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ON GENERALIZED WEIGHTED MEANS AND COMPACTNESS OF MATRIX OPERATORS

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Abstract. This paper mainly divided into two parts. The first part gives same facts about topological properties of certain linear topological spaces, inclusion relations and matrix mappings. The second part establishes some identities or estimates for the matrix operator norms and the Hausdorff measures of noncompactness of certain matrix operators, characterize some classes of compact operators on these spaces.

1. Introduction

In the literature, by using the matrix domain over the paranormed spaces, many authors have defined new sequence spaces. Some of them as follows: Choudhary and Mishra [5] have studied sequence space $\overline{\ell(p)}$, where $\overline{\ell(p)}$ consists of the sequences whose $S-$trasforms are in $\ell(p)$, Başar and Altay [14] have defined the spaces $\lambda(u,v;p) = \{\lambda(p)\}_G$ for $\lambda \in \{\ell_\infty, c_0, c\}$. The same authors also have defined the spaces [4] $r^t_\infty(p) = \{\ell_\infty(p)\}_{R^t}, r^t_0(p) = \{c_0(p)\}_{R^t}, r^t_c(p) = \{c(p)\}_{R^t}$. Recently, Karakaya et all. have introduced and studied the spaces [33] $\gamma(\lambda;p) = \{\gamma(p)\}_\Lambda$ for $\gamma \in \{\ell_\infty, c_0, c\}$. R^t, G, Λ and S denote Riesz, generalized difference, lambda and summation matrix, respectively. Also, the information on matrix domain of sequence spaces can be found (see [6, 7, 17, 19, 28]).

The main purpose of the present paper is to introduce the sequence spaces $\mu(u,v,p;B)$ of non-absolute type and derive some related results. We also establish some inclusion relations. Furthermore, we determine the $\alpha-$, $\beta-$ and $\gamma-$duals of those spaces and construct their basis. Also, we characterize some classes of infinite matrices concerning the spaces $\mu(u,v,p;B)$, where $\mu \in \{\ell_\infty, c_0, c\}$.

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Finally, we establish some identities or estimates for the operator norms and the Hausdorff measures of noncompactness of certain matrix operators on the spaces $c_0(u,v,p;B)$, $\ell_\infty(u,v,p;B)$ and by using the Hausdorff measure of noncompactness, we characterize some classes of compact operators on these spaces.

2. Notations and Auxiliary Facts

By ω, we shall denote the space of all real or complex valued sequences. Any vector subspace of ω is called sequence space. We shall write ℓ_∞, c and c_0 for the spaces of all bounded, convergent and null sequence respectively. Also by ℓ_1 and ℓ_p $(1 < p < \infty)$ we denote the spaces of all absolutely and $p-$ absolutely convergent series, respectively. Further, we shall write bs, cs for the spaces of all sequences associated with bounded and convergent series.

Let μ and γ be two sequence spaces and $A = (a_{nk})$ be an infinite matrix of real or complex numbers a_{nk}, where $n,k \in \mathbb{N}$. Then, we say that A defines a matrix mapping from μ into γ, and we denote it by writing $A : \mu \to \gamma$, if for every sequence $x = (x_k) \in \mu$ the sequence $Ax = \{(Ax)_n\}$, the $A-$transform of x is in γ; where

$$(Ax)_n = \sum_k a_{nk}x_k \quad (n \in \mathbb{N}). \tag{2.1}$$

The notation $(\mu : \gamma)$ denotes the class of all matrices A such that $A : \mu \to \gamma$. Thus, $A \in (\mu : \gamma)$ if and only if the series on the right hand side of (2.1) converges for each $n \in \mathbb{N}$. The matrix domain μ_A of an infinite matrix A in a sequence space μ is defined by

$$\mu_A = \{x = (x_k) \in \omega : Ax \in \mu\}. \tag{2.2}$$

A linear topological space X over the real field $\mathbb{R}$ is said to be a paranormed space if there exists subadditive function $h : X \longrightarrow \mathbb{R}$ such that $h(\theta) = 0$, $h(-x) = h(x)$ and scalar multiplication is continuous, i.e., $|\alpha_n - \alpha| \longrightarrow 0$ and $h(x_n - x) \longrightarrow 0$ imply $h(\alpha_n x_n - \alpha x) \longrightarrow 0$ for all α's in $\mathbb{R}$ and all x's in X, where θ is the zero vector in the linear space X.

Assume here and after that $p = (p_k)$ be a bounded sequence of strictly positive real numbers with $\sup p_k = H$ and $M = \max\{1, H\}$. Then, the linear spaces $\ell_\infty(p)$, $c_0(p)$ and $c(p)$ were defined by Maddox [32](see also Simons [27] and Nakano [15])

$$\begin{aligned}
\ell_\infty(p) &= \left\{x = (x_k) \in \omega : \sup_{k\in\mathbb{N}} |x_k|^{p_k} < \infty\right\}, \\
c_0(p) &= \left\{x = (x_k) \in \omega : \lim_{k\to\infty} |x_k|^{p_k} = 0\right\}, \\
c(p) &= \left\{x = (x_k) \in \omega : \exists l \in \mathbb{R} \text{ such that } \lim_{k\to\infty} |x_k - l|^{p_k} = 0\right\}
\end{aligned}$$

which are the complete paranormed by

$$h(x) = \sup_{k\in\mathbb{N}} |x_k|^{p_k/M}.$$

Throughout the article, by $\mathcal{F}$ and $\mathbb{N}_k$, respectively, we denote the collection of all subsets of $\mathbb{N}$ and the set of all $n \in \mathbb{N}$ such that $n \geq k$. Also we write $e = (1, 1, 1, \ldots)$ and $e^{(n)}$ is the sequence whose only non-zero term is 1 in the n^{th} place for each $n \in \mathbb{N}$, where $\mathbb{N} = \{0, 1, 2, \ldots\}$.

Let U denotes the set of all sequences $u = (u_k)$ such that $u_k \neq 0$ for all $k \in \mathbb{N}$ and $u, v \in U$, and define the matrix $G(u, v) = (g_{nk})$ by

$$g_{nk} = \begin{cases} u_n v_k, & (k < n), \\ u_n v_n, & (k = n), \\ 0, & (k > n) \end{cases}$$

for all $k \in \mathbb{N}$, where u_n depends only on n and v_k only on k. The matrix $G(u, v)$ which is called as generalized weighted mean or factorable matrix.

Let r and s be non-zero real numbers and define the generalized difference matrix $B(r, s) = \{b_{nk}(r, s)\}$ by

$$b_{nk}(r, s) = \begin{cases} r, & k = n, \\ s, & k = n - 1, \\ 0, & \text{otherwise} \end{cases} \tag{2.3}$$

for all $n, k \in \mathbb{N}$. We note that the matrix $B(r, s)$ can be reduced to the difference matrix $\triangle$ in case $r = 1$ and $s = -1$. So, the results related to the domain of the matrix $B(r, s)$ are more general and more comprehensive than the consequences of the domain of the matrix $\triangle$ and include them.

3. Certain Main Results

3.1. **Basic Topological Properties.** This part is devoted to examination of the basic topological properties of the sets $\mu(u, v, p; B)$, where $\mu \in \{\ell_\infty, c_0, c\}$. For $u \in U$ and $\frac{1}{u} = (\frac{1}{u_k})$.

