



EKELAND'S VARIATIONAL PRINCIPLE AND CRITICAL POINTS OF DYNAMICAL SYSTEMS IN LOCALLY COMPLETE SPACES

C. BOSCH¹ * , C.L. GARCÍA¹ , F. GARIBAY-BONALES², C. GÓMEZ-WULSCHNER¹
AND R. VERA²

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ABSTRACT. Ekeland's variational principle and the existence of critical points of dynamical systems, also known as multiobjective optimization, have been proved in the setting of locally complete spaces. In this article we prove that these two properties can be deduced one from the other under certain convexity conditions.

1. INTRODUCTION AND PRELIMINARIES

One of the most important tools to solve problems in optimization, optimal control, game theory, nonlinear equations, dynamical systems, etc., is the Ekeland's variational principle [5]. Since the discovery of such a principle there have also appeared many extensions or equivalent formulations of the principle [2], or [10]. In [2] C.Bosch, A. García, C.L.García, established the variational principle in the setting of locally complete spaces.

In 1983 G.Isaac [8] obtained a general existence result considering a set as a critical point for a dynamical system and using the completeness of the space. By adapting some ideas of the cited paper, C. Bosch, A. García in [3] extended that multiobjective optimization result to locally convex spaces by assuming that some related sets are locally complete.

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* Corresponding author.

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In the setting of locally complete spaces we prove that Ekeland's variational principle and multiobjective optimization can be derived from each other, making them, in this sense equivalent.

Throughout this paper (E, τ) will denote a locally convex space with topology τ , generated by a family of seminorms $\{\rho_\alpha : \alpha \in \Lambda\}$ with Λ a set of indices. A *disk* B in E is a closed, bounded and absolutely convex set. We denote by (E_B, ρ_B) the linear span of B endowed with the topology defined by the Minkowski functional associated with B . When B is bounded, ρ_B is a norm, and the norm topology is finer than the topology inherited from E . If (E_B, ρ_B) is a Banach space we say that B is a Banach disk. E is a locally complete locally convex space (briefly locally complete space) if each closed, bounded disk is a Banach disk. There are many examples of locally complete spaces, in fact every sequentially complete space is locally complete. On the other side one can also find examples of locally complete spaces that are not sequentially complete. A sequence (x_n) in E is said to be locally convergent or *Mackey convergent* to an element x of E , if there exists a disc B in E such that the sequence converges to x in E_B with respect to ρ_B . A sequence is called *locally Cauchy* or *Mackey Cauchy* if it is a ρ_B -Cauchy sequence in E_B for a certain disk in E . Let C be a non void subset of E . A point x is a *local limit point* of C if there is a sequence in C that is locally convergent to x . A set C is *locally closed* if every local limit point of C belongs to C . A subset A of a space E is said to be *locally complete* if every locally Cauchy sequence in A converges locally to a point of A . It is clear that every locally complete subset of a space is locally closed.

In [9] [2-14 p.20] locally complete spaces are also given the name c^∞ - or convenient spaces. It should be noted that the structure of c^∞ - (locally complete) spaces have become important in recent years due to the use of such spaces in nonlinear distribution theory. More detail about applications of these spaces can be found in [9] and the references therein. In [6] one can also see how theorems on existence and uniqueness for integro-differential equations are now extended to locally complete spaces.

The class Φ of perturbations we will use, is defined as the family of functions $\varphi : [0, \infty) \rightarrow [0, \infty)$ which are subadditive, strictly increasing, continuous and such that $\varphi(0) = 0$, and $\lim_{x \rightarrow \infty} \varphi(x) = \infty$. Clearly the inverse of φ exists and is superadditive, strictly increasing and continuous, $\varphi^{-1}(0) = 0$. Here φ is said to be subadditive if $\varphi(s + t) \leq \varphi(s) + \varphi(t)$, for every $s, t \in [0, \infty)$, and φ^{-1} is said to be superadditive if $\varphi^{-1}(s + t) \geq \varphi^{-1}(s) + \varphi^{-1}(t)$, for every $s, t \in [0, \infty)$. Functions like $\varphi(t) = t$, $\varphi(t) = \sqrt[3]{t}$, $\varphi(t) = \ln(1 + t)$, are examples of elements in Φ .

