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ON DIVERGENCE OF ANY ORDER CESÀRO MEAN OF LOTKA–VOLTERRA OPERATORS

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ABSTRACT. Based on some numerical calculations, S.M. Ulam has conjectured that the ergodic theorem holds true for any quadratic stochastic operator acting on the finite dimensional simplex. However, M.I. Zakharevich showed that Ulam's conjecture is false in general. Later, N.N. Ganikhodjaev and D.V. Zanin have generalized Zakharevich's example in the class of quadratic stochastic Volterra operators acting on 2D simplex. In this paper, we provide a class of Lotka–Volterra operators for which any order Cesàro mean diverges. This class of Lotka–Volterra operators encompasses all previously presented operators in this context.

1. Introduction

A mapping $V : S^{m-1} \to S^{m-1}$, $V(x) = x'$ such that

$$x'_k = \sum_{i,j=1}^{m} p_{ijk} x_i x_j, \quad \forall\, k = \overline{1, m}$$

is called a quadratic stochastic operator, where $p_{ijk} = p_{jik} \geq 0$, $\sum\limits_{k=1}^{m} p_{ijk} = 1$ for all i, j, k and $S^{m-1} = \{x = (x_1, \cdots, x_m) \in \mathbb{R}^m : x_i \geq 0,\ \sum\limits_{i=1}^{m} x_i = 1\}$ is the $(m-1)$-dimensional standard simplex. Based on some numerical calculations, S.M. Ulam conjectured [7] that the ergodic theorem holds true for any quadratic stochastic

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operator V, that is the limit

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}V^{(i)}(x) \tag{1.1}$$

exists for any $x\in S^{m-1}$ where $V^{(i+1)}=V\circ V^{(i)}$. However, Zakharevich showed [8] that Ulam's conjecture is false in general. He showed that the limit (1.1) does not exist for the following quadratic stochastic operator $V: S^2\to S^2$ for any $x\in \mathrm{Int}S^2\setminus\{(\frac{1}{3},\frac{1}{3},\frac{1}{3})\}$ where $V(x)=x'$ and

$$x'_1=x_1^2+2x_1x_2,\quad x'_2=x_2^2+2x_2x_3,\quad x'_3=x_3^2+2x_1x_3. \tag{1.2}$$

Some years later, Ganikhodjaev and Zanin [2] have generalized Zakharevich's example in the class of quadratic stochastic Volterra operators acting on 2D simplex. Namely they proved that the limit (1.1) does not exist for the following quadratic stochastic operator $V_{a,b,c}: S^2\to S^2$ for any $x\in\mathrm{Int}S^2\setminus\{P_0\}$ where $V_{a,b,c}(x)=x'$

$$x'_1=x_1(1+ax_2-bx_3),\quad x'_2=x_2(1-ax_1+cx_3),\quad x'_3=x_3(1+bx_1-cx_2) \tag{1.3}$$

and $P_0=(\frac{c}{a+b+c},\frac{b}{a+b+c},\frac{a}{a+b+c})$ with nonzero parameters $a,b,c\in[-1,1]$ having the same sign. If one has that $a=b=c=1$ then $V_{1,1,1}$ is Zakharevich's example given by (1.2).

We define the k-th order Cesàro mean by the following formula

$$Ces_k^{(n)}(x,V)=\frac{1}{n}\sum_{i=0}^{n-1}Ces_{k-1}^{(i)}(x,V),\quad k\in\mathbb{N} \tag{1.4}$$

and $Ces_0^{(i)}(x,V)=V^{(i)}(x)$.

It is clear that the first order Cesàro mean $Ces_1^{(n)}(x,V)$ is nothing but $\frac{1}{n}\sum\limits_{i=0}^{n-1}V^{(i)}(x)$. Based on these notations, the previous results say that the first order Cesàro mean $\left\{Ces_1^{(n)}(x,V_{a,b,c})\right\}_{n=0}^{\infty}$ of the operator $V_{a,b,c}$ given by (1.3) diverges for any $x\in\mathrm{Int}S^2\setminus\{P_0\}$. Surprisingly, it was proven in [6] that any order Cesàro mean $\left\{Ces_k^{(n)}(x,V_{a,b,c})\right\}_{n=0}^{\infty}$ of the operator $V_{a,b,c}$ diverges for any $x\in\mathrm{Int}S^2\setminus\{P_0\}$. The reader may refer to [3, 5] for the resent development of this subject.