Now, we introduce the new sequence spaces $\ell_\infty(u, v, p; B)$, $c_0(u, v, p; B)$ and $c(u, v, p; B)$ as follows:

$$\ell_\infty(u, v, p; B) = \left\{ x = (x_k) \in \omega : \sup_{n \in \mathbb{N}} \left| u_n \sum_{k=0}^{n} v_k (r x_k + s x_{k-1}) \right|^{p_n} < \infty \right\},$$

$$c_0(u, v, p; B) = \left\{ x = (x_k) \in \omega : \lim_{n \to \infty} \left| u_n \sum_{k=0}^{n} v_k (r x_k + s x_{k-1}) \right|^{p_n} = 0 \right\},$$

$$c(u, v, p; B) = \left\{ x = (x_k) \in \omega : \exists l \in \mathbb{R} \text{ such that } \lim_{n \to \infty} \left| u_n \sum_{k=0}^{n} v_k (r x_k + s x_{k-1}) \right|^{p_n} = l \right\}.$$

By the notation of (2.2), we can redefine the spaces $\mu(u, v, p; B)$ as follows:

$$\mu(u, v, p; B) = [\mu(u, v; p)]_B,$$

where B denotes the generalized difference matrix $B(r, s) = \{b_{nk}(r, s)\}$ defined by (2.3). On the other hand, we define the triangle matrix

$$\widehat{G}(u, v, B) = G(u, v) B(r, s) = (\widehat{g}_{nk})$$

$$\widehat{g}_{nk} = \begin{cases} u_n(rv_k + sv_{k+1}), & k < n, \\ ru_n v_n, & k = n, \\ 0, & k > n \end{cases}$$

for all $n, k \in \mathbb{N}$. Then, we have following special cases.

(i) If $r = 1$ and $s = -1$, then $\mu(u, v, p; B) = \mu(u, v, p; \triangle)$, where $\mu \in \{\ell_\infty, c_0, c\}$ (see [3]).

(ii) Let $\lambda = (\lambda_k)$ is a strictly increasing sequence of positive reals tending to ∞. If $v = (\lambda_k - \lambda_{k-1})$ and $u = (1/\lambda_n)$ with $p = e$, then $c_0(u, v, p; B) = c_0^\lambda(B)$ and $c(u, v, p; B) = c^\lambda(B)$ (see[1]).

(iii) If $v = (\lambda_k - \lambda_{k-1})$ and $u = (1/\lambda_n)$ with $p = e$ and $r = 1, s = -1$, then $c_0(u, v, p; \triangle) = c_0^\lambda(\Delta)$ and $c(u, v, p; B) = c^\lambda(\Delta)$ (see [24]).

(iv) If $v = (1 + r^k)$ and $u = (1/n + 1)$ with $p = e$ and $r = 1, s = -1$, then $c_0(u, v, p; B) = a_0^r(\triangle)$ and $c(u, v, p; B) = a_c^r(\Delta)$ (see [9]).

(v) If $v = (q_k)$ and $u = (1/\sum_{k=0}^n q_k)$, then $\ell_\infty(u, v, p; B) = r_\infty^q(p, B)$ and $c_0(u, v, p; B) = r_0^q(p, B)$ as well as $c(u, v, p; B) = r_c^q(p, B)$ (see [18]).

Define the sequence $y = (y_k)$, which will be frequently used, as the $\widehat{G}-$transform of a sequence $x = (x_k)$, i.e.

$$y_k = \sum_{i=0}^{k} u_k v_i (rx_i + sx_{i-1}) \tag{3.1}$$

and every $x = (x_k) \in \omega$ which leads us together with (2.2) to the fact that

$$\mu(u, v, p; B) = [\mu(p)]_{\widehat{G}}.$$

Also we derive that equality of (3.1)

$$y_0 = ru_0v_0 \text{ and } y_k = u_k \left(\sum_{i=0}^{k-1} (rv_i + sv_{i+1})x_i + ru_k v_k x_k \right) \text{ for all } k \geq 1.$$

Theorem 3.1. *We have the following*

(a): *$\mu(u, v, p; B)$ is the complete linear metric space paranormed by g, defined by*

$$g(x) = \sup_n \left| \sum_{k=0}^{n-1} u_n(rv_k + sv_{k+1})x_k + u_n v_n x_n \right|^{\frac{p_n}{M}}$$

where $M = \max\{1, \sup p_n\}$ and $0 < p_n \leq H < \infty$ for all $n \in \mathbb{N}$.

(b): *Then, $\mu(u, v, p; B) = \mu_{\widehat{G}}$ is a $BK-$space with the norm $\|x\|_{\mu(u,v,p;B)} = \|\widehat{G}x\|_\infty$. That is,*

$$\|x\|_{\mu(u,v,p;B)} = \sup_n |(\widehat{G}x)_n|. \tag{3.2}$$

Proof. (a) We prove the theorem only for the space $c_0(u,v,p;B)$. The linearity of $c_0(u,v,p;B)$ with respect to the coordinatewise addition and scalar multiplication follows from the following inequalities which are satisfied for $z,x \in c_0(u,v,p;B)$ [25, p.30]

$$\begin{aligned} & \sup_n \left| \sum_{k=0}^{n-1} u_n(rv_k + sv_{k+1})(x_k+z_k) + u_n v_n (x_n+z_n) \right|^{\frac{p_n}{M}} \\ \leq \; & \sup_n \left| \sum_{k=0}^{n-1} u_n(rv_k + sv_{k+1})x_k + u_n v_n x_n \right|^{\frac{p_n}{M}} \\ & + \sup_n \left| \sum_{k=0}^{n-1} u_n(rv_k + sv_{k+1})z_k + u_n v_n z_n \right|^{\frac{p_n}{M}} . \end{aligned} \tag{3.3}$$

For any $\alpha \in \mathbb{R}$ (see[32]), we get

$$|\alpha|^{p_k} \leq \max\{1, |\alpha|^M\}. \tag{3.4}$$

It is clear that $g(\theta)=0$ and $g(x)=g(-x)$ for all $x \in c_0(u,v,p;B)$. The inequalities (3.3) and (3.4) again yield the subadditivity of g and

$$g(\alpha x) \leq \max\{1, |\alpha|^M\} g(x). \tag{3.5}$$

Let $\{x^n\}$ be any sequence of points in $c_0(u,v,p;B)$ such that $g(x^n - x) \longrightarrow 0$ and (α_n) also be any sequence of scalars that $\alpha_n \longrightarrow \alpha$. Then, we obtain

$$g(\alpha_n x^n - \alpha x) \leq g\left[(\alpha_n - \alpha)(x^n - x)\right] + g\left[\alpha(x^n - x)\right] + g\left[x(\alpha^n - \alpha)\right]. \tag{3.6}$$

It follows from $\alpha_n \longrightarrow \alpha$ as $n \longrightarrow \infty$ that $|\alpha_n - \alpha| < 1$ for all sufficient large n. Therefore,

$$\lim_{n\to\infty} g\left[(\alpha_n - \alpha)(x^n - x)\right] \leq \lim_{n\to\infty} g\left[(x^n - x)\right] = 0. \tag{3.7}$$

By (3.5), we have

$$\lim_{n\to\infty} g\left[\alpha(x^n - x)\right] \leq \max\{1, |\alpha|^M\} \lim_{n\to\infty} g\left[(x^n - x)\right] = 0. \tag{3.8}$$

Furthermore, we get

$$\lim_{n\to\infty} g\left[x(\alpha^n - \alpha)\right] \leq \lim_{n\to\infty} |\alpha_n - \alpha| g(x) = 0. \tag{3.9}$$

Then, we obtain from (3.6)-(3.9) that $g(\alpha_n x^n - \alpha x) \longrightarrow 0$ as $n \longrightarrow \infty$. This shows that g is a paranorm on $c_0(u,v,p;B)$.