A closed, pointed convex cone in a locally convex space is a nonempty subset $K \subset E$ such that:

- (1) K is a closed subset,
- (2) $K + K \subseteq K$
- (3) $\lambda K \subseteq K$, for all $\lambda \in \mathbf{R}$
- (4) $K \cap (-K) = \{0\}$

If a closed, pointed convex cone $K \subset E$ is given we can define an ordering in E by $x \preceq y$ if and only if $y - x \in K$.

If $A \subset E$, is a nonempty subset we say that $a \in A$ is an efficient (maximal) point of A , with respect to K , if $A \cap (a + K) = \{a\}$. We denote by $E(A; K)$ the set of efficient points of A with respect to K .

We say that $\Gamma : A \rightarrow 2^A$ is a dynamical system if for every $x \in A$, and $\Gamma(x)$ is a nonempty subset of A and $x^* \in A$ is a critical point for Γ if $\Gamma(x^*) = \{x^*\}$. We can see easily that $\Gamma_A(x) = A \cap (x + K)$ for every $x \in A$ is a dynamical system. The reader can verify that an element $x^* \in A$ is an efficient point of A , if and only if x^* is a critical point of Γ_A .

A function $\Psi : A \rightarrow \mathbf{R} \cup \{+\infty\}$ is proper if it is not identically $+\infty$. A proper function $\Psi : A \rightarrow \mathbf{R} \cup \{+\infty\}$ is (locally) lower semicontinuous if $\{x \in A : \Psi(x) \leq r\}$ is (locally) closed in E for every r in \mathbf{R} or equivalently for each $x' \in A$ and every sequence (x_n) in A , (locally) convergent to x' we have that $\Psi(x') \leq \liminf \Psi(x_n)$.

2. MAIN RESULTS

In this section we present an equivalence between Ekeland's variational principle for locally complete spaces [2] and critical points of dynamical systems or multiobjective optimization [3]. To do that we need a technical condition on certain sets. First we will state two theorems, the third one will give the conditions under which they are equivalent.

Theorem 2.1. *Let (E, τ) be a locally complete space and $f : E \rightarrow \mathbf{R} \cup \{\infty\}$ be a proper, lower semicontinuous and bounded below function. Let φ be in Φ and x_0 be a point in $\text{Dom}(f)$, that is $f(x_0) < \infty$. Then for any Banach disk B in E such that $x_0 \in E_B$ there exists $x^* \in E_B$ such that :*

- (a) $f(x^*) + \varphi(\rho_B(x^* - x_0)) \leq f(x_0)$ and
- (b) $f(x^*) < \varphi(\rho_B(x^* - x)) + f(x)$ for all $x \in E_B \setminus \{x^*\}$

S. Al Homidan, Q.H. Ansari and J.C. Yao established in [7] [corollary 3.1] a similar result to the previous theorem but as they pointed out in their remark 3.3 the results are not comparable, they are of different kind.

Theorem 2.2. *Let (E, τ) be a locally convex space and φ be in Φ . Let A be a nonempty locally complete subset of E . Let $\Gamma : A \rightarrow 2^A$ be a dynamical system. Suppose that there exists $\Psi : A \rightarrow \mathbf{R}$ locally lower semicontinuous and bounded below. If there exists $\{c_\alpha \in \mathbf{R}^+ : \alpha \in \Lambda\}$ such that:*

- (a) $D = \bigcap_\alpha \{x \in E : c_\alpha \rho_\alpha(x) \leq 1\}$ is a non zero Banach disc.
- (b) For every $x \in A$ and for every $u \in \Gamma(x)$ we have $\varphi(c_\alpha \rho_\alpha(x - u)) \leq \Psi(x) - \Psi(u)$ for every $\alpha \in \Lambda$.

Then Γ has a critical point $x^ \in A$.*

Theorem 2.3. *Theorem 2.1 and 2.2 are equivalent if A is absolutely convex.*

Proof. First let us assume that Theorem 2.1 holds and that the hypothesis of Theorem 2.2 are satisfied with A being absolutely convex. Let $E_A = \bigcup_{r>0} rA$. Now we have two possibilities $\overline{D \cap E_A} \neq \{0\}$ or $D \cap E_A = \{0\}$.

