator T is said to be a p-hyponormal operator if and only if $(T^*T)^p\geq (TT^*)^p$ for a positive number p, and it is said to be a log-hyponormal if it is invertible and satisfies the following relation $\log T^*T\geq \log TT^*$, [26]. The class of p-hyponormal operators and the class of log-hyponormal operators were defined as the extension of hyponormal operators, i.e, $T^*T\geq TT^*$. An operator T is called M-hyponormal if there is a constant $M>0$ such that $M\|Tx\|\geq \|T^*x\|$ for all $x\in H$, (see [5]). It is well known that every p-hyponormal operator is a q-hyponormal operator for $p\geq q>0$, by the Löwner-Heinz theorem "$A\geq B\geq 0$ ensures $A^\alpha\geq B^\alpha$ for any $\alpha\in[0,1]$," and every invertible p-hyponormal operator is a log-hyponormal operator since log is an operator monotone function. An operator T is paranormal

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* Corresponding author.
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if $\|T^2x\| \geq \|Tx\|^2$, for every unit vector $x \in H$, [9]. We will say that T is a $*$-paranormal operator if it satisfies the following relation: $\|T^2x\| \geq \|T^*x\|^2$, for every unit vector $x \in H$, [3]. An operator T is said to be M-paranormal if $M\|T^2x\| \geq \|Tx\|^2$, for every unit vector $x \in H$(see [4]). An operator T is said to be M^*-paranormal if $M\|T^2x\| \geq \|T^*x\|^2$, for every unit vector $x \in H$(see [2]).

In [10], Furuta, Ito and Yamazaki introduced the class $\mathcal{A}$ of operators, respectively class $\mathcal{A}(k)$ of operators defined as follows: For each $k > 0$, an operator T is a class $\mathcal{A}(k)$ operator if

$$\left(T^*|T|^{2k}T\right)^{\frac{1}{k+1}} \geq |T|^2,$$

(for $k = 1$ it defines the class $\mathcal{A}$ operators) which includes the class of log-hyponormal operators (see Theorem 2, in [10]) and it is included in the class of paranormal operators, in case where $k = 1$ (see Theorem 1 in [10]). In the same paper the absolute-k-paranormal operators were introduced as follows: For each $k > 0$, an operator T is an absolute-k-paranormal operator if

$$\left\||T|^kTx\right\| \geq \|Tx\|^{k+1},$$

for every unit vector $x \in H$. The class $\mathcal{A}(k)$ of operators is included in the absolute-k-paranormal operators for any $k > 0$ (see Theorem 2 in [10]).

In this paper we will show the behavior of the class $\mathcal{A}(k^*)$ which is defined as follows:

$$\left(T^*|T|^{2k}T\right)^{\frac{1}{k+1}} \geq |T^*|^2,$$

for every $k > 0$. In case where $k = 1$ it defines the class $\mathcal{A}^*$ operators. Every class $\mathcal{A}^*$ operator is a $*$-paranormal operator, Theorem 1.3 in [8].

In paper [22] the absolute-k^*-paranormal class of operators was introduced as follows:

$$\mathcal{A}_{(k^*)}P = \{T \in B(H) : \||T|^kTx\| \geq \|T^*x\|^{k+1}, x \in H, \|x\| = 1\}$$

for any $k > 0$. For each $k > 0$, every class $\mathcal{A}(k^*)$ operator is an absolute-k^*-paranormal operator, Theorem 2.4 in [22].

Also, we will show the behavior of the class $M - \mathcal{A}(k^*)$ of operators which is defined as follows: For each $k > 0, M > 0$ an operator T is a class $M - \mathcal{A}(k^*)$ operator if

$$\left(T^*|T|^{2k}T\right)^{\frac{1}{k+1}} \geq M|T^*|^2,$$

and an absolute-$k^* - M$-paranormal operator, if for each $k > 0$, $M > 0$

$$\left\||T|^kTx\right\| \geq M\|T^*x\|^{k+1},$$

for every unit vector $x \in H$.

This paper deals with some classes of bounded linear operators on Hilbert spaces. The main emphasis is put onto the classes $\mathcal{A}(k^*)$ and $\mathcal{A}_{(k^*)}P$, $k > 0$. Some additional results are given for other classes, like $P\mathcal{A}(k^*)$, $M - \mathcal{A}(k^*)$ and spectral properties of operators belonging to $\mathcal{A}(k^*)$ are considered. We also describe under what conditions a matrix-operator $T_{A,B}$ belongs to $\mathcal{A}(k^*)$, $\mathcal{A}_{(k^*)}P$ or $P\mathcal{A}(k^*)$.

2. Class $\mathcal{A}(k^*)$ of operators

Now we will denote by:

$$P\mathcal{A}(k^*) = \{T \in B(H) : T^*(T^*T)^kT \geq (TT^*)^{k+1}\}.$$

Theorem 2.1. *Holds*

(1) *Every class $P\mathcal{A}(k^*)$ operator is a class $\mathcal{A}(k^*)$ operator for $k > 0$,*
(2) *Every $*$-paranormal operator is an absolute-k^*-paranormal operator for $k \geq 1$.*

Proof. Let $T \in B(H)$.

1). Let us suppose that T is a class $P\mathcal{A}(k^*)$ operator, it means that the following relation

$$T^*(T^*T)^kT \geq (TT^*)^{k+1},$$

holds for every $k > 0$. Now applying Löwner–Heinz inequality we get that $T \in \mathcal{A}(k^*)$ class operator.