A mapping $V_{\mathbf{F}}: S^{m-1}\to S^{m-1}$ $V_{\mathbf{F}}(x)=x'$ such that

$$x'_k=x_k(1+f_k(x)),\quad k=\overline{1,m}$$

is called a Lotka–Volterra operator where the mapping $\boldsymbol{F}\equiv(f_1,f_2,\dots,f_m): S^{m-1}\to\mathbb{R}^m$ satisfies the following conditions:

1^0 The mapping $\boldsymbol{F}$ is continuous;
2^0 One has that $f_k(x)\geq -1$ for all $x\in S^{m-1}$ and $k=\overline{1,m}$;
3^0 One has that $\sum\limits_{k=1}^{m}x_kf_k(x)=0$ for all $x\in S^{m-1}$;
4^0 For every $\alpha\subset\{1,2,\cdots,m\}$ one has that $f_k(x)>-1$, $\forall\, x\in\mathrm{Int}\Gamma_\alpha$, $k\in\alpha$.

where $\Gamma_\alpha = conv\{e_k\}_{k\in\alpha}$ and $\{e_k\}_{i=1}^m$ is the standard basis of $\mathbb{R}^m$.

A Lotka–Volterra operator is a discrete analogy of a generalized predator-prey model. It is worth of mentioning that a quadratic stochastic operator is the Lotka–Volterra operator if and only if $p_{ijk} = 0$ whenever $k \notin \{i, j\}$.

In this paper, we present a new class of Lotka–Volterra operators (not quadratic stochastic operator) defined on the two dimensional simplex for which the ergodic theorem will fail.

2. Non-Ergodic Lotka–Volterra Operators

Let $f : S^2 \to [-1, 1]$ be any C^1-smooth functional (having the first order continuous partial derivatives). We define the Lotka–Volterra operator $V_f : S^2 \to S^2$, $V_f(x, y, z) = (x', y', z')$ as follows

$$V_f : \begin{cases} x' = x[1 + (ay - bz)f(x, y, z)] \\ y' = y[1 + (cz - ax)f(x, y, z)] \\ z' = z[1 + (bx - cy)f(x, y, z)] \end{cases} \tag{2.1}$$

where $a, b, c \in [-1, 1]$.

It is clear that if $f \equiv const$ then the Lotka–Volterra operator V_f is the quadratic stochastic operator given by (1.3).

Throughout this paper, we always assume that the non-zero parameters a, b, c have the same sign and $f : S^2 \to [-1, 1]$ is a non-vanishing C^1-smooth functional. Moreover, we present the proofs of all results in the case $a, b, c > 0$ and $f : S^2 \to [m, M]$ where $0 < m \le M \le 1$. In other cases, the proofs are similar.

It is easy to check that the fixed points of V_f are only points $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, $e_3 = (0, 0, 1)$, and $P_0 = (\frac{c}{a+b+c}, \frac{b}{a+b+c}, \frac{a}{a+b+c})$. Moreover, P_0 is repelling and e_1, e_2, e_3 are saddle points. The trajectory of V_f starting from any initial point $P \in \partial S^2 \setminus \{e_1, e_2, e_3\}$ moves along the boundary of the simplex as $e_1 \to e_3 \to e_2 \to e_1$.