It remains to prove the completeness of the space $c_0(u,v,p;B)$. Let $\{x^j\}$ be any Cauchy sequence in the space $c_0(u,v,p;B)$, where $x^i = (x_0^i, x_2^i, x_2^i, \ldots)$. Then for a given ε there exists a positive integer $n_0(\varepsilon)$ such that

$$g(x^j - x^i) < \varepsilon$$

for all $i,j \geq n_0(\varepsilon)$. Using definition of g, we obtain for each fixed $n \in \mathbb{N}$ that

$$\left|(\widehat{G}x^j)_n - (\widehat{G}x^i)_n\right|^{\frac{p_n}{M}} \leq \sup_n \left|(\widehat{G}x^j)_n - (\widehat{G}x^i)_n\right|^{\frac{p_n}{M}} < \frac{\varepsilon}{2} \tag{3.10}$$

for every $i, j \geq n_0(\varepsilon)$, which lead us to fact that $\{(\widehat{G}x^0)_k, (\widehat{G}x^1)_k, (\widehat{G}x^2)_k, \ldots\}$ is a Cauchy sequence of real numbers for every fixed $n \in \mathbb{N}$. Since $\mathbb{R}$ is complete, it converges, say $(\widehat{G}x^j)_k \longrightarrow (\widehat{G}x)_k$ as $j \longrightarrow \infty$. Using these infinitely many limits, we may write the sequence $\{(\widehat{G}x)_0, (\widehat{G}x)_1, (\widehat{G}x)_2, \ldots\}$. Using (3.10) as $i \longrightarrow \infty$ and for all $j \geq n_0(\varepsilon)$, we have

$$\left|(\widehat{G}x^j)_n - (\widehat{G}x)_n\right| < \frac{\varepsilon}{2} \tag{3.11}$$

for every fixed $n \in \mathbb{N}$. Since $x^j = (x_k^{(j)}) \in c_0(u, v, p; B)$ for each $j \in \mathbb{N}$, there exists $n_0(\varepsilon)$ such that $|(\widehat{G}x^j)_n|^{\frac{p_n}{M}} < \frac{\varepsilon}{2}$ for every $j \geq n_0(\varepsilon)$. We obtain by (3.11)

$$\left|(\widehat{G}x^j)_n\right|^{\frac{p_n}{M}} \leq \left|(\widehat{G}x^j)_n - (\widehat{G}x)_n\right|^{\frac{p_n}{M}} + |(\widehat{G}x^j)_n|^{\frac{p_n}{M}} < \varepsilon$$

for every $j \geq n_0(\varepsilon)$. Thus, we get $x \in c_0(u, v, p; B)$. Since (x^j) was an arbitrary Cauchy sequence, the space $c_0(u, v, p; B)$ is complete.

(b) Since the sequence space μ endowed with the norm $\|.\|_\infty$ is BK-space (see[26, Example 7.3.2(b),(c)]) and the matrix $\widehat{G}$ is a triangle, Theorem 4.3.2 of Wilansky [2, p.61] gives the fact that the spaces $\mu(u, v, p; B)$ are BK-space with the norm in (3.2). □

One can easily check that the absolute property does not hold on the space $\mu(u, v, p; B)$, that is $\|x\|_{\mu(u,v,p;B)} \neq \||x|\|_{\mu(u,v,p;B)}$ for at least one sequence in the space $\mu(u, v, p; B)$ and this tells us that $\mu(u, v, p; B)$ none-absolute type, where $|x| = (|x_k|)$.

Theorem 3.2. *The sequence spaces $\ell_\infty(u, v, p; B)$, $c_0(u, v, p; B)$ and $c(u, v, p; B)$ of none-absolute type is linearly isomorphic to the spaces $\ell_\infty(p)$, $c_0(p)$ and $c(p)$, respectively, where $0 < p_k \leq H < \infty$.*

Proof. To prove the fact that $c_0(u, v, p; B) \cong c_0(p)$ we should show the existence of a linear bijection between the spaces $c_0(u, v, p; B)$ and $c_0(p)$. Consider the transformation T defined with the notation of (2.2) from $c_0(u, v, p; B)$ to $c_0(p)$ by $x \mapsto y = Tx = \widehat{G}x$. The linearity of T is clear. Further, it is clear that $x = \theta$ whenever $Tx = \theta$ and hence T is injective.

Let $y = (y_k) \in c_0(p)$ and define the sequence $x = (x_k)$ by

$$x_k = \sum_{j=0}^{k-1} \left(\frac{-s}{r}\right)^{k-j} \left[\frac{1}{rv_j} + \frac{1}{sv_{j+1}}\right] \frac{1}{u_j} y_j + \frac{1}{ru_k v_k} y_k \quad \text{for each } k \in \mathbb{N}. \tag{3.12}$$

Then, we have

$$g(x) = \sup_n \left|\sum_{k=0}^{n-1} u_n(rv_k + sv_{k+1})x_k + ru_n v_n x_n\right|^{\frac{p_n}{M}} = \sup_n |y_n|^{\frac{p_n}{M}} = h(y).$$

Therefore, we have $x \in c_0(u, v, p; B)$. As a result, T is surjective. This implies T is linear bijection. So, the space $c_0(u, v, p; B)$ and $c_0(p)$ are linearly isomorphic as desired. This completes the proof. □

Now, we define the Schauder basis of a paranormed sequence space and then give the basis of the sequence spaces $c(u,v,p;B)$ and $c_0(u,v,p;B)$. If a sequence space γ paranormed by g_1 contains a sequence (b_k) with the property that for every $x \in \gamma$, there is a unique of scalars (α_k) such that

$$\lim_{n\to\infty} g_1\left(x - \sum_{k=0}^{n} \alpha_k b_k\right) = 0,$$

then (b_k) is called a Schauder basis (or briefly basis) for γ. The series $\sum \alpha_k b_k$ which has the sum x is called the expansion of x with respect to (b_k), and written as $x = \sum \alpha_k b_k$. It is known from Theorem 2.3 of Jarrah and Malkowsky [22] that the domain ν_T of an infinite matrix T in a sequence space ν has a basis if and only if ν has a basis, if T is a triangle. As a direct consequence of this fact, we have:

Corollary 3.3. *Let $\alpha_k = (\widehat{G}x)_k$ for all $k \in \mathbb{N}$ and $l = \lim_{k\to\infty}(\widehat{G}x)_k$. Define the sequence $b^{(k)} = \{b_n^{(k)}\}$ for every fixed $n \in \mathbb{N}$ by*

$$b_n^{(k)} \begin{cases} \left(\frac{-s}{r}\right)^{n-k}\left[\frac{1}{rv_k} + \frac{1}{sv_{k+1}}\right]\frac{1}{u_k}, & k < n, \\ \frac{1}{ru_nv_n}, & k = n, \\ 0, & k > n \end{cases}$$

Then, the following statements hold.
(a) The sequence $(b_n^{(k)})$ is a basis for the space $c_0(u,v,p;B)$ and every $x \in c_0(u,v,p;B)$ has a unique representation of the form $x = \sum_k \alpha_k b^{(k)}$.
(b) The sequence space $\{b, b^{(0)}, b^{(1)}, \ldots\}$ is a basis for the space $c(u,v,p;B)$ and any $x \in c(u,v,p;B)$ has a unique representation of the for $x = lb + \sum_k[\alpha_k - l]b_k$, where

$$b = (b_k) = \left\{\sum_{j=0}^{k-1}\left(\frac{-s}{r}\right)^{k-j}\left[\frac{1}{rv_j} + \frac{1}{sv_{j+1}}\right]\frac{1}{u_j} + \frac{1}{ru_kv_k}\right\}_{k=0}^{\infty}.$$

3.2. **Inclusion Relations.** In this part, we give some inclusion relations concerning the spaces $\mu(u,v,p;B)$.

Theorem 3.4. *The inclusions $c_0(u,v,p;B) \subset c(u,v,p;B) \subset \ell_\infty(u,v,p;B)$ strictly hold.*

Proof. Let $x = (x_k) \in c_0(u,v,p;B)$. This implies $\widehat{G}x \in c_0(p)$. Then, since the inclusion $c_0(p) \subset c(p)$ holds, $\widehat{G}x \in c(p)$ which means that $x \in c(u,v,p;B)$. Further consider the sequence as follows:

$$x_k = \frac{1}{2}\left[\sum_{j=0}^{k-1}\left(\frac{-s}{r}\right)^{k-j}\left[\frac{1}{rv_j} + \frac{1}{sv_{j+1}}\right]\frac{1}{u_j} + \frac{1}{ru_kv_k}\right],\ p_n = \frac{2n+5}{n+1} \text{ for each } n \in \mathbb{N}.$$

Then, we have

$$|\widehat{G}_n(x)|^{p_n} = \frac{1}{2^{\frac{2n+5}{n+1}}}. \tag{3.13}$$

From (3.13), we get x is in $c(u,v,p;B)$ but not in $c_0(u,v,p;B)$. Since the inclusion $c(p) \subset \ell_\infty(p)$ is strict, one can find at least a sequence $\widehat{G}x \in \ell_\infty(p) \setminus c(p)$ which shows that $\ell_\infty(u,v,p;B) \setminus c(u,v,p;B))$ is not empty, as desired. □

Theorem 3.5. *Following statements are hold.*

(i): *If $p_n > 1$ for all $n \in \mathbb{N}$, then the inclusion $\mu(u,v,B) \subset \mu(u,v,p;B)$ holds*

(ii): *If $p_n < 1$ for all $n \in \mathbb{N}$, then the inclusion $\mu(u,v,p;B) \subset \mu(u,v,B)$ holds*

where $\mu \in \{\ell_\infty, c_0, c\}$.