2). Let us consider that T is a $*$-paranormal operator. Then we get:

$$\|T^2x\| \geq \|T^*x\|^2,$$

for every unit vector $x \in H$, respectively

$$\|T^2x\|^2 \geq (\|T^*x\|^2)^2.$$

By properties of real quadratic forms, this gives

$$t^2 - 2t\|T^*x\|^2 + \|T^2x\|^2 \geq 0,$$

for every $t > 0$. Hence

$$T^{*2}T^2 - 2tTT^* + t^2 \geq 0. \tag{2.1}$$

Based on [22, Theorem 2.5], the relation (2.1) defines the absolute-1^*-paranormal operator. It is a known fact that the function $f(k) = \||T|^kTx\|^{\frac{1}{k+1}}$ (see Theorem 4 in [10]) is increasing from which we get that the operator T belongs to the class $\mathcal{A}_{(k^*)}P$, for every $k \geq 1$. □

Remark 2.2. Based on the Theorem 2.1 and Theorem 2.4 in [22] we have this relation between the above classes of operators:

$$P\mathcal{A}(k^*) \subset \mathcal{A}(k^*) \subset \text{absolute-}k^*\text{-paranormal} \tag{2.2}$$

In what follows by examples, we will prove that the converse in the relation (2.2) is not true in general. The following fact gives rise in the possibilities to compare the class of operators defined above. For that let us define the following operators: Suppose H is a direct sum of denumerable copies of two dimensional Hilbert spaces $\mathbb{R} \times \mathbb{R}$. Let A and B be any two positive operators on $\mathbb{R} \times \mathbb{R}$. We define the operator $T = T_{A,B}$ on H as follows:

$$\mathbf{T_{A,B}} = \begin{pmatrix} \ddots & & & & & & \\ & B & 0 & 0 & & & \\ & 0 & B & \boxed{0} & & & \\ & 0 & 0 & B & & & \\ & & & & A & & \\ & & & & & A & \\ & & & & & & \ddots \end{pmatrix}$$

where $\boxed{}$ shows the place of the $(0,0)$ matrix element.

The following Theorem 2.3 is obtained by easy calculations, so we omit to describe this calculations.

Theorem 2.3. *The following assertions holds:*

(1) *For each $k > 0$, $T_{A,B}$ is a class $P\mathcal{A}(k^*)$ operator if and only if*
$$BA^{2k}B \geq B^{2(k+1)},$$
and
$$A^{2(k+1)} \geq B^{2(k+1)}.$$
(2) *For each $k > 0$, $T_{A,B}$ is a class $\mathcal{A}(k^*)$ operator if and only if*
$$(BA^{2k}B)^{\frac{1}{k+1}} \geq B^2,$$
and
$$A^2 \geq B^2.$$
(3) *$T_{A,B}$ is a class $\mathcal{A}^*$ operator if and only if*
$$(BA^2B)^{\frac{1}{2}} \geq B^2,$$
and
$$A^2 \geq B^2.$$
(4) *For each $k > 0$, $T_{A,B}$ is a class $\mathcal{A}_{(k^*)}P$ operator if and only if*
$$BA^{2k}B - (k+1)\lambda^k B^2 + k\lambda^{k+1} \geq 0,$$
and
$$A^{2(k+1)} - (k+1)\lambda^k B^2 + k\lambda^{k+1} \geq 0,$$
for all $\lambda > 0$.

(5) *$T_{A,B}$ is a $*$-paranormal operator if and only if*
$$BA^2B - 2\lambda B^2 + \lambda^2 \geq 0,$$
and
$$A^4 - 2\lambda B^2 + \lambda^2 \geq 0,$$
for all $\lambda > 0$.

Given a bounded sequence of complex numbers $\{\alpha_n : n \in \mathbb{Z}\}$(called weights), let T be the bilateral weighted shift on an infinite dimensional Hilbert space operator $H = l_2$, with the canonical orthonormal basis $\{e_n : n \in \mathbb{Z}\}$, defined by $Te_n = \alpha_n e_{n+1}$ for all $n \in \mathbb{Z}$. Based on the definition of the class $\mathcal{A}(k^*)$ operators the following facts for bilateral weighted shift operators are valid:

Lemma 2.4. *Let T be a bilateral weighted shift operator with weights $\{\alpha_n : n \in \mathbb{Z}\}$. Then T is a class $\mathcal{A}(k^*)$ operator if and only if*

$$|\alpha_n|^2 \cdot |\alpha_{n+1}|^{2k} \geq |\alpha_{n-1}|^{2(k+1)},$$

for all $n \in \mathbb{Z}$.

Proof. Let us define by $Te_n = \alpha_n e_{n+1}$ for all $n \in \mathbb{Z}$, the bilateral weighted shift operator with weights $\{\alpha_n : n \in \mathbb{Z}\}$. Then it follows that $T^*(e_n) = \alpha_{n-1}e_{n-1}$. And after some calculations we get:

$$(T^*|T|^{2k}T)(e_n) = |\alpha_n|^2 \cdot |\alpha_{n+1}|^{2k}(e_n),$$

respectively

$$|\alpha_n|^2 \cdot |\alpha_{n+1}|^{2k} \geq |\alpha_{n-1}|^{2(k+1)},$$

for all $n \in \mathbb{Z}$. □

Lemma 2.5. *Let T be a non-singular bilateral weighted shift operator with weights $\{\alpha_n : n \in \mathbb{Z}\}$. Then T^{-1} is a class $\mathcal{A}(k^*)$ operator if and only if*

$$|\alpha_{n-1}|^2 \cdot |\alpha_{n-2}|^{2k} \leq |\alpha_n|^{2(k+1)},$$

for all $n \in \mathbb{Z}$.

Proof of this Lemma is omitted, because it is similar to the previous Lemma.

Considering the above lemmas, we will further show that if $T \in \mathcal{A}(k^*)$ class, then it does not follow that $T^{-1} \in \mathcal{A}(k^*)$ class operator. This fact is shown by the following example.

Example 2.6. Let us denote by T the bilateral weighted shift operator, with weighted sequence $\{\alpha_n : n \in \mathbb{Z}\}$, given by the relation:

$$\alpha_n = \begin{cases} 1 & \text{if } n \leq 1 \\ 2 & \text{if } n = 2 \\ 1 & \text{if } n = 3 \\ 4 & \text{if } n = 4 \\ 1 & \text{if } n = 5 \\ 16 & \text{if } n \geq 6. \end{cases}$$

Following Lemma 2.4 it follows that $T \in \mathcal{A}(2^*)$, but $T^{-1} \notin \mathcal{A}(2^*)$ which follows from Lemma 2.5, for $n = 3$.

By the following example we will show that classes $\mathcal{A}(k^*)$ and $\mathcal{A}_{(k^*)}P$ are different from each other. We will follow the ideas given in example 8 on paper [10].