Case 1 : $C = \overline{D \cap E_A}^\tau \neq \{0\}$.

By the definition C , is τ -closed, bounded since D is bounded, balanced and convex since it is the intersection of two balanced and convex sets D and E_A . Now since $C \neq \{0\}$ there exists $y_o \neq 0$ such that $y_o \in D \cap E_A$, then there is an r_o such that $y_o \in D \cap r_o A$. If $r_o \leq 1$ then $y_o \in D \cap A$ and we denote $x_o = y_o \in D \cap A \subset C$. If $1 < r_o$ we will take $x_o = \frac{1}{r_o} y_o \in D \cap A \subset C$.

Since D is closed and $C \subset D$ we have that $E_C \subset E_D$ and $\rho_D(x) \leq \rho_C(x)$ for every $x \in E_C$.

Claim 1: If (E_D, ρ_D) is complete then (E_C, ρ_C) is also complete.

Let $\{x_n\} \subset E_C$ be ρ_C -Cauchy, then it is also ρ_D -Cauchy and since we are assuming that (E_D, ρ_D) is complete, there exists $x \in E_D$ such that $x_n \rightarrow x$ with respect to ρ_D . We have that $\{x_n\} \subset E_C$ is ρ_C -Cauchy so for $\varepsilon > 0$ there exists $N \in \mathbf{N}$ such that if $n, m \geq N$ then $x_n - x_m \in \varepsilon C$. We use that the topology induced by ρ_D in E_D is stronger than the topology induced in E_D by the topology τ and the fact that $C \subset E_D$ is τ -closed to conclude that $\overline{C}^{\rho_D} = C$. Then $\rho_C(x_n - x) < \varepsilon$ for every $n \geq N$, that is, $x_n \rightarrow x \in (E_C, \rho_C)$, which proves the claim.

Claim 2: $\rho_D(x) = \rho_C(x)$ for every $x \in E_C \cap E_A$.

Take x such that $\rho_D(x) < t$ then $x \in tD \cap E_A = t(D \cap E_A) = tC$ so $\rho_C(x) < t$ which means that $\rho_C(x) \leq \rho_D(x)$. Now using the fact that $\rho_D(x) \leq \rho_C(x)$ for every $x \in E_C$, we have the equality of the seminorms $\rho_D(x) = \rho_C(x)$.

Now apply Theorem 2.1 to the complete space (E_C, ρ_C) , with $B = C$, $x_o \in D \cap A \subset C$ and the function $g : E_C \rightarrow R \cup \{\infty\}$ defined by

$$g(x) = \begin{cases} \Psi(x), & x \in A \\ \infty, & x \in E_C \setminus A. \end{cases}$$

g is a lower semicontinuous and bounded below function, since Ψ is locally lower semicontinuous in A and then locally lower semicontinuous (E_C, ρ_C) , furthermore Ψ is bounded below in A . Then there exists $z_o \in E_C$ such that

- (a) $g(z_o) + \rho_C(x_o - z_o) \leq g(x_o) = \Psi(x_o) < \infty$ and
- (b) $g(z_o) < g(x) + \rho_C(x - z_o)$ for all $x \in E_C$, $x \neq z_o$.

Using (a) we have $g(z_o) < \infty$ which says that $z_o \in A$.

Claim 3 : $\Gamma(z_o) \cap E_C = \{z_o\}$.

Take $x \in \Gamma(z_o) \cap E_C$ and suppose that $x \neq z_o$ then $\Psi(x) + \varphi(\rho_D(x - z_o)) \leq \Psi(z_o) < \infty$ using the function g we have $g(x) + \varphi(\rho_C(x - z_o)) \leq g(z_o)$ since for $x \in \Gamma(z_o) \subset g(x) = \Psi(x)$ and for $x - z_o \in E_C \rho_D(x - z_o) = \rho_C(x - z_o)$. Finally since $B = C$ using item (b) from Theorem 2.1 we get $g(z_o) < \varphi(\rho_B(z_o - x)) + g(x)$ which gives a contradiction and then $x = z_o$.