Proof. (i) If $p_n = p$ for all $n \in \mathbb{N}$, then we can write $c_0(u,v,B)$ instead of $c_0(u,v,p;B)$. Let $x \in c_0(u,v,B)$. It is clear that $\widehat{G}x \in c_0$. We can find $m \in \mathbb{N}$ such that $|\widehat{G}x| < 1$ for all $n \geq m$. By our assumption (i), we have $|\widehat{G}x|^{p_n} < |\widehat{G}x|$ for all $n \geq m$. Therefore, we get $x \in c_0(u,v,p;B)$.

(ii) Let $x \in c_0(u,v,p;B)$. Then $\widehat{G}x \in c_0(p)$ and there exists $m \in \mathbb{N}$ such that $|\widehat{G}x|^{p_n} < 1$ for all $n \geq m$. Now, consider following inequality:

$$|\widehat{G}x| = (|\widehat{G}x|^{p_n})^{1/p_n} < |\widehat{G}x|^{p_n}$$

for all $n \geq m$. So we have $x \in c_0(u,v,B)$. This completes the proof. □

3.3. **Duals.** In this part, we state and prove certain theorems to determine the $\alpha-$, $\beta-$ and γ duals of the spaces $\mu(u,v,p;B)$ for $\mu \in \{\ell_\infty, c_0, c\}$. We start with the definition of the alpha, beta and gamma duals. If x and y are sequences and X and Y are subsets of ω, then we write $x \cdot y = (x_k y_k)_{k=0}^\infty$, $x^{-1} * Y = \{a \in \omega : a \cdot x \in Y\}$ and

$$M(X,Y) = \bigcap_{x \in X} x^{-1} * Y = \{a : a \cdot x \in Y \text{ for all } x \in X\}$$

for the multiplier space of X and Y. One can easily observe for a sequence space Z with $Y \subset Z$ and $Z \subset X$ that inclusions $M(X,Y) \subset M(X,Z)$ and $M(X,Y) \subset M(Z,Y)$ hold respectively. The $\alpha-$, $\beta-$and $\gamma-$duals of a sequence space, which are respectively denoted by X^α, X^β and X^γ are defined by

$$X^\alpha = M(X,\ell_1),\ X^\beta = M(X,cs) \text{ and } X^\gamma = M(X,bs).$$

It is obvious that $X^\alpha \subset X^\beta \subset X^\gamma$. Also, it can easily be seen that the inclusions $X^\alpha \subset Y^\alpha$, $X^\beta \subset Y^\beta$ and $X^\gamma \subset Y^\gamma$ hold whenever $Y \subset X$.

Theorem 3.6. *Define the matrix $D = (d_{nk})$ by*

$$d_{nk} = \begin{cases} \left(\frac{-s}{r}\right)^{n-k}\left[\frac{1}{rv_k} + \frac{1}{sv_{k+1}}\right]\frac{1}{u_k}a_n, & k < n, \\ \frac{1}{ru_nv_n}a_n, & k = n, \\ 0, & k > n \end{cases}$$

for all $n, k \in \mathbb{N}$. Then,

$$\mu^\alpha(u,v,p;B) = \{a = (a_k) \in w : D \in (\mu(p) : \ell_1)\}.$$

Proof. Let $a = (a_n) \in w$. Then by using (3.12), we immediately derive for every $n \in \mathbb{N}$ that

$$a_n x_n = \frac{1}{r}\sum_{k=0}^{n}\left(\frac{-s}{r}\right)^{n-k}\left[\frac{1}{rv_k}+\frac{1}{sv_{k+1}}\right]\frac{1}{u_k}y_k a_n + \frac{1}{ru_n v_n}y_n a_n = (Dy)_n. \quad (3.14)$$

Thus, we observe by (3.14) that $ax = (a_n x_n) \in \ell_1$ whenever $x = (x_k) \in \mu(u,v,p;B)$ if and only if $D_n(y) \in \ell_1$ whenever $y = (y_k) \in \mu(p)$ which implies $a = (a_k) \in \{\mu(u,v,p;B)\}^{\alpha}$ if and only if $D \in (\mu(p) : \ell_1)$. □

Theorem 3.6 corresponds in the special case $q_n = 1$ for all $n \in \mathbb{N}$ to Part (1-3) of Theorem 5.1 of [16].

As a direct consequence of Theorem 3.6, we have following.

Corollary 3.7. *Let $K^* = K \cap \{n \in \mathbb{N} : n-1 \le k \le n\}$ for $K \subset \mathcal{F}$ and $M \in \mathbb{N}_2$. Define the sets $t_1(p), t_2(p), t_3(p)$ as follows:*

$$\begin{aligned}
t_1(p) &:= \bigcap_{M>1}\left\{a = (a_k) \in \omega : \sup_{K\in\mathcal{F}}\sum_n \left|\sum_{k\in K^*} d_{nk}M^{1/p_k}\right| < \infty\right\},\\
t_2(p) &:= \bigcup_{M>1}\left\{a = (a_k) \in \omega : \sum_n \left|\sum_k d_{nk}\right| < \infty\right\},\\
t_3(p) &:= \bigcup_{M>1}\left\{a = (a_k) \in \omega : \sup_{K\in\mathcal{F}}\sum_n \left|\sum_{k\in K^*} d_{nk}M^{-1/p_k}\right| < \infty\right\}.
\end{aligned}$$

Then, $\ell_\infty^{\alpha}(u,v,p;B) = t_1(p)$, $c_0^{\alpha}(u,v,p;B) = t_3(p)$ and $c^{\alpha}(u,v,p;B) = t_2(p) \cap t_3(p)$.

Theorem 3.8. *Let $u, v \in U$ and r,s be non-zero real numbers. Define the matrix $C = (c_{nk})$ by*

$$c_{nk} = \begin{cases} \widehat{a}_k(n), & 0 \le k \le n-1,\\ \frac{1}{ru_n v_n}a_n, & k = n,\\ 0, & k > n \end{cases} \quad (3.15)$$

for all $n, k \in \mathbb{N}$, where

$$\widehat{a}_k(n) = \frac{1}{u_k}\left[\frac{a_k}{rv_k} + \left(\frac{1}{rv_k}+\frac{1}{sv_{k+1}}\right)\sum_{j=k+1}^{n}\left(\frac{-s}{r}\right)^{k-j}a_j\right] \quad \textit{for } k < n.$$

$$\mu^{\beta}(u,v,p;B) = \{a = (a_k) \in w : C \in (\mu(p) : c)\},$$

$$\mu^{\gamma}(u,v,p;B) = \{a = (a_k) \in w : C \in (\mu(p) : \ell_\infty)\}.$$

Proof. Let us consider following equation

$$\begin{aligned}
\sum_{k=0}^{n} a_k x_k &= \sum_{k=0}^{n} \left\{ \sum_{j=0}^{k-1} \left(\frac{-s}{r} \right)^{k-j} \left[\frac{1}{rv_j} + \frac{1}{sv_{j+1}} \right] \frac{1}{u_j} y_j + \frac{1}{u_k v_k} y_k \right\} a_k \\
&= \sum_{k=0}^{n-1} \frac{1}{u_k} \left[\frac{a_k}{rv_k} + \left(\frac{1}{rv_k} + \frac{1}{sv_{k+1}} \right) \sum_{j=k+1}^{n} \left(\frac{-s}{r} \right)^{k-j} a_j \right] y_k + \frac{a_n}{ru_n v_n} y_n \\
&= \sum_{k=0}^{n-1} \widehat{a}_k(n) y_k + \frac{a_n}{ru_n v_n} y_n \\
&= (Cy)_n \text{ for all } \ n \in \mathbb{N}, \qquad (3.16)
\end{aligned}$$

where $C = (c_{nk})$ defined by (3.15). We deduce from (3.16) that $ax = (a_n x_n) \in cs$ or bs whenever $x = (x_k) \in \mu(u, v, p; B)$ if and only if $Cy \in c$ or ℓ_∞ whenever $y = (y_k) \in \mu(p)$. This means that $a = (a_k) \in \{\mu(u, v, p; B)\}^\beta$ or $a = (a_k) \in \{\mu(u, v, p; B)\}^\gamma$ if and only if $C \in (\mu(p) : c)$ or $C \in (\mu(p) : \ell_\infty)$. This completes the proof. □

As a direct consequence of Theorem 3.8, we have following.