Example 2.7. Let $K = \oplus_{n=-\infty}^{\infty} H_n$ where $H_n \cong \mathbb{R}^2$. For given positive operators A, B on H, define the operator $T_{A,B}$ on K as in Theorem 2.3. Then we have the following example:

(a) An example of non-class $\mathcal{A}(2^*)$, absolute-2^*-paranormal operator. Let us denote by

$$B = \begin{pmatrix} 4 & 0 \\ 0 & 20 \end{pmatrix}^{\frac{1}{4}} \text{ and } A = \frac{1}{2}\begin{pmatrix} 1+\sqrt{3} & 1-\sqrt{3} \\ 1-\sqrt{3} & 1+\sqrt{3} \end{pmatrix}.$$

Then

$$(AB^4A)^{\frac{1}{3}} - A^2,$$

is negative and from this it follows that $T_{A,B} \notin \mathcal{A}(2^*)$. In what follows we will show that $T_{A,B}$ is an absolute-2^*-paranormal operator. To prove this (from Theorem 2.3, condition 4) it is enough to prove the following two relations:

$$BA^4B - 3\lambda^2B^2 + 2\lambda^3 \geq 0$$

and

$$A^6 - 3\lambda^2B^2 + 2\lambda^3 \geq 0,$$

for all $\lambda > 0$. The first one is proved in example 8-3, in [10]. We will prove just the second one. Let us denote by $A(\lambda)$ the trace of the matrix $A^6 - 3\lambda^2B^2 + 2\lambda^3$, so

$$A(\lambda) = Tr(A^6 - 3\lambda^2B^2 + 2\lambda^3) = 4\lambda^3 - 12\lambda^2 + 8 + 20\sqrt{20}$$

and the minimal value of this expression is achieved for $\lambda = 12$ from which follows that

$$A(\lambda) \geq A(12) = 5281.443,$$

respectively $A(\lambda) \geq 0$, for all $\lambda > 0$. Let us denote by $B(\lambda)$ the determinant for the matrix $A^6 - 3\lambda^2B^2 + 2\lambda^3$,

$$\mathrm{Det}(A^6 - 3\lambda^2B^2 + 2\lambda^3) = B(\lambda).$$

The function $B(\lambda)$ has the minimum value for $\lambda = 0$, and from this we get that $B(\lambda) \geq B(0) = 160\sqrt{20} > 0$, for every $\lambda > 0$. Calling again Theorem 2.3, property 4 it follows that $T_{A,B} \in \mathcal{A}_{(2^*)}P$.

(b) An example of non $*$-paranormal, absolute-2^*-paranormal operator. Let us denote by

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix} \text{ and } B = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then

$$BA^2B - 2\lambda B^2 + \lambda^2,$$

is negative for $\lambda = 4$, where it follows that $T_{A,B}$ is a non $*$-paranormal operator. In the following we will show that the following two relations are valid:

$$BA^4B - 3\lambda^2B^2 + 2\lambda^3 \geq 0$$

and

$$A^6 - 3\lambda^2B^2 + 2\lambda^3 \geq 0,$$

for all $\lambda > 0$. The first relation is valid(see example 8-4 in [10]), we will prove the second. From the above facts it follows that:

$$Tr(A^6 - 3\lambda^2 B^2 + 2\lambda^3) = 4\lambda^3 - 12\lambda^2 + 1584 \geq Tr(\lambda = 12) = 6768$$

from this relation we get:

$$Tr(A^6 - 3\lambda^2 B^2 + 2\lambda^3) \geq 0,$$

for all $\lambda > 0$. In the same way we show that

$$\mathrm{Det}(A^6 - 3\lambda^2 B^2 + 2\lambda^3) = 4\lambda^6 - 24\lambda^5 + 3168\lambda^3 - 16224\lambda^2 + 64 \geq 0,$$

for every $\lambda > 0$. Based on Theorem 2.3, property 5, we get that $T_{A,B}$ is an absolute-2^*-paranormal operator.

(c) An example of non-$P\mathcal{A}(3^*)$, which is $\mathcal{A}(3^*)$ operator. Let us denote by

$$A = \begin{pmatrix} 6 & -5 \\ -5 & 6.2 \end{pmatrix} \text{ and } B = \begin{pmatrix} 3 & -2 \\ -2 & 3 \end{pmatrix}.$$

Then $A^2 - B^2 \geq 0$ and $(BA^6B)^{\frac{1}{4}} \geq B^2$. From which follows that $T_{A,B} \in \mathcal{A}(3^*)$. But relation

$$BA^6B \geq B^8,$$

is not valid, hence we conclude that $T_{A,B} \notin P\mathcal{A}(3^*)$.

An operator $T \in B(H)$ is normaloid if $\|T^n\| = \|T\|^n$ for all positive integers n. It is known that every normal operator is normaloid(see [7, Theorem 4.5.12]) and every paranormal operator is normaloid (see [13], [9]). In what follows we will show that every absolute-k^*-paranormal operator is normaloid for $0 < k \leq 1$.

Theorem 2.8. *Let T be an absolute-1^*-paranormal operator. Then T is normaloid.*

Proof. From the fact that T is from $\mathcal{A}_{(1^*)}P$ and Theorem 2.5 in [22] we get the following relation:

$$T^*|T|^{2k}T - (k+1)\lambda^k|T^*|^2 + k \cdot \lambda^{k+1} \geq 0, \tag{2.3}$$

for every $k > 0$ and $\lambda > 0$. If we put $k = 1$ in the relation (2.3), we establish the necessary and sufficient condition (see [2]) under which an operator T is $*$-paranormal. Now the proof of the theorem follows from Theorem 1.1 given in [3]. □

Now we show the general case of the Theorem 2.8.

Theorem 2.9. *Let T be an absolute-k^*-paranormal operator for some $0 < k < 1$. Then T is normaloid.*

Proof. Suppose that T is an absolute-k^*-paranormal operator. In the case where $0 < k < 1$ based on Theorem 2.6 in [22] it follows that T is a $*-$paranormal operator, then by Theorem 1.1 given in [3] it follows that T is normaloid. □

Example 2.10. An example of non-absolute-k^*-paranormal operator which is normaloid. Let us denote by

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Then $\|T^n\| = \|T\|^n$ for all positive integers n. However, the relation

$$\left\||T|^k Tx\right\| \geq \|T^*x\|^{k+1}$$

does not hold for the unit vector $e_3 = (0,0,1)$, by which it is proved that T is a non-absolute-k^*-paranormal operator, but it is normaloid.