Lemma 2.1. *The omega limiting set $\omega(P)$ of the trajectory of V_f starting from any initial point $P \in IntS^2 \setminus \{P_0\}$ is infinite and lies on the boundary ∂S^2 of the simplex S^2.*

Proof. We set $P = (x, y, z)$, $\varphi(P) = x^c y^b z^a$ and

$$\psi(P) = [1 + (ay - bz)f(P)]^c [1 + (cz - ax)f(P)]^b [1 + (bx - cy)f(P)]^a.$$

Due to Young's inequality, we get that

$$\psi(P) \le \left(1 + \frac{c(ay - bz)f(P) + b(cz - ax)f(P) + a(bx - cy)f(P)}{a + b + c}\right)^{a+b+c} = 1.$$

On the other hand, we obtain that

$$\varphi\left(V_f^{(n+1)}(P)\right) = \varphi\left(V_f^{(n)}(P)\right)\psi\left(V_f^{(n)}(P)\right) \le \varphi\left(V_f^{(n)}(P)\right),$$

i.e. $\left\{\varphi\left(V_f^{(n)}(P)\right)\right\}_{n=0}^{\infty}$ is a decreasing sequence and converges to some limit λ. Since $P \ne P_0$, we have that $0 \le \lambda < \varphi(P) < \varphi(P_0)$.

If $\lambda > 0$ then $1 = \lim_{n\to\infty} \frac{\varphi\left(V_f^{(n+1)}(P)\right)}{\varphi\left(V_f^{(n)}(P)\right)} = \lim_{n\to\infty} \psi\left(V_f^{(n)}(P)\right) \leq 1$. Consequently, for any $P^* \in \omega(P)$, one has that $1 = \psi(P^*) = \max \psi(P)$, i.e, $P^* = P_0$. However, this contradicts to $\varphi(P^*) = \lambda < \varphi(P_0)$. It shows that $\lambda = 0$, i.e.,

$$\omega(P) \subset \partial S^2 = \{x \in S^2 : x_1 x_2 x_3 = 0\}.$$

Obviously, $\omega(P) \neq \{e_1, e_2, e_3\}$. Since $V_f(\omega(P)) = \omega(P)$, the set $\{V_f^{(n)}(P^*)\}_{n=0}^{\infty} \subset \omega(P)$ for $P^* \in \omega(P) \setminus \{e_1, e_2, e_3\}$ is infinite and so is also $\omega(P)$. This completes the proof. $\square$

Let us introduce the following subsets of the simplex: $G_1 = \{\frac{x}{c} \geq \frac{y}{b} \geq \frac{z}{a}\}$, $G_2 = \{\frac{x}{c} \geq \frac{z}{a} \geq \frac{y}{b}\}$, $G_3 = \{\frac{z}{a} \geq \frac{x}{c} \geq \frac{y}{b}\}$, $G_4 = \{\frac{z}{a} \geq \frac{y}{b} \geq \frac{x}{c}\}$, $G_5 = \{\frac{y}{b} \geq \frac{z}{a} \geq \frac{x}{c}\}$, $G_6 = \{\frac{y}{b} \geq \frac{x}{c} \geq \frac{z}{a}\}$. The notation $G_i \hookrightarrow G_j$ stands for $V_f(G_i) \subset G_i \cup G_j$.

Lemma 2.2. *In a long run time, the trajectory of V_f moves along the periodic itinerary $G_1 \hookrightarrow G_2 \hookrightarrow G_3 \hookrightarrow G_4 \hookrightarrow G_5 \hookrightarrow G_6 \hookrightarrow G_1$.*

Proof. Let us show that $G_1 \hookrightarrow G_2$. The rest is similar to this case. Since $\omega(P) \subset \partial S^2$ for any $P \in IntS^2 \setminus \{P_0\}$, in a long run time, one has that $V_f^{(n)}(P) \in \partial S^2_\varepsilon = \{P \in S^2 : \mathrm{dist}(P, \partial S^2) < \varepsilon\}$ for sufficiently small ε. So, we want to show that $V_f\left(G_1 \cap \partial S^2_\varepsilon\right) \subset G_1 \cup G_2$.