Corollary 3.9. *Define the sets $d_1(p), d_2(p), d_3(p), d_4(p)$ and $d_5(p)$ as follows:*

$$\begin{aligned}
d_1(p) &= \bigcap_{B>1} \left\{ a = (a_k) \in \omega : \sum_k |\widehat{a}_k(n)| \, B^{1/p_k} \ \text{convergent uniformly in } n \right\}, \\
d_2(p) &= \left\{ a = (a_k) \in \omega : \left(\frac{1}{ru_n v_n} a_k B^{1/p_k} \right) \in c_0 \right\}, \\
d_3(p) &= \bigcap_{B>1} \left\{ a = (a_k) \in \omega : \sum_k |\widehat{a}_k(n)| \, B^{1/p_k} < \infty \right\}, \\
d_4(p) &= \bigcup_{B>1} \left\{ a = (a_k) \in \omega : \sum_k |\widehat{a}_k(n)| \, B^{-1/p_k} < \infty \right\}, \\
d_5(p) &= \bigcap_{B>1} \left\{ a = (a_k) \in \omega : \{\widehat{a}_k(n)\} \, B^{1/p_k} \in \ell_\infty \right\},
\end{aligned}$$

$\{\ell_\infty(u, v, p; B)\}^\beta = d_1(p) \cap d_2(p)$, $\{c_0(u, v, p; B)\}^\beta = \{c_0(u, v, p; B)\}^\gamma = d_4(p)$, $\{c(u, v, p; B)\}^\beta = d_4(p) \cap cs$, $\{c(u, v, p; B)\}^\gamma = d_4(p) \cap bs$ and $\{\ell_\infty(u, v, p; B)\}^\gamma = d_3(p) \cap d_5(p)$.

3.4. **Certain Matrix Mappings.** In this part, we characterize the matrix mappings from the sequence space $\mu(u, v, p; B)$ into any given sequence space, where $\mu \in \{c_0, c, \ell_\infty\}$. For an infinite matrix $A = (a_{nk})$, we write for brevity that

$$\overline{a}_{nk}(m) = \frac{1}{u_k} \left[\frac{a_{nk}}{rv_k} + \left(\frac{1}{rv_k} + \frac{1}{sv_{k+1}} \right) \sum_{j=k+1}^{m} \left(\frac{-s}{r} \right)^{k-j} a_{nj} \right] \ (k < m)$$

and

$$\overline{a}_{nk} = \frac{1}{u_k}\left[\frac{a_{nk}}{rv_k} + \left(\frac{1}{rv_k} + \frac{1}{sv_{k+1}}\right)\sum_{j=k+1}^{\infty}\left(\frac{-s}{r}\right)^{k-j} a_{nj}\right], \tag{3.17}$$

for all $k, n, m \in \mathbb{N}$ provided the convergence of the series. Now, we give the characterization of the classes $(\mu(u, v, p; B) : \nu)$ and $(\nu : \mu(u, v, p; B))$, where ν any given sequence space.

Theorem 3.10. *$A = (a_{nk}) \in (\mu(u, v, p; B) : \nu)$ if and only if $D = (d_{nk}) \in (\mu(p) : \nu)$ and*

$$E^{(n)} \in (\mu(p) : c) \tag{3.18}$$

for every fixed $n \in \mathbb{N}$, where $d_{nk} = \overline{a}_{nk}$ and $E^{(n)} = (e^{(n)}_{mk})$ with

$$e^n_{mk} = \begin{cases} \overline{a}_{nk}(m), & 0 \leq k \leq m-1, \\ \frac{1}{ru_m v_m} a_{mk}, & k = m, \\ 0, & k > m \end{cases}$$

for all $k, n, m \in \mathbb{N}$.

Proof. Assume that ν is any given sequence space and keep in mind that the spaces $\mu(u, v, p; B)$ and $\mu(p)$ are paranorm isomorphic.
Let $A = (a_{nk}) \in (\mu(u, v, p; B) : \nu)$ and $x \in \mu(u, v, p; B)$. Then we obtain the equality

$$\sum_{k=0}^{m} a_{nk}x_k = \sum_{k=0}^{m-1} \overline{a}_{nk}(m)y_k + \frac{a_{nm}}{ru_m v_m} y_m = \sum_{k=0}^{m} e^n_{mk} y_k \tag{3.19}$$

for all $m, n \in \mathbb{N}$. Since Ax exists, $E^{(n)}$ must belong to the class $(\mu(p) : c)$. Letting $m \longrightarrow \infty$ in the equality (3.19) we have that $Ax = Dy$. Since $Ax \in \nu$, then $Dy \in \nu$. That is $D = (d_{nk}) \in (\mu(p) : \nu)$.

Conversely, let $D \in (\mu(p) : \nu)$ and (3.18) holds, and take any $x \in \mu(u, v, p; B)$. Then, we have $(d_{nk})_{k\in\mathbb{N}} \in \mu^{\beta}(p)$ which gives together with (3.18) that $(a_{nk})_{k\in\mathbb{N}} \in [\mu(u, v, p; B)]^{\beta}$ for each $n \in \mathbb{N}$. Thus, Ax exists and we obtain from equality (3.19) as $m \longrightarrow \infty$ that $Dy = Ax$ and which means that $A = (a_{nk}) \in (\mu(u, v, p; B) : \nu)$. This completes the proof. □

Theorem 3.11. *Suppose that the entries of the infinite matrices $A = (a_{nk})$ and $B = (b_{nk})$ are connected with the relation*

$$b_{nk} = \sum_{j=0}^{n-1}(rv_j + sv_{j+1})a_{jk} + ru_n v_n a_{nk}$$

for all $k, n \in \mathbb{N}$ and ν be any given sequence space. Then, $A \in (\nu : \mu(u, v, p; B))$ if and only if $B \in (\nu : \mu(p))$.

Proof. Let $x = (x_k) \in \nu$ and consider the equality

$$\begin{aligned}\sum_{k=0}^{m} b_{nk}x_k &= \sum_{k=0}^{m}\left(\sum_{j=0}^{n-1}(rv_j + sv_{j+1})a_{jk} + ru_nv_na_{nk}\right)x_k \\ &= \sum_{j=0}^{n-1}(rv_j + sv_{j+1})\sum_{k=0}^{m} a_{jk}x_k + ru_nv_n\sum_{k=0}^{m} a_{nk}x_k\end{aligned}$$

for all $k, m, n \in \mathbb{N}$ which yields as $m \to \infty$ that $Bx = \widehat{G}(Ax)$. Therefore, $Ax \in \mu(u, v, p; B)$ whenever $x \in \nu$ if and only if $Bx \in \mu(p)$ whenever $x \in \nu$. This step completes the proof. □