3. Tensor product for $\mathcal{A}(k^*)$

Let H and K denote the Hilbert spaces. For given non zero operators $T \in B(H)$ and $S \in B(K)$, $T \otimes S$ denotes the tensor product on the product space $H \otimes K$. The normaloid property is invariant under tensor products, [24]. There exist paranormal operators T and S, such that $T \otimes S$ is not paranormal, [1]. $T \otimes S$ is normal if and only if T and S are normal, [25]. This result was extended to the class $\mathcal{A}$ operators, class $\mathcal{A}(k)$ operators, and $*$-class $\mathcal{A}$ operators in [14], [15], and [8], respectively. In this section, we prove an analogues result for $\mathcal{A}(k^*)$ operators.

Let $T \in B(H)$ and $S \in B(K)$ be non zero operators. Then $(T \otimes S)^*(T \otimes S) = T^*T \otimes S^*S$ holds. By the uniqueness of positive square roots, we have $|T \otimes S|^r = |T|^r \otimes |S|^r$ for any positive rational number r. From the density of the rationales in the real, we obtain $|T \otimes S|^p = |T|^p \otimes |S|^p$ for any positive real number p.

Lemma 3.1. [25] *Let $T_1, T_2 \in B(H)$, $S_1, S_2 \in B(K)$ be non-negative operators. If T_1 and S_1 are non-zero, then the following assertions are equivalent:*

(1) $T_1 \otimes S_1 \leq T_2 \otimes S_2$,
(2) *There exists $c > 0$, such that $T_1 \leq cT_2$, and $S_1 \leq c^{-1}S_2$.*

Lemma 3.2. [19, Hölder-McCarthy inequality] *Let T be a positive operator. Then the following inequalities hold for all $x \in H$:*

(1) $\langle T^r x, x\rangle \leq \langle Tx, x\rangle^r \|x\|^{2(1-r)}$ *for* $0 < r < 1$,
(2) $\langle T^r x, x\rangle \geq \langle Tx, x\rangle^r \|x\|^{2(1-r)}$ *for* $r \geq 1$.

Theorem 3.3. *Let $T \in B(H)$ and $S \in B(K)$ be non-zero operators. Then $T \otimes S$ belongs to class $\mathcal{A}(k^*)$ if and only if one of the following holds:*

(1) *T and S are $\mathcal{A}(k^*)$ operators.*
(2) $|T^*|^2 = 0$ *or* $|S^*|^2 = 0$.

Proof. Consider

$$\left((T \otimes S)^*|T \otimes S|^{2k}(T \otimes S)\right)^{\frac{1}{k+1}} - |(T \otimes S)^*|^2 =$$

$$\left((T^* \otimes S^*)(|T|^{2k} \otimes |S|^{2k})(T \otimes S)\right)^{\frac{1}{k+1}} - (|T^*|^2 \otimes |S^*|^2) =$$

$$\left(T^*|T|^{2k}T\right)^{\frac{1}{k+1}} \otimes \left(S^*|S|^{2k}S\right)^{\frac{1}{k+1}} - (|T^*|^2 \otimes |S^*|^2) =$$

$$(T^*|T|^{2k}T)^{\frac{1}{k+1}} \otimes (S^*|S|^{2k}S)^{\frac{1}{k+1}} - |T^*|^2 \otimes (S^*|S|^{2k}S)^{\frac{1}{k+1}} +$$
$$|T^*|^2 \otimes (S^*|S|^{2k}S)^{\frac{1}{k+1}} - (|T^*|^2 \otimes |S^*|^2) =$$
$$\left((T^*|T|^{2k}T)^{\frac{1}{k+1}} - |T^*|^2\right) \otimes (S^*|S|^{2k}S)^{\frac{1}{k+1}} + |T^*|^2 \otimes \left((S^*|S|^{2k}S)^{\frac{1}{k+1}} - |S^*|^2\right).$$

Hence, if either (1) T and S are class $\mathcal{A}(k^*)$ operators, or (2) $|T^*|^2 = 0$ or $|S^*|^2 = 0$, then $T \otimes S$ is a class $\mathcal{A}(k^*)$ operator.

Conversely, suppose that $T \otimes S$ is class $\mathcal{A}(k^*)$. It suffices to show that if the statement (1) does not hold, then the statement (2) holds. Assume that $|T^*|^2$ and $|S^*|^2$ are non-zero operators. Since $T \otimes S$ is a class $\mathcal{A}(k^*)$ operator, then

$$(T^*|T|^{2k}T)^{\frac{1}{k+1}} \otimes (S^*|S|^{2k}S)^{\frac{1}{k+1}} \geq |T^*|^2 \otimes |S^*|^2.$$

Therefore, by Lemma 3.1 there exists a positive real number c for which

$$c(T^*|T|^{2k}T)^{\frac{1}{k+1}} \geq |T^*|^2 \text{ and } c^{-1}(S^*|S|^{2k}S)^{\frac{1}{k+1}} \geq |S^*|^2.$$

Consequently, for arbitrary $x, y \in H$, by Hölder–McCarthy inequality we have

$$\begin{aligned}
&\|T^*\|^2 \\
&= \sup_{\|x\|=1} \langle |T^*|^2 x, x\rangle \leq \sup_{\|x\|=1} \left\langle c(T^*|T|^{2k}T)^{\frac{1}{k+1}} x, x\right\rangle \\
&\leq c\left(\sup_{\|x\|=1} \langle T^*|T|^{2k}Tx, x\rangle\right)^{\frac{1}{k+1}} = c\|T^*|T|^{2k}T\|^{\frac{1}{k+1}} \\
&\leq c\|T\|^2 = c\|T^*\|^2.
\end{aligned}$$

So, $\|T^*\|^2 \leq c\|T^*\|^2$. In the same way we show $\|S^*\|^2 \leq c^{-1}\|S^*\|^2$. Thus $c = 1$, and hence T and S are class $\mathcal{A}(k^*)$ operators. □

4. Behavior of class $M - \mathcal{A}(k^*)$ operators

For each $k > 0$, an operator T is a class $M - P\mathcal{A}(k^*)$ operator, for some $M > 0$, if

$$\left(T^*|T|^{2k}T\right) \geq M^{k+1}|T^*|^{2(k+1)},$$

(for $M = 1$ it's equal to the class $P\mathcal{A}(k^*)$ of operators). T is a class $M - \mathcal{A}(k^*)$ operator, for some $M > 0$, if

$$\left(T^*|T|^{2k}T\right)^{\frac{1}{k+1}} \geq M|T^*|^2,$$

(for $M = 1$ it's equal to the class $\mathcal{A}(k^*)$ of operators).