Obviously, for $P = (x, y, z) \in G_1 \cap \partial S^2_\varepsilon$, one has that $z < \varepsilon$. Since $ay \geq bz$ and $ax \geq cz$, one has that $\frac{x'}{c} \geq \frac{x}{c} \geq \frac{y}{b} \geq \frac{y'}{b}$. On the other side, we have that

$$\frac{z'}{a} \leq \frac{2z}{a} \leq \frac{2\varepsilon}{a} \leq \frac{\min\{a,b,c\}}{(a+b+c)c} \leq \frac{x}{c} \leq \frac{x'}{c}$$

here, ε is sufficiently small, say $\varepsilon < \frac{\min\{a,b,c\}}{2(a+b+c)} \min\{\frac{a}{b}, \frac{b}{a}, \frac{a}{c}, \frac{c}{a}, \frac{b}{c}, \frac{c}{b}\}$, which is suitable for all cases. This means that $\frac{x'}{c} \geq \max\{\frac{y'}{b}, \frac{z'}{a}\}$, i.e., $V_f(P) = P' \in G_1 \cup G_2$. This completes the proof. $\square$

Let us choose a neighborhood U_0 of P_0 such that $U_0 \subset IntS^2$ and $U_1 = (G_1 \cup G_2) \setminus U_0$, $U_2 = (G_3 \cup G_4) \setminus U_0$, $U_3 = (G_5 \cup G_6) \setminus U_0$ are convex sets which satisfy $U_1 \cap U_2 \cap U_3 = \emptyset$.

Lemma 2.3. *Let $P \notin U$, $V_f^{(k)}(P) \in U$ for all $k = \overline{1,n}$ and $V_f^{(n+1)}(P) \notin U$, where U is one of the sets U_i, $i = 1, 2, 3$ and $P_0 \neq P \in IntS^2$. Then, $n > A \log \frac{B}{\varphi(V_f(P))}$ where A, B are some positive constants.*

Proof. Let $P = (x, y, z)$ and $V_f^{(k)}(P) = (x_k, y_k, z_k)$. Without loss of generality, we suppose that $U = U_1$. Since $P \notin U_1$, one has that $P \in G_6$, or equivalently, $\frac{y}{b} \geq \frac{x}{c} \geq \frac{z}{a}$. Since $V_f(P) \in U_1$, one has that $V_f(P) \in G_1$, or equivalently, $\frac{x_1}{c} \geq \frac{y_1}{b} \geq \frac{z_1}{a}$. Since $V_f^{(n+1)}(P) \notin U_1$, one has that $V_f^{(n+1)}(P) \in G_3$, or equivalently, $\frac{z_{n+1}}{a} \geq \frac{x_{n+1}}{c} \geq \frac{y_{n+1}}{b}$. Therefore, $y \geq \alpha$, $x_1 \geq \alpha$, and $z_{n+1} \geq \alpha$ where $\alpha = \frac{\min\{a,b,c\}}{a+b+c}$. We then obtain that

(*i*) $y_1 = y[1 + (cz - ax)f(P)] = (1 - af(P))xy + y^2 + (1 + cf(P))yz \geq y^2 \geq \alpha^2$;

(*ii*) $\frac{z_{n+1}}{z_1} = \prod_{k=1}^{n} \frac{z_{k+1}}{z_k} = \prod_{k=1}^{n} [1 + (bx_k - cy_k)f(x_k, y_k, x_k)] \leq 2^n$;

(iii) $\dfrac{z_{n+1}}{z_1} \geq \dfrac{\alpha}{z_1} = \left(\dfrac{\alpha^a}{z_1^a}\right)^{1/a} = \left(\dfrac{x_1^c y_1^b \alpha^a}{\varphi(V_f(P))}\right)^{1/a} \geq \left(\dfrac{\alpha^{a+2b+c}}{\varphi(V_f(P))}\right)^{1/a}$;

(iv) $2^n \geq \left(\dfrac{\alpha^{a+2b+c}}{\varphi(V_f(P))}\right)^{1/a}$, or equivalently, $n > A\log\dfrac{B}{\varphi(V_f(P))}$ for some constants A, B. This completes the proof.