The necessary and sufficient conditions characterizing the matrix mapping between the sequence spaces $\ell_\infty(p)$, $c_0(p)$ and $c(p)$ of Maddox are determined by Grosse-Erdmann [16]. Let N and K denote subsets of $\mathbb{N}$, L and M also denote the natural numbers and define the sets K_1 and K_2 by $K_1 = \{k \in \mathbb{N} : p_k \leq 1\}$, $K_2 = \{k \in \mathbb{N} : p_k > 1\}$. Before giving the theorems, let us suppose that (q_n) is non-decreasing sequence of positive real numbers and consider following condition:

$$\exists M, \quad \sup_K \sum_n \left|\sum_{k\in K} a_{nk}M^{-1/p_k}\right|^{q_n} < \infty \quad (q_n \geq 1), \tag{3.20}$$

$$\sum_n \left|\sum_k a_{nk}\right|^{q_n} < \infty, \quad (q_n \geq 1), \tag{3.21}$$

$$\forall M, \quad \sup_K \sum_n \left|\sum_{k\in K} a_{nk}M^{-1/p_k}\right|^{q_n} < \infty \quad (q_n \geq 1), \tag{3.22}$$

$$\lim_{n\to\infty} |a_{nk}|^{q_n} = 0 \quad \text{for all } k, \tag{3.23}$$

$$\forall L, \quad \sup_{n\in\mathbb{N}} \sup_{k\in K_1} \left|a_{nk}L^{1/q_n}\right|^{p_k} < \infty, \tag{3.24}$$

$$\forall L,\ \exists M, \quad \sup_{n\in K_1} \sum_{k\in K_2} \left|a_{nk}L^{1/q_n}M^{-1}\right|^{p'_k} < \infty, \tag{3.25}$$

$$\forall L,\ \exists M, \quad \sup_{n\in\mathbb{N}} L^{1/q_n} \sum_k |a_{nk}|M^{-1} < \infty, \tag{3.26}$$

$$\lim_{n\to\infty} \left|\sum_k a_{nk}\right|^{q_n} = 0 \tag{3.27}$$

$$\forall M, \quad \lim_{n\to\infty} \left(\sum_k |a_{nk}|M^{1/p_k}\right)^{q_n} = 0, \tag{3.28}$$

$$\sup_{n\in\mathbb{N}} \sup_{k\in K_1} |a_{nk}|^{p_k}, \tag{3.29}$$

$$\exists M, \quad \sup_{k\in\mathbb{N}} \sum_{k\in K_2} \left|a_{nk}M^{-1}\right|^{p'_k} < \infty, \tag{3.30}$$

$$\exists(\alpha_k), \quad \lim_{n\to\infty} |a_{nk}-\alpha_k|^{q_n} = 0 \quad \text{for all } k, \tag{3.31}$$

$$\exists(\alpha_k),\ \forall L, \quad \sup_{n\in\mathbb{N}} \sup_{k\in K_1} \left(|a_{nk}-\alpha_k| L^{1/q_n}\right)^{p_k} < \infty, \tag{3.32}$$

$$\exists(\alpha_k),\ \forall L,\ \exists M,\ \sup_{n\in\mathbb{N}} \left(|a_{nk}-\alpha_k| L^{1/q_n} M^{-1}\right)^{p'_k} < \infty, \tag{3.33}$$

$$\exists M, \quad \sup_{k\in\mathbb{N}} \sum_{k\in K_2} \left|a_{nk}M^{-1}\right|^{p_k} < \infty, \tag{3.34}$$

$$\exists M, \quad \sup_{n\in\mathbb{N}} \sum_{k} |a_{nk}| M^{-1/p_k} < \infty, \tag{3.35}$$

$$\exists \alpha, \quad \lim_{n\to\infty} \left|\sum_k a_{nk} - \alpha_k\right| < \infty, \tag{3.36}$$

$$\forall M, \quad \sup_{n\in\mathbb{N}} \sum_{k} |a_{nk}| M^{1/p_k} < \infty, \tag{3.37}$$

$$\exists(\alpha_k),\ \exists M,\ \lim_{n\to\infty} \left(\sum_k |a_{nk}-\alpha_k| M^{-1/p_n}\right) = 0, \tag{3.38}$$

$$\exists L, \quad \sup_{n\in\mathbb{N}} \sup_{k\in K_1} \left|a_{nk}L^{-1/q_n}\right|^{p_k} < \infty, \tag{3.39}$$

$$\exists L, \quad \sup_{n\in\mathbb{N}} \sum_{k\in K_2} \left|a_{nk}L^{-1/q_n}\right|^{p_k} < \infty, \tag{3.40}$$

$$\exists M, \quad \sup_{n\in\mathbb{N}} \left(\sum_{k\in K} |a_{nk}| M^{-1/p_k}\right)^{q_n} < \infty, \tag{3.41}$$

$$\sup_{n\in\mathbb{N}} \left|\sum_k a_{nk}\right|^{q_n} < \infty, \tag{3.42}$$

$$\forall M, \quad \sup_{n\in\mathbb{N}} \left(\sum_k |a_{nk}| M^{1/p_k}\right)^{q_n} < \infty. \tag{3.43}$$

Corollary 3.12. *(i) $A = (a_{nk}) \in (c_0(u,v,p;B) : \ell(q))$ if and only if (3.20) holds with $\overline{a}_{nk}$ instead of a_{nk} and (3.18) also holds with $\mu = c_0$.*
(ii) $A = (a_{nk}) \in (c_0(u,v,p;B) : c(q))$ if and only if (3.31), (3.34) and (3.35) hold with $\overline{a}_{nk}$ instead of a_{nk} and (3.18) also holds with $\mu = c_0$.
(iii) $A = (a_{nk}) \in (c_0(u,v,p;B) : \ell(q))$ if and only if (3.41) holds with $\overline{a}_{nk}$ instead of a_{nk} and (3.18) also holds with $\mu = c_0$.

Corollary 3.13. *(i) $A = (a_{nk}) \in (c(u,v,p;B) : \ell(q))$ if and only if (3.20) and (3.21) hold with $\overline{a}_{nk}$ instead of a_{nk} and (3.18) also holds with $\mu = c$.*
(ii) $A = (a_{nk}) \in (c(u,v,p;B) : c(q))$ if and only if (3.31), (3.34)-(3.36) hold with $\overline{a}_{nk}$ instead of a_{nk} and (3.18) also holds with $\mu = c$.

(iii) $A = (a_{nk}) \in (c(u, v, p; B) : \ell(q))$ *if and only if (3.41) and (3.42) hold with* $\overline{a}_{nk}$ *instead of* a_{nk} *and (3.18) also holds with* $\mu = c$.

Corollary 3.14. *(i)* $A = (a_{nk}) \in (\ell_\infty(u, v, p; B) : \ell(q))$ *if and only if (3.22) holds with* $\overline{a}_{nk}$ *instead of* a_{nk} *and (3.18) also holds with* $\mu = \ell_\infty$.
(ii) $A = (a_{nk}) \in (\ell_\infty(u, v, p; B) : c(q))$ *if and only if (3.28) hold with* $\overline{a}_{nk}$ *instead of* a_{nk} *and (3.18) also holds with* $\mu = \ell_\infty$.
(iii) $A = (a_{nk}) \in (\ell_\infty(u, v, p; B) : \ell(q))$ *if and only if (3.37) and (3.38) hold with* $\overline{a}_{nk}$ *instead of* a_{nk} *and (3.18) also holds with* $\mu = \ell_\infty$.
(iv) $A = (a_{nk}) \in (\ell_\infty(u, v, p; B) : \ell_\infty(q))$ *if and only if (3.43) holds with* $\overline{a}_{nk}$ *instead of* a_{nk} *and (3.18) also holds with* $\mu = \ell_\infty$.

Corollary 3.15. *(i)* $A = (a_{nk}) \in (c_0(p) : c(u, v, p; B))$ *if and only if (3.31), (3.34) and (3.35) hold with* b_{nk} *instead of* a_{nk} .
(ii) $A = (a_{nk}) \in (c_0(p) : \ell_\infty(u, v, p; B))$ *if and only if (3.41) holds with* b_{nk} *instead of* a_{nk} .

Corollary 3.16. *(i)* $A = (a_{nk}) \in (c(p) : c(u, v, p; B))$ *if and only if (3.31), (3.34)-(3.36) hold with* b_{nk} *instead of* a_{nk} .
(ii) $A = (a_{nk}) \in (c(p) : \ell_\infty(u, v, p; B))$ *if and only if (3.41) and (3.42) hold with* b_{nk} *instead of* a_{nk} .