In what follows we will introduce the class of absolute-$k^* - M$-paranormal operators, as follows: For each $k > 0$, an operator T is an absolute-$k^* - M$-paranormal (class $M - \mathcal{A}_{(k^*)}P$) operator if

$$\left\||T|^k Tx\right\| \geq M\|T^*x\|^{k+1},$$

for every unit vector $x \in H$.

The class $\mathcal{A}(k^*)$ of operators is included in the absolute-k^*-paranormal operators for any $k > 0$ (see Remark 2.2). The same fact is valid for the class $M - \mathcal{A}(k^*)$ of operators, which is included in absolute-$k^* - \sqrt{M^{k+1}}$-paranormal operators. The following Lemma holds.

Lemma 4.1. *For each $k > 0$ and each $M > 0$, every class $M - \mathcal{A}(k^*)$ operator is absolute-$k^* - \sqrt{M^{k+1}}$-paranormal operator.*

The proof of the lemma is similar to Theorem 2 in [10] and is done by taking into consideration the Hölder–McCarthy inequality.

The following lemma is valid:

Lemma 4.2. *If T is an operator from Hilbert space H, then the following relation holds:*

$$M - P\mathcal{A}(k^*) \subset M - \mathcal{A}(k^*), \text{ for every } k > 0.$$

Remark 4.3. In the assertion given in Lemma 4.1 and Lemma 4.2 the equality between the mentioned class of operators is not valid.

It is known that there exists a linear operator T for which an exponent of it, let us say T^n, is a compact operator but T itself is not compact. In this context we will show that in the case where an operator T is from class $M - \mathcal{A}(k^*)$ and if its exponent T^n is compact, for some n, then T is compact too. In Lemma 4.1 it was proved that every class $M - \mathcal{A}(k^*)$ operator is an absolute-$k^* - \sqrt{M^{k+1}}$-paranormal operator. Using this fact we will show the following theorem.

Theorem 4.4. *If T is an operator from $M - \mathcal{A}(k^*)$ and if T^n is compact for some $n \in \mathbb{N}$, then it follows that T is compact too.*

Proof. If $T \in B(H)$ we have

$$\begin{aligned} &\|T^*T^{n-1}x\| \\ &\qquad = \left\| T^*T\frac{T^{n-2}x}{\|T^{n-2}x\|} \right\| \|T^{n-2}x\| \\ &\qquad \geq \left\| T\frac{T^{n-2}x}{\|T^{n-2}x\|} \right\|^2 \|T^{n-2}x\| = \frac{\|T^{n-1}x\|^2}{\|T^{n-2}x\|}. \end{aligned}$$

To prove theorem it is enough to prove that T^{n-1} is compact. Let us consider the unit vector $\frac{T^{n-1}x}{||T^{n-1}x||} \in H$, for $n \geq 2$. Since T is an absolute-$k^* - \sqrt{M^{k+1}}$-paranormal operator, and from above relation, we have

$$\begin{aligned} &\sqrt{M^{k+1}}\frac{\|T^{n-1}x\|^{2(k+1)}}{\|T^{n-2}x\|^{k+1}||T^{n-1}x||^{k+1}} \\ &\qquad \leq \sqrt{M^{k+1}} \left\| T^* \frac{T^{n-1}x}{||T^{n-1}x||} \right\|^{k+1} \\ &\qquad \leq \left\| |T|^k T \frac{T^{n-1}x}{||T^{n-1}x||} \right\|. \end{aligned}$$

Then,

$$\begin{aligned} &\sqrt{M^{k+1}} \left\|T^{n-1}x\right\|^{k+2} \\ &\qquad \leq ||T^{n-2}x||^{k+1} \left\||T|^k T T^{n-1}x\right\| \\ &\qquad \leq ||T^{n-2}x||^{k+1} \left\||T|^k\right\| \|T^n x\|. \end{aligned}$$

Let (x_m) be any unit sequence of vectors from H, such that $x_m \xrightarrow{w} 0, m \to \infty$. Now from compactness of T^n and above relation it follows that

$$\left\|T^{n-1}x_m\right\|^{k+2} \to 0,\ m \to \infty,$$

hence T^{n-1} is a compact operator. □

Corollary 4.5. *If T, S are operators from $M - \mathcal{A}(k^*)$, and if T^n and S^m are compact operators for some $n, m \in \mathbb{N}$, then it follows that $T \oplus S$ is compact too.*

Proof. Let us suppose that T^n and S^m are compact operators for some $n, m \in \mathbb{N}$. Then from Theorem 4.4 it follows that T and S are compact operators. Without lose of generality we can consider that $n = m$. Let (x_n, y_n) be any unit vector sequence from $H \oplus H$ such that $(x_n, y_n) \xrightarrow{w} (0, 0)$. Then

$$||(Tx_n, Sy_n)|| = ||Tx_n|| + ||Sy_n|| \to 0,$$

when $n \to \infty$. □

Corollary 4.6. *If $T_i, i \in \mathbb{N}$ are operators from $M - \mathcal{A}(k^*)$, and if T_i^n are compact for each $i \in \mathbb{N}$ and for some $n \in \mathbb{N}$, then it follows that $(\oplus_i T_i)_p^m$, for $1 \leq p \leq \infty$ is compact too.*

A norm on the algebraic tensor product $H \otimes H$ is called a tensor norm or a cross norm if $||x \otimes y|| = ||x|| \cdot ||y||$ for all decomposable tensors $x \otimes y$ (for further details see [18], [6]).