□

Lemma 2.4. *Let U be one of the sets U_i, $i = 1, 2, 3$ and $P_0 \neq P \in IntS^2$. Let $(n_i, m_i)_{i=1}^{\infty}$ be a sequence of natural numbers such that $V_f^{(n_i)}(P) \notin U$, $V_f^{(n_i+k)}(P) \in U$ for $k = \overline{1, m_i}$, and $V_f^{(n_i+m_i+1)}(P) \notin U$. Then there exists a constant c such that $m_i > cn_i$.*

Proof. Due to Lemma 2.1 and 2.2, one has that $n_i \to \infty$. Let $\rho = \max\limits_{P\in S^2\setminus U_0} \psi(P)$. It is clear that $\rho < 1$. Then, $\varphi(V_f(P)) = \varphi(P)\psi(P) < \rho\varphi(P)$ for any $P \in S^2 \setminus U_0$. Due to Lemma 2.3, we have that $m_i > A\log\frac{B}{\varphi(V_f^{(n_i)}(P))} > A'\log\frac{B}{\rho^{n_i}\varphi(P_0)} > cn_i$ □

Corollary 2.5. *Let $U_{1\varepsilon}, U_{2\varepsilon}, U_{3\varepsilon}$ be a sufficiently small disjoint neighborhoods of the vertices of the simplex S^2 and $U_\varepsilon = U_{1\varepsilon} \cup U_{2\varepsilon} \cup U_{3\varepsilon}$. Let $\Lambda_n = U_\varepsilon \bigcap \left\{V_f^{(k)}(x)\right\}_{k=0}^{n}$ and $\#(\Lambda_n)$ be a number of elements of Λ_n. Then one has that $\lim\limits_{n\to\infty} \frac{\#(\Lambda_n)}{n} = 1$.*

Let $\Upsilon(x, V_f : U)$ be a function such that

$$\Upsilon(x, V_f : U) = \begin{cases} 1 & \text{if} \quad V_f(x) \in U, \\ 0 & \text{if} \quad V_f(x) \notin U. \end{cases}$$

Let $U_{1\varepsilon}, U_{2\varepsilon}, U_{3\varepsilon}$ be a sufficiently small disjoint neighborhoods of the vertices of the simplex S^2 and U be one of the sets $U_{i\varepsilon}$, $i = 1, 2, 3$. Suppose that $x \in U$. We define the following sequence $\left\{\Upsilon\left(x, V_f^{(n)} : U\right)\right\}_{n=0}^{\infty}$, i.e.,

$$\underbrace{1, 1, \cdots, 1}_{p_1}, \underbrace{0, 0, \cdots, 0}_{q_1}, \underbrace{1, 1, \cdots, 1}_{p_2}, \underbrace{0, 0, \cdots, 0}_{q_2}, \cdots \tag{2.2}$$

Corollary 2.6. *Let $\{p_n, q_n\}_{n=1}^{\infty}$ be a sequence defined above. There exists a constant c mentioned in Lemma 2.4 such that for any $n \geq 2$ one has that*

$$p_n > c\sum_{i=1}^{n-1}(p_i + q_i), \quad q_n > c\left(\sum_{i=1}^{n-1}(p_i + q_i) + p_n\right). \tag{2.3}$$

Theorem 2.7. *Let $V_f : S^2 \to S^2$ be the Lotka–Volterra operator given by (2.1). If non-zero parameters a, b, c have the same sign and $f : S^2 \to [-1, 1]$ is a non-vanishing C^1-smooth functional then the limit (1.1) does not exist for any $x \in IntS^2 \setminus \{P_0\}$.*

Proof. Let us assume that $\lim\limits_{n\to\infty} \dfrac{1}{n}\sum\limits_{k=0}^{n-1} V_f^{(k)}(P) = P^*$ for any $P \in IntS^2$ such that $P \neq P_0$. Suppose that $P^* \notin U_1$ and (n_i, m_i) are as in Lemma 2.4. Let