Corollary 3.17. *(i)* $A = (a_{nk}) \in (\ell_\infty(p) : c_0(u, v, p; B))$ *if and only if (3.28) holds with* b_{nk} *instead of* a_{nk}.
(ii) $A = (a_{nk}) \in (\ell_\infty(p) : \ell_\infty(u, v, p; B))$ *if and only if (3.43) holds with* b_{nk} *instead of* a_{nk} .
(iii) $A = (a_{nk}) \in (\ell_\infty(p) : c(u, v, p; B))$ *if and only if (3.37) and (3.38) hold and* b_{nk} *instead of* a_{nk}.

Corollary 3.18. *(i)* $A = (a_{nk}) \in (\ell(p) : c_0(u, v, p; B))$ *if and only if (3.23) and (3.25) hold with* b_{nk} *instead of* a_{nk}.
(ii) $A = (a_{nk}) \in (\ell(p) : c(u, v, p; B))$ *if and only if (3.29) and (3.33) hold with* b_{nk} *instead of* a_{nk} .
(iii) $A = (a_{nk}) \in (\ell(p) : \ell_\infty(u, v, p; B))$ *if and only if (3.39) and (3.40) hold and* b_{nk} *instead of* a_{nk}.

4. COMPACTNESS OF MATRIX OPERATORS

In the section, we establish some identities or estimates for the operator norms and the Hausdorff measures of noncompactness of certain matrix operators on the spaces $c_0(u, v; B)$ and $\ell_\infty(u, v, ; B)$. Further, by using the Hausdorff measure of noncompactness, we characterize certain some classes of compact operators on these spaces. It is quite natural to find condition for a matrix map between

$BK-$space to define a compact operator since a matrix transformation between $BK-$spaces are continuous. This can be achieved by applying the Hausdorff measure of noncompactness. Recently, several authors characterized classes of compact operators given by infinite matrices on some sequence spaces by using this method. For example, in [20, 21], Mursaleen and Noman, Malkowsky and Rakočević [10], Djolović and Malkowsky [30], and Karaisa [8] established some identities or estimate for the operator norms and Hausdorff measure of noncompactness of the linear operator given by infinite matrices that map an arbitrary $BK-$space or the matrix domain of triangles in arbitrary $BK-$space. Further, They characterized some classes of compact operators on these spaces by using the Hausdorff measure of noncompactness.

4.1. **Notations and Auxiliary Facts.** Now, we give some related definitions, notations and preliminary result.

Let X and Y be Banach space. Then, we write $\mathcal{B}(X,Y)$ for the set of all bounded linear operators $L : X \longrightarrow Y$, which is a Banach space with the operator norm given by $\|L\| = \sup_{x\in S_X} \|L(x)\|_Y$ for all $L \in \mathcal{B}(X,Y)$, where S_X denotes the unit sphere in X, the sequence $(L(x_n))$ has a subsequence which converges in Y. By $\mathcal{C}(X,Y)$, we denote the class of all compact operator in $\mathcal{B}(X,Y)$. An operator $L \in \mathcal{B}(X,Y)$ is said to be of finite rank if $dimR(L) < \infty$, where $R(L)$ denotes range of L. An operator of finite rank is clearly compact.

If $(\|.\|, X)$ is a normed sequence space then we write

$$\|a\|_X^* = \sup_{x\in S_X} \sum_{k=0}^{\infty} |a_k x_k| \tag{4.1}$$

for $a \in w$ provided the expression on the right-hand side exists and is finite which the case whenever X is a $BK-$space and $a \in X^\beta$ [11]. Let S and M be subsets of metric space (X,d) and $\varepsilon > 0$. Then S is called an $\varepsilon-$net of M in X if every $x \in M$ there exists $s \in S$ such that $d(x,s) < \varepsilon$. Further the set S is finite, then the $\varepsilon-$net S of M is called a finite $\varepsilon-$net of M, and we say that M has a finite $\varepsilon-$net in X. A subset of a metric space is said to be totally bounded if it has a finite $\varepsilon-$net for every $\varepsilon > 0$. By $\mathcal{M}_X$, we denote the collection of all bounded subsets of a metric space (X,d). If $Q \in \mathcal{M}_X$, then the Hausdorff measure of noncompactness of the set Q, denotes by $\chi(Q)$, is defined by

$$\chi(Q) = \inf\{\varepsilon > 0 : Q \text{ has a finite } \varepsilon\text{-net in } X\}.$$

The function $\chi : \mathcal{M}_X \longrightarrow [0,\infty)$ is called the Hausdorff measure of noncompactness [11, p. 387].

The basic properties of the Hausdorff measure of noncompactness can be found in [12, Lemma 2]; for example if Q, Q_1 and Q_2 are bounded subsets of a metric space (X,d), then

$$\chi(Q) = 0 \text{ if and only if } Q \text{ is totally bounded,}$$
$$Q_1 \subset Q_2 \text{ implies } \chi(Q_1) \le \chi(Q_2).$$

Further, if X is a normed space, then the function χ has some additional properties connected with the linear structure, that is

$$\chi(Q_1+Q_2)\le\chi(Q_1)+\chi(Q_2),$$
$$\chi(\alpha Q)=|\alpha|\chi(Q) \text{ for all } \alpha\in\mathbb{C}.$$

We shall need the following known results for our investigation.

Lemma 4.1. [12, Lemma 15(a)] *Let $\varphi\supset X$ and Y be a $BK-$space. Then, we have $(X,Y)\subset\mathcal{B}(X,Y)$, that is, every matrix $A\in(X,Y)$ defines an operator $L_A\in\mathcal{B}(X,Y)$ by $L_A(x)=Ax$ for all $x\in X$.*

Lemma 4.2. [13, Theorem 3.8)] *Let T be a triangle. Then, we have*
(a) For arbitrary subsets X and Y of ω, $A\in(X,Y_T)$ if and only if $B=TA\in(X,Y)$.
(b) Further, if X and Y are BK spaces and $A\in(X,Y_T)$, then $\|L_A\|=\|L_B\|$.

Lemma 4.3. [31, Lemma 5.2] *Let $\varphi\supset X$ be a BK space and Y be any of the spaces c_0, c or ℓ_∞. If $A\in(X,Y)$, then we have*

$$\|L_A\|=\|A\|_{(X,\ell_\infty)}=\sup_n|A_n|_X^*<\infty.$$

Lemma 4.4. [13, Theorem 1.29] *Let X be any of the spaces c, c_0 or ℓ_∞. Then, we have $X^\beta=\ell_1$ and $\|a\|_X^*=\|a\|_{\ell_1}$ for all $a\in\ell_1$.*

Lemma 4.5. *Let X be denote any of the spaces $c_0(u,v;B)$ and $\ell_\infty(u,v;B)$. If $a=(a_k)\in X^\beta$, then we $\widetilde{a}=(\widetilde{a}_k)\in\ell_1$ and equality*

$$\sum_{k=0}^{\infty}a_kx_k=\sum_{k=0}^{\infty}\widetilde{a}_ky_k \tag{4.2}$$

holds for every $x=(x_k)\in X$, where $y=\widehat{G}(x)$ is the associated sequence defined by (3.1) and

$$\widetilde{a}_k = \frac{1}{u_k}\left[\frac{a_k}{rv_k}+\left(\frac{1}{rv_k}+\frac{1}{sv_{k+1}}\right)\sum_{j=k+1}^{\infty}\left(\frac{-s}{r}\right)^{k-j}a_j\right].$$

Theorem 4.6. *Let X be denote any of the spaces $c_0(u,v;B)$ and $\ell_\infty(u,v;B)$. Then, we have*

$$\|a\|_X=\|\widetilde{a}\|_{\ell_1}=\sum_{k=0}^{\infty}|\widetilde{a}_k|<\infty.$$

for all $a=(a_k)\in X^\beta$, where $\widetilde{a}=(\widetilde{a}_k)$ is as in Lemma 4.5.