Theorem 4.7. *If T, R are operators from $M - \mathcal{A}(k^*)$ and if T^n is compact for some $n \in \mathbb{N}$ or R^m is compact for some $m \in \mathbb{N}$, then it follows that $T \otimes R$ is compact too.*

Proof. Let us suppose that T^n is a compact operator for some $n \in \mathbb{N}$. Then from Theorem 4.4 it follows that T is a compact operator. Let $x_n \otimes y_n$ be any unit vector sequence from $H \otimes H$ such that $x_n \otimes y_n \xrightarrow{w} 0$. Then

$$||Tx_n \otimes Ry_n|| = ||Tx_n|| \cdot ||Ry_n|| \to 0,$$

when $n \to \infty$, completing the proof. □

Lemma 4.8. *Let T be a bilateral weighted shift operator with weights $\{\alpha_n : n \in \mathbb{Z}\}$. Then T is a class $M - \mathcal{A}(k^*)$ operator if and only if*

$$|\alpha_n|^2 \cdot |\alpha_{n+1}|^{2k} \geq M^{k+1}|\alpha_{n-1}|^{2(k+1)},$$

for all $n \in \mathbb{Z}$.

Lemma 4.9. *Let T be a non-singular bilateral weighted shift operator with weights $\{\alpha_n : n \in \mathbb{Z}\}$. Then T^{-1} is a class $M - \mathcal{A}(k^*)$ operator if and only if*

$$|\alpha_{n-1}|^2|\alpha_{n-2}|^{2k}M^{k+1} \leq |\alpha_n|^{2(k+1)},$$

for all $n \in \mathbb{Z}$.

Considering the above lemmas, we will further show that if $T \in M - \mathcal{A}(k^*)$ class, then it does not follow that $T^{-1} \in M - \mathcal{A}(k^*)$ class operator. This fact is shown by the following example.

Example 4.10. Let us denote by T the bilateral weighted shift operator, with weighted sequence $\{\alpha_n : n \in \mathbb{Z}\}$, given by the relation:

$$\alpha_n = \begin{cases} 1 & \text{if } n \leq 0 \\ \frac{n}{n+1} & \text{if } n \geq 1. \end{cases}$$

Following Lemma 4.8 it follows that $T \in \frac{1}{3} - \mathcal{A}(1^*)$ but $T^{-1} \notin \frac{1}{3} - \mathcal{A}(1^*)$ which follows from Lemma 4.9, for $n = 1$.

5. Spectral properties

A complex number $\lambda \in \mathbb{C}$ is said to be in the point spectrum $\sigma_p(T)$ of the operator T if there is a vector $x \neq 0$ satisfying $(T - \lambda)x = 0$.

If $T \in B(H)$, we shall write $N(T)$ and $R(T)$ for the null space and the range of T, respectively. Also, let $\sigma(T)$ and $\sigma_a(T)$ denote the spectrum and the approximate point spectrum of T, respectively. An operator T is called Fredholm if $R(T)$ is closed, $\alpha(T) = \dim N(T) < \infty$ and $\beta(T) = \dim H/R(T) < \infty$. Moreover if

$$\operatorname{ind}(T) = \alpha(T) - \beta(T) = 0,$$

then T is called Weyl operator. The essential spectrum $\sigma_e(T)$ and the Weyl spectrum $\sigma_w(T)$ are defined by

$$\sigma_e(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Fredholm}\}$$

and

$$\sigma_w(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Weyl}\},$$

respectively. It is known that $\sigma_e(T) \subset \sigma_w(T) \subset \sigma_e(T) \cup \operatorname{acc}\sigma(T)$ where we write acc K for the set of all accumulation points of $K \subset \mathbb{C}$.

An operator $T \in B(H)$ is said to have finite ascent if $N(T^m) = N(T^{m+1})$ for some positive integer m, and finite descent if $R(T^n) = R(T^{n+1})$ for some positive integer n. The operator T is called Browder if it is Fredholm of finite ascent and descent. The Browder spectrum of T is given by

$$\sigma_b(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Browder}\}.$$

We say that Browder's theorem holds for T if

$$\sigma_w(T) = \sigma_b(T).$$

T is said to have the single valued extension property, abbreviated, T has SVEP if $f(z)$ is an analytic vector valued function on some open set $D \subset \mathbb{C}$ such that $(T - z)f(z) = 0$ for all $z \in D$, then $f(z) = 0$ for all $z \in D$.

In this section we study spectral properties of absolute-k^*-paranormal operators. It is shown that if T is an absolute-k^*-paranormal operator, then the spectral mapping theorem holds for the essential approximate point spectrum and for Weyl spectrum. It is also shown that a-Browder's theorem holds for absolute-k^*-paranormal operators. Spectral properties of different classes of operators are given in [11, 21].

Theorem 5.1. *Let $T \in B(H)$ be an absolute-k^*-paranormal operator for $0 < k \leq 1$. If $(T - \lambda)x = 0$, then $(T - \lambda)^*x = 0$ for all $\lambda \in \mathbb{C}$.*

Proof. We have to show that $N(T-\lambda) \subseteq N(T^*-\overline{\lambda})$.

Let $\lambda = 0$ and assume that $x \in N(T)$, then $Tx = 0$. Since T is absolute-k^*-paranormal, from $\|T^*x\|^{k+1} \le \||T|^kTx\|$, since $Tx = 0$, we have $\|T^*x\|^{k+1} = 0$, therefore $T^*x = 0$, so $x \in N(T^*)$.

Let $\lambda \neq 0$ and assume that $x \in N(T-\lambda)$. Then $Tx = \lambda x$ and since T is absolute-k^*-paranormal, we have

$$\begin{aligned}\|T^*x\|^{k+1} &\le \||T|^kTx\| = |\lambda|\langle |T|^{2k}x, x\rangle^{\frac{1}{2}} \le |\lambda|\langle |T|^2x, x\rangle^{\frac{k}{2}} \\ &= |\lambda|||Tx||^k = |\lambda|^{k+1}.\end{aligned}$$

Then $\|T^*x\| \le |\lambda|$ for all $x \in N(T-\lambda)$ with $\|x\| = 1$. So if $x \in N(T-\lambda)$, then

$$\begin{aligned}&\langle (T-\lambda)^*x, (T-\lambda)^*x\rangle \\ &\qquad = \quad ||T^*x||^2 - \langle x, \overline{\lambda}Tx\rangle - \langle \overline{\lambda}Tx, x\rangle + |\lambda|^2\|x\|^2 \\ &\qquad \le \quad |\lambda|^2 - |\lambda|^2 - |\lambda|^2 + |\lambda|^2 = 0.\end{aligned}$$