$\delta = \operatorname{dist}(P^*, U_1)$ and $\lambda_i = \dfrac{m_i}{n_i}$. Then $\lambda_i > c$ and $\operatorname{dist}\left(\dfrac{1}{n}\sum_{k=0}^{n-1} V_f^{(k)}(P), P^*\right) < \dfrac{\delta}{K}$ for sufficiently large K. We let

$$P' = \frac{1}{1+\lambda_i}\left(\frac{1}{n_i}\sum_{k=0}^{n_i-1} V_f^{(k)}(P)\right) + \frac{\lambda_i}{1+\lambda_i}\left(\frac{1}{m_i}\sum_{k=n_i}^{n_i+m_i-1} V_f^{(k)}(P)\right).$$

Then, we have that

$$\begin{aligned}
\operatorname{dist}(P', P^*) &\geq \operatorname{dist}\left(\frac{1}{n_i}\sum_{k=0}^{n_i-1} V_f^{(k)}(P), P'\right) - \frac{\delta}{K} \\
&= \frac{\lambda_i}{1+\lambda_i}\operatorname{dist}\left(\frac{1}{n_i}\sum_{k=0}^{n_i-1} V_f^{(k)}(P), \frac{1}{m_i}\sum_{k=n_i}^{n_i+m_i-1} V_f^{(k)}(P)\right) - \frac{\delta}{K} \\
&\geq \frac{c}{1+c}\operatorname{dist}\left(\frac{1}{n_i}\sum_{k=0}^{n_i-1} V_f^{(k)}(P), U_1\right) - \frac{\delta}{K} \\
&\geq \frac{c}{1+c}\left(\operatorname{dist}(P^*, U_1) - \frac{\delta}{K}\right) - \frac{\delta}{K} \\
&\geq \left(\frac{c}{1+c} - \frac{1}{K}\frac{2+c}{1+c}\right)\delta > \frac{\delta}{K}
\end{aligned}$$

for sufficiently large n. This is a contradiction. This completes the proof □

It turns out that any order Cesàro mean $\left\{Ces_k^{(n)}(x, V_f)\right\}_{n=0}^{\infty}$ of the Lotka–Volterra operator V_f diverges for any $x \in \operatorname{Int}S^2 \setminus \{P_0\}$. For that purpose, we need one auxiliary result which was proven in [6].

Lemma 2.8. *Let $\{1^{p_n}0^{q_n}\}_{n=1}^{\infty}$ be a sequence defined by (2.2) in which p_n, q_n satisfy the inequality (2.3). Then any order Cesàro mean of the sequence $\{1^{p_n}0^{q_n}\}_{n=1}^{\infty}$ diverges.*

Proof. For the sake of completeness, we present the proof of this technical lemma in the case when the sequence $\{p_n, q_n\}_{n=1}^{\infty}$ generates a geometric progression $\{q^n\}_{n=0}^{\infty}$ for some integer q. In the general case, the technique is the same.

Consequently, we want to show that any (say k) order Cesàro mean of the sequence $\{a_n\}_{n=0}^{\infty} = \{1^{q^n}0^{q^{n+1}}\}_{n=0}^{\infty}$ diverges. To do so, it is enough to show that the sequence $\{\gamma_n^{(k)}\}_{n=0}^{\infty}$ diverges (see [1, 4]), where

$$\gamma_n^{(k)} = \frac{C_{n+k-1}^{k-1}a_0 + C_{n+k-2}^{k-1}a_1 + \cdots + C_{k-1}^{k-1}a_n}{C_{n+k}^{k}}$$

and $C_n^m = \frac{n!}{m!(n-m)!}$ is the binomial coefficient. It is worth of mentioning that

$$C_{n+k}^{k} = C_{n+k-1}^{k-1} + C_{n+k-2}^{k-1} + \cdots + C_{k-1}^{k-1}.$$

We shall consider two subsequences of the sequence $\{\gamma_n^{(k)}\}_{n=0}^{\infty}$.