Claim 4: $\Gamma(z_o) \subset E_C$.

Take $x \in \Gamma(z_o) \subset A$ then $x - z_o \in E_D$ and $x \in z_o + E_D \subset E_C + E_D = E_D$, so there exists $r \geq 1$ such that $x \in rD$ which means that $\frac{1}{r}x \in D \cap A \subset E_C$, and then $x \in rE_C = E_C$.

To finish the proof note that claim 3 and claim 4 give that $\Gamma(z_o) = \{z_o\}$ so in the case that $C = \overline{D \cap E_A}^\tau \neq \{0\}$, Theorem 2.2 holds by taking $x^* = z_o$.

Case 2: $D \cap E_A = \{0\}$.

Again, A is balanced and convex. Take $x_o \in A$ and let $C = \{x \in A \mid \rho_\alpha(x) \leq \rho_\alpha(x_o) \forall \alpha \in \Lambda\}$, C is bounded, balanced, convex and τ -closed, furthermore $x_o \in C \subset A$.

Claim 5: (E_C, ρ_C) is complete.

Take $\{x_n\} \subset E_C$ such that $\{x_n\}$ is ρ_C -Cauchy then it is ρ_C -bounded so there exists $r > 0$ such that $\{x_n\} \subset rC \subset rA$ then $\{\frac{1}{r}x_n\} \subset A$ is locally Cauchy (ρ_C -Cauchy) and then there exists $y \in A \cap E_C$ such that $\{\frac{1}{r}x_n\} \rightarrow y$ with respect to ρ_C so we have $\{x_n\} \rightarrow x = ry$, with respect to ρ_C .

Now apply Theorem 2.1 to the complete space (E_C, ρ_C) , with $B = C, x_o \in B$ and the function $g : E_C \rightarrow R \cup \{\infty\}$ defined by

$$g(x) = \begin{cases} \Psi(x), & x \in A \\ \infty, & x \in E_C \setminus A \end{cases}$$

g is a lower semicontinuous and bounded below function, since Ψ has the necessary properties. Then there exists $z_o \in E_C$ such that

- (a) $g(z_o) + \rho_C(x_o - z_o) \leq g(x_o) = \Psi(x_o) < \infty$
- (b) $g(z_o) < g(x) + \rho_C(x - z_o)$ for every $x \in E_C \setminus \{z_o\}$.

The property (a) says that $z_o \in A$.

Claim 6: $\Gamma(z_o) \cap E_C = \{z_o\}$.

Take $x \in \Gamma(z_o) \cap E_C$ then $x - z_o \in E_D \cap E_C$ and $x - z_o \in r_1D \cap r_2C$, choose $r = \max(r_1, r_2)$ so we have $x - z_o \in rD \cap rC \subset rD \cap rA = r(D \cap A)$. This can be rewritten as $\frac{1}{r}(x - z_o) \in D \cap A \subset D \cap E_A = \{0\}$ and we can conclude that $x = z_o$.

Claim 7: $\Gamma(z_o) \subset E_C$

Take $x \in \Gamma(z_o) \subset A$ then $x - z_o \in E_D$ so $x \in z_o + E_D \subset E_C + E_D = E_D$, now take $r \geq 1$ such that $x \in rD$ and we have $\frac{1}{r}x \in D \cap A \subset E_C$, and conclude that $x \in rE_C = E_C$.

To finish the proof note that claim 6 and claim 7 give that $\Gamma(z_o) = \{z_o\}$ so in the case that $D \cap E_A = \{0\}$, Theorem 2.2 hold by taking $x^* = z_o$.

Now we prove that Theorem 2.2 implies Theorem 2.1.