Therefore

$$\|T^* - \overline{\lambda}x\| = 0,$$

and so $x \in N(T^*-\overline{\lambda})$. Hence $N(T-\lambda) \subseteq N(T^*-\overline{\lambda})$. □

Lemma 5.2. *If T is an absolute-k^*-paranormal operator for $0 < k \le 1$, then $T-\lambda$ has finite ascent for each $\lambda \in \mathbb{C}$.*

Proof. Since T is an absolute-k^*-paranormal operator, it follows that $N(T-\lambda) \subset N(T^*-\overline{\lambda})$, for each $\lambda \in \mathbb{C}$ by Theorem 5.1. Thus we can represent $T-\lambda$ as the following 2x2 operator matrix with respect to the decomposition $N(T-\lambda) \oplus N(T-\lambda)^{\perp}$:

$$T-\lambda = \begin{pmatrix} 0 & 0 \\ 0 & S \end{pmatrix}.$$

Let $x \in N((T-\lambda)^2)$, and let's write $x = y + z$, where $y \in N(T-\lambda)$ and $z \in N(T-\lambda)^{\perp}$. Then $0 = (T-\lambda)^2x = (T-\lambda)^2z$, so that

$$(T-\lambda)z \in N(T-\lambda) \cap N(T-\lambda)^{\perp} = \{0\},$$

which implies that $z \in N(T-\lambda)$, and hence $x \in N(T-\lambda)$. Therefore $N(T-\lambda) = N(T-\lambda)^2$. □

Corollary 5.3. *If T is an absolute-k^*-paranormal operator for $0 < k \le 1$, then T has SVEP.*

Proof. Proof of the corollary follows directly from Lemma 5.2 and Proposition 1.8 in [16]. □

Corollary 5.4. *If T^* is absolute-k^*-paranormal for $0 < k \le 1$, then $\beta(T-\lambda) \le \alpha(T-\lambda)$ for all $\lambda \in \mathbb{C}$.*

Proof. It is obvious from Theorem 5.1. □

Now we will show that the spectral mapping theorem holds for Weyl's spectrum.

Theorem 5.5. *If T or T^* is absolute-k^*-paranormal for $0 < k \le 1$, then $\sigma_w(f(T)) = f(\sigma_w(T))$ for every $f \in Hol(\sigma(T))$, where $Hol(\sigma(T))$ denotes the set of all analytic functions on some open neighborhood of $\sigma(T)$.*

Proof. Since $\sigma_w(f(T)) \subseteq f(\sigma_w(T))$ holds for any operator, we need only to prove that

$$f(\sigma_w(T)) \subseteq \sigma_w(f(T)). \tag{5.1}$$

Note that (5.1) clearly holds if f is constant on some open neighborhood of $\sigma(T)$.

Let $\lambda \notin \sigma_w(f(T))$, we may assume that $f(z)-\lambda$ has only a finitely number of zeros in some open neighborhood G of $\sigma(T)$. Now we write

$$f(z)-\lambda = (z-\lambda_1)(z-\lambda_2)\cdot \ldots \cdot (z-\lambda_n)g(z),$$

where $\lambda_j, j = 1, \cdots, n$ are the zeros of $f(z)-\lambda$ in G, listed according to multiplicity, and $g(z) \neq 0$ for all $z \in G$. Thus

$$f(T)-\lambda = (T-\lambda_1)(T-\lambda_2)\cdot \ldots \cdot (T-\lambda_n)g(T). \tag{5.2}$$

Clearly, $\lambda \in f(\sigma_w(T))$ if and only if $\lambda_j \in \sigma_w(T)$ for some j. Therefore, to prove (5.1), it suffices to show that $\lambda_j \notin \sigma_w(T)$ for all $j = 1, 2, \cdots, n$. First, suppose that T is absolute-k^*-paranormal. Since $f(T)-\lambda$ is Weyl and the operators $T-\lambda_1, T-\lambda_2, \cdots, T-\lambda_n$ commute, each $T-\lambda_j$ is Fredholm. Moreover, since

$$N(T-\lambda_j) \subseteq N(f(T)-\lambda) \text{ and } N((T-\lambda_j)^*) \subseteq N((f(T)-\lambda)^*),$$

both $N(T-\lambda_j)$ and $N((T-\lambda_j)^*)$ are finite dimensional. Then $\text{ind}(T-\lambda_j) \leq 0$ by Theorem 5.1. Then $\text{ind}(f(T)-\lambda) = \text{ind}(g(T)) = 0$, it follows from (5.2) that $\text{ind}(T-\lambda_j) = 0$ for all $j = 1, 2, \cdots, n$. Consequently, $T-\lambda_j$ is Weyl, and $\lambda_j \notin \sigma_w(T)$, for all $j = 1, 2, \cdots, n$.

Now assume that T^* is absolute-k^*-paranormal. Then by Corollary 5.4 $\text{ind}(T-\lambda) \geq 0$ for each $j = 1, 2, \cdots, n$. However,

$$\sum_{i=1}^{n} \text{ind}(T-\lambda_j) = \text{ind}(f(T)-\lambda) = 0,$$

and so $T-\lambda_j$ is Weyl for each $j = 1, 2, \cdots, n$. Hence $\lambda \notin f(\sigma_w(T))$. Therefore $\sigma_w(f(T)) = f(\sigma_w(T))$. □

Let $T \in B(H)$. The essential approximate point spectrum $\sigma_{ea}(T)$ is defined by

$$\sigma_{ea}(T) = \cap\{\sigma_a(T+K) : K \text{ is a compact operator}\}.$$

We consider the set

$$\Phi_+^-(H) = \{T \in B(H) : T \text{ is left semi-Fredholm and } \text{ind}(T) \leq 0\}.$$

V. Rakočević [23], proved that

$$\sigma_{ea}(T) = \{\lambda \in \mathbb{C} : T-\lambda \notin \Phi_+^-(H)\}$$

and the inclusion $\sigma_{ea}(f(T)) \subset f(\sigma_{ea}(T))$ holds for every $f \in Hol(\sigma(T))$. The next theorem shows the spectral mapping theorem on the essential approximate point spectrum of absolute-k^*-paranormal operator.