CASE I. Let $n = 1 + q + \cdots + q^{2m-1} = \frac{q^{2m}-1}{q-1}$. We then obtain that

$$\gamma_n^{(k)} = \frac{C_{n+k-1}^{k-1} + \sum_{i=1}^{m-1}\left[C^k_{n+k-\frac{q^{2i}-1}{q-1}} - C^k_{n+k-\frac{q^{2i+1}-1}{q-1}}\right]}{C^k_{n+k}} = \frac{C^{k-1}_{n+k-1}}{C^k_{n+k}} + \beta_n^{(k)}$$

Since $\lim_{n\to\infty} \frac{C^{k-1}_{n+k-1}}{C^k_{n+k}} = 0$, it is enough to study the sequence $\{\beta_n^{(k)}\}_{n=0}^\infty$.

Let $r_i = \frac{q^{2i}-1}{q-1}$ and $r_i' = \frac{q^{2i+1}-1}{q-1}$. It is easy to check that

$$\beta_n^{(k)} = \frac{\sum_{i=1}^{m-1}\left[(n-r_i+1)\cdots(n-r_i+k) - (n-r_i'+1)\cdots(n-r_i'+k)\right]}{(n+1)\cdots(n+k)}$$

Let $p_i = q^{2i}$ and $\delta_j = n - r_i + j$ for $j = \overline{1,k}$. Since $r_i' = r_i + p_i$, we get that

$$(n-r_i+1)\cdots(n-r_i+k) - (n-r_i'+1)\cdots(n-r_i'+k) =$$
$$= (-1)^{k+1}\left[p_i^k - \left(\sum_j \delta_j\right)p_i^{k-1} + \cdots + (-1)^{k-1}\left(\sum_{j_1\cdots j_{k-1}} \delta_{j_1}\cdots\delta_{j_{k-1}}\right)p_i\right]$$

We know that

$$\lim_{m\to\infty}\frac{1}{n^k}\sum_{i=1}^{m-1} p_i^k = \frac{(q-1)^k}{q^{2k}-1}, \quad \lim_{m\to\infty}\frac{1}{n^{l+r}}\sum_{i=1}^{m-1} p_i^l = 0.$$

Moreover, it is easy to check that

$$\sum_{j_1\cdots j_t} \delta_{j_1}\cdots\delta_{j_t} = C_k^t(n-r_i)^t + o(n^t).$$

Consequently, after some algebraic manipulations, we obtain that

$$\lim_{m\to\infty}\gamma_n^{(k)} = \Gamma_1^{(k)} \equiv (-1)^{k+1}\sum_{t=0}^{k-1}\frac{(-1)^t C_k^t}{q^{k-t}+1}.$$

CASE II. Let $n = 1 + q + \cdots + q^{2m} = \frac{q^{2m+1}-1}{q-1}$. In the similar manner, we obtain that

$$\lim_{m\to\infty}\gamma_n^{(k)} = \Gamma_2^{(k)} \equiv (-1)^{k+1}\sum_{t=0}^{k-1}\frac{(-1)^t q^{k-t} C_k^t}{q^{k-t}+1}.$$

It is clear that $\Gamma_1^{(k)} + \Gamma_2^{(k)} = 1$ for any k. Moreover, it is easy to check that $\Gamma_1^{(k)} \neq \frac{1}{2}$ and $\Gamma_2^{(k)} \neq \frac{1}{2}$. Therefore, $\Gamma_1^{(k)} \neq \Gamma_2^{(k)}$ for any k. This completes the proof. □

Lemma 2.8 immediately implies the following result which is the generalization of Theorem 2.7.

Theorem 2.9. *Let $V_f : S^2 \to S^2$ be the Lotka–Volterra operator given by (2.1). If non-zero parameters a, b, c have the same sign and $f : S^2 \to [-1, 1]$ is a non-vanishing C^1-smooth functional then any order Cesàro mean defined by (1.4) of the Lotka–Volterra operator V_f diverges for any $x \in IntS^2 \setminus \{P_0\}$.*

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