Since Theorem 2.2 holds, suppose (E, τ) is a locally convex space, $\{\rho_\lambda : \lambda \in \Lambda\}$ the family of semi-norms that define the topology τ and φ is in Φ . Let $f : E \rightarrow \mathbf{R} \cup \{\infty\}$ be a proper, lower semicontinuous and bounded below function, such that $f(x_0) < \infty$ for some $x_0 \in E_B$. Take B a disc in E , since (E, τ) is a locally complete then E_B is complete, B is a Banach disc, it is bounded and then for each $\lambda \in \Lambda$ there exists $r_\lambda \in \mathbf{R}^+$ such that $\rho_\lambda(x) \leq r_\lambda$ for every $x \in B$ or $c_\lambda \rho_\lambda(x) \leq 1$ where $c_\lambda = 1/r_\lambda$. Let $D = \{x \in E : \rho_\lambda(x) \leq r_\lambda \text{ for every } \lambda \in \Lambda\} = \cap_\lambda \{x \in E : c_\lambda \rho_\lambda(x) \leq 1\}$, so by part (a) of theorem 2.2, D is a disc and E_D is complete. Furthermore $B \subset D$, then $E_B \subset E_D$ and $\rho_D(x) \leq \rho_B(x)$ for $x \in E_B$. Define $\Gamma : S_B(x_0) \rightarrow 2^{S_B(x_0)}$ by $\Gamma(x) = S_B(x) \subset S_B(x_0) = A$ where $S_B(x) = \{y \in E_B : f(y) + \varphi(\rho_B(y - x)) \leq f(x)\}$. If $S_B(x_0) = \{x_0\}$ then take $x^* = x_0$ and Theorem 2.1 holds. If not take $x \in E_B$ and $u \in \Gamma(x)$, then $c_\lambda \rho_\lambda(x) \leq \sup\{c_\lambda \rho_\lambda(x) : \lambda \in \Lambda\} = \rho_D(x) \leq \rho_B(x)$ we then have $\varphi(c_\lambda \rho_\lambda(x - u)) \leq \varphi(\rho_D(x - u)) \leq \varphi \rho_B(x - u) \leq f(x) - f(u)$. By defining $f|_A = \Psi$ we have that Ψ is locally lower semicontinuous since f is lower semicontinuous in A then it is locally lower semicontinuous in A , now $\varphi(c_\lambda \rho_\lambda(x - u)) \leq \Psi(x) - \Psi(u)$ so by Theorem

2.2 we can conclude that Γ has a critical point $x^* \in A$ then $\Gamma(x^*) = \{x^*\}$, and Theorem 2.1 holds by using x^* .

Note that Theorem 2.2 implies Theorem 2.1 whether A is absolutely convex or not. \square

3. FINAL COMMENTS

Over more than three decades a good deal of effort has been made to look for equivalent formulations or generalizations of Ekeland's Variational Principle (EVP). In [10] and the references therein one can see the flavor of these generalizations. The main result in the seminal article of Ekeland [5] says, roughly, that for a lower semicontinuous bounded below function f on a complete metric space X , a slight perturbation of the function has a strict minimum. In the category of normed spaces, the perturbation is taken as a multiple of the norm. EVP was proved to hold for spaces that are not necessary complete metric spaces but that enjoy completeness and generalize distances or weaker form of completeness in locally convex spaces, these two situations as pointed out before can not be compared. The EVP is proved with the weaker completeness, locally complete spaces in [2]. Furthermore in [4] it is shown that in locally complete spaces EVP is equivalent to Caristi–Kirk type fixed point theorem, Takahashi type minimization theorem and an equilibrium version of EVP. Multiobjective optimization theorem deals with the existence of efficient points. Theorem 2.2 asserts, in a quite general setting, the existence of efficient points. In locally complete spaces we have proved that multiobjective optimization and EVP can be deduced from each other and from the equilibrium version of EVP. In the proof of Theorem 2.3 one can see the relation between the efficient points and the point where the perturbed function attains its minimum. Relations between EVP and the multiobjective optimization have been studied in different contexts, a good reference is [1].

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¹ INSTITUTO TECNOLÓGICO AUTÓNOMO DE MÉXICO, MÉXICO D.F. 01080, MÉXICO.
E-mail address: bosch@itam.mx; clgarcia@itam.mx; claudiag@itam.mx

² UNIVERSIDAD MICHOACANA DE SAN NICOLÁS DE HIDALGO, MORELIA 58030, MÉXICO
E-mail address: rigovera@gmail.com; fgaribay@gmail.com