Theorem 5.6. *Let T or T^* be absolute-k^*-paranormal for $0 < k \leq 1$. Then $\sigma_{ea}(f(T)) = f(\sigma_{ea}(T))$ for every $f \in Hol(\sigma(T))$.*

Proof. The proof is similar to Theorem 2.9 given in [12], based on Corollary 5.4 and Lemma 5.2. □

Recall [17] that $S, T \in B(H)$ are said to be quasi-similar if there exist injections $X, Y \in B(H)$ with dense range such that $XS = TX$ and $YT = SY$, respectively, and this relation is denoted by $S \sim T$.

Theorem 5.7. *Let $T \in B(H)$ be absolute-k^*-paranormal for $0 < k \leq 1$. If $S \sim T$, then S has SVEP.*

Proof. Since T is absolute-k^*-paranormal, it follows from Corollary 5.3 that T has SVEP. Let U be any open set and $f : U \to H$ be any analytic function such that $(S - \lambda)f(\lambda) = 0$ for all $\lambda \in U$. Since $S \sim T$, there exists an injective operator A with dense range such that $AS = TA$. Thus $A(S - \lambda) = (T - \lambda)A$ for all $\lambda \in U$. Since $(S - \lambda)f(\lambda) = 0$ for all $\lambda \in U$,

$$A(S - \lambda)f(\lambda) = 0 = (T - \lambda)Af(\lambda),$$

for all $\lambda \in U$. But T has SVEP, hence $Af(\lambda) = 0$ for all $\lambda \in U$. Since A is injective, $f(\lambda) = 0$ for all $\lambda \in U$. Thus S has SVEP. $\square$

Now we will show that a-Browder's theorem holds for absolute-k^*-paranormal operators. For this we need the following definitions. The Browder essential approximate point spectrum $\sigma_{ab}(T)$ of T is defined by

$$\sigma_{ab}(T) = \cap\{\sigma_a(T + K) : TK = KT,\ K \text{ is a compact operator}\}.$$

We say that a-Browder's theorem holds for T if

$$\sigma_{ea}(T) = \sigma_{ab}(T).$$

It is well known that

$$a\text{-Browder's theorem} \Rightarrow \text{Browder's theorem}.$$

Theorem 5.8. *Let $T \in B(H)$ be an absolute-k^*-paranormal operator for $0 < k \leq 1$. Then T obeys a-Browder's theorem.*

Proof. Since an absolute-k^*-paranormal operator has SVEP, Theorem 2.8 in [20] implies that T obeys a-Browder's theorem. $\square$

Theorem 5.9. *Let $T \in B(H)$ be an absolute-k^*-paranormal operator for $0 < k \leq 1$. Then a-Browder's theorem holds for $f(T)$ for every $f \in Hol(\sigma(T))$.*

Proof. Since $\sigma_{ea}(f(T)) = f(\sigma_{ea}(T))$, the rest of the proof follows as in the proof of Corollary 2.3 in [20]. $\square$

Theorem 5.10. *Let $T \in B(H)$ be absolute-k^*-paranormal for $0 < k \leq 1$. If $S \sim T$, then a-Browder's theorem holds for $f(S)$ for every $f \in Hol(\sigma(T))$.*

Proof. Since a-Browder's theorem holds for S, we have

$$\sigma_{ab}(f(S)) = f(\sigma_{ab}(S)) = f(\sigma_{ea}(S)) = \sigma_{ea}(f(S)).$$

Hence a-Browder's theorem holds for $f(S)$. $\square$

A complex number λ is said to be in the approximate point spectrum, $\sigma_a(T)$, of T if there is a sequence $\{x_n\}_{n=1}^{\infty}$ of unit vectors in H such that $(T - \lambda)x_n \to 0$. If in addition, $(T - \lambda)^* x_n \to 0$, then λ is said to be in the approximate joint point spectrum, $\sigma_{ja}(T)$, of T.

Theorem 5.11. *Let $T \in B(H)$ be absolute-k^*-paranormal with $0 < k \leq 1$. If $\lambda \in \sigma_a(T)$, then $\lambda \in \sigma_{ja}(T)$.*

Proof. Assume that $\lambda \in \sigma_a(T)$, then there is a sequence $\{x_n\}_{n=1}^{\infty}$ of unit vectors in H such that $(T-\lambda)x_n \to 0$. Since T is absolute-k^*-paranormal, $\|T^*x_n\|^{k+1} \leq \||T|^kTx_n\|$ for every unit sequence $\{x_n\}_{n=1}^{\infty} \subset H$. Hence

$$\begin{aligned}\|T^*x_n\|^{k+1} \leq \||T|^kTx_n\| & \\ &= \langle |T|^{2k}Tx_n, Tx_n\rangle^{\frac{1}{2}} \\ &\leq \langle |T|^2Tx_n, Tx_n\rangle^{\frac{k}{2}}\|Tx_n\|^{1-k} \\ &= \|T^2x_n\|^k\|Tx_n\|^{1-k}\end{aligned}$$

Since $(T-\lambda)x_n \to 0$, then $\lim_{n\to\infty}\|T^*x_n\| \leq |\lambda|$, and we have

$$\begin{aligned}\|(T^*-\overline{\lambda})x_n\|^2 & \\ &= \langle (T-\lambda)^*x_n, (T-\lambda)^*x_n\rangle \\ &= \|T^*x_n\|^2 - \langle x_n, \overline{\lambda}Tx_n\rangle - \langle \overline{\lambda}Tx_n, x_n\rangle + |\lambda|^2\|x_n\|^2.\end{aligned}$$

Therefore $\|(T^*-\overline{\lambda})x_n\|^2 \to 0$ as $n \to \infty$ and so $(T^*-\overline{\lambda})x_n \to 0$ as $n \to \infty$. ☐

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[1] Department of Mathematics and Computer Sciences, University of Prishtina, Avenue "Mother Theresa" 5, Prishtinë, 10000, Kosova.

E-mail address: nbraha@yahoo.com; ilmihoxha011@gmail.com

[2] Taibah University, College of Science, Department of Mathematics, P. O. Box 30002, Al-Madinah-AlMunawarah, Saudi-Arabia.

E-mail address: mecherisalah@hotmail.com