Proof. Let Y be the respective one of the space c_0 or ℓ_∞, and take any $a=(a_k)\in X^\beta$. Then, we have by Lemma 4.5 that $\widetilde{a}=(\widetilde{a}_k)\in\ell_1$ and the equality (4.2) holds for all sequences $x=(x_k)\in X$ and $y=(y_k)\in Y$ which are connected by the relation (3.1). Further, it follows by (3.2) that $x\in S_X$ if and only if $y\in S_Y$. Therefore, we derive from (4.1) and (4.2) that

$$\|a\|_X=\sup_{x\in S_X}\left|\sum_{k=0}^{\infty}a_kx_k\right|=\sup_{y\in S_Y}\left|\sum_{k=0}^{\infty}\widetilde{a}_ky_k\right|=\|\widetilde{a}\|_Y.$$

Since $\widetilde{a} \in \ell_1$, we obtain from Lemma 4.4 that

$$\|a\|_X^* = \|\widetilde{a}\|_Y^* = \|\widetilde{a}\|_{\ell_1}^* < \infty.$$

□

Lemma 4.7. *Let X be any of the space $c_0(u,v;B)$ or $\ell_\infty(u,v;B)$, Y the respective one of the spaces c_0 or ℓ_∞, Z a sequence space and $A = (a_{nk})$ an infinite matrix. If $A \in (X,Z)$, then $\overline{A} \in (Y,Z)$ such that $Ax = \overline{A}y$ for all sequences $x \in X$ and $y \in Y$ which are connected by the relation (3.1), where $\overline{A} = (\overline{a}_{nk})$ is the associated matrix defined by (3.17).*

Proof. It can be similarly proved by the same technique in [21, Lemma 2.3]. □

Lemma 4.8. *Let X be any of the space $c_0(u,v;B)$ or $\ell_\infty(u,v;B)$, $A = (a_{nk})$ an infinite matrix and $\overline{A} = (\overline{a}_{nk})$ is the associated matrix. If A is in any of the classes (X,c_0), (X,c) or (X,ℓ_∞), then*

$$\|L_A\| = \|A\|_{(X,\ell_\infty)} = \sup_n \left(\sum_n^\infty |\overline{a}_{nk}| \right) < \infty. \tag{4.3}$$

Proof. This is immediate by combining Lemmas 4.3 and 4.6. □

The following results shows how to compute the Hausdorff measure of noncompactness in the space c_0.

Lemma 4.9. [29, Theorem 3.3] *Let $Q \in \mathcal{M}_{c_0}$ and $p_r : c_0 \longrightarrow c_0$ $(r \in \mathbb{N})$ be operator defined by $p_r(x) = (x_0, x_1, \ldots, x_r, 0, 0, \ldots)$ for all $x = (x_k) \in c_0$. Then we have*

$$\chi(Q) = \lim_{r\to\infty} \left(\sup_{x\in Q} \|(I - p_r)(x)\|_{\ell_\infty} \right),$$

where I is the identity operator on c_0.

Further, we know by [13, Theorem 1.10] *that every $z = (z_k) \in c$ has a unique representation $z = \overline{z}e + \sum_n^\infty (z_n - \overline{z})e^{(n)}$, where $\overline{z} = \lim_{n\to\infty} z_n$. Thus, we define the projectors $p_r : c \longrightarrow c$ $(r \in \mathbb{N})$ by*

$$p_r = \overline{z}e + \sum_{n=0}^r (z_n - \overline{z})e^{(n)}; \quad (r \in \mathbb{N})$$

for all $z = (z_k) \in c$ with $\overline{z} = \lim_{n\to\infty} z_n$. In this situation, the following result gives an estimate for the Hausdorff measure of noncompactness in the space c.

Lemma 4.10. [12, Theorem 5(b)] *Let $Q \in \mathcal{M}_c$ and $p_r : c \longrightarrow c$ $(r \in \mathbb{N})$ be the projector onto the linear span of $(e^{(0)}, e^{(1)}, \ldots, e^{(r)})$. Then, we have*

$$\frac{1}{2} \lim_{r\to\infty} \left(\sup_{x\in Q} \|(I - p_r)(x)\|_{\ell_\infty} \right) \leq \chi(Q) \leq \lim_{r\to\infty} \left(\sup_{x\in Q} \|(I - p_r)(x)\|_{\ell_\infty} \right), \tag{4.4}$$

where I is the identity operator on c.

The next lemma is related to the Hausdorff measure of noncompactness of a bounded linear operator.

Lemma 4.11. [13, Theorem 2.25] *Let X and Y be Banach spaces and $L \in B(X,Y)$. Then, we have*

$$\|L_A\|_\chi = \chi(L(S_X)) \tag{4.5}$$

and

$$L \in \mathcal{C}(X,Y) \quad \text{if and only if} \quad \|L_A\|_\chi = 0. \tag{4.6}$$

4.2. **Compact Operators on the Spaces $c_0(u,v;B)$ and $\ell_\infty(u,v;B)$.** In this part, we establish some identities or estimates for the Hausdorff measures of noncompactness of certain matrix operators on the spaces $c_0(u,v;B)$ and $\ell_\infty(u,v;B)$. Further, we apply our results to characterize some classes of compact operators on those spaces. We begin with the following lemmas which will be used in proving our results.

Lemma 4.12. [20, Lemma 3.1] *Let X be any of the space c_0 or ℓ_∞. If $A \in (X,c)$*

$$\begin{aligned}
&\alpha_k = \lim_{n\to\infty} a_{nk} \quad \text{exists for every} \quad k \in \mathbb{N},\\
&\alpha = (\alpha_k) \in \ell_1,\\
&\sup_n \left(\sum_k^\infty |a_{nk} - \alpha_k| \right) < \infty,\\
&\lim_{n\to\infty} A_n = \sum_k^\infty \alpha_k x_k \quad \text{for all} \quad x = (x_k) \in X.
\end{aligned}$$

Lemma 4.13. [23, Theorem 3.7] *Let $X \supset \phi$ be a $BK-$space. Then we have*
(a) If $A \in (X,c_0)$, then

$$\|L_A\|_\chi = \limsup_{n\to\infty} \|A_n\|_X^*.$$

(b) If $A \in (X,\ell_\infty)$, then

$$0 \le \|L_A\|_\chi \le \limsup_{n\to\infty} \|A_n\|_X^*.$$

Theorem 4.14. *Let X denote any of the spaces $c_0(u,v;B)$ and $\ell_\infty(u,v;B)$. Then we have*
(a) If $A \in (X,c_0)$, then

$$\|L_A\|_\chi = \limsup_{n\to\infty} \left(\sum_{k=0}^\infty |\overline{a}_{nk}| \right) \tag{4.7}$$

and

$$L_A \quad \text{compact if and only if} \quad \lim_{n\to\infty} \left(\sum_{k=0}^\infty |\overline{a}_{nk}| \right) = 0. \tag{4.8}$$

(b) If $A \in (X, \ell_\infty)$, then

$$0 \leq \|L_A\|_\chi \leq \limsup_{n\to\infty} \left(\sum_{k=0}^{\infty} |\overline{a}_{nk}| \right)$$

and

$$L_A \ \textit{ compact if and only if } \ \lim_{n\to\infty} \left(\sum_{k=0}^{\infty} |\overline{a}_{nk}| \right) = 0.$$

Proof. Let $A \in (X, c_0)$. Since $A_n \in X^\beta$ for all $n \in \mathbb{N}$, we have from Lemma 4.6 that

$$\|A_n\|_X = \|\overline{A}_n\|_{\ell_1} = \sum_{k=0}^{\infty} |\overline{a}_{nk}| < \infty \,. \tag{4.9}$$

Thus, we get (4.7) from (4.9), (4.3) and Lemma 4.13(a). We derived (4.8) from (4.6). Part (b) can be proved similarly by using Lemma 4.13(b). □

Theorem 4.15. *Let X denote any of the spaces $c_0(u, v; B)$ and $\ell_\infty(u, v; B)$. If $A \in (X, c)$, then we have*

$$\frac{1}{2} \limsup_{n\to\infty} \left(\sum_{k=0}^{\infty} |\overline{a}_{nk} - \overline{\alpha}_k| \right) \leq \|L_A\|_\chi \leq \limsup_{n\to\infty} \left(\sum_{k=0}^{\infty} |\overline{a}_{nk} - \overline{\alpha}_k| \right) \tag{4.10}$$

and

$$L_A \ \textit{ compact if and only if } \ \lim_{n\to\infty} \left(\sum_{k=0}^{\infty} |\overline{a}_{nk} - \overline{\alpha}_k| \right) = 0, \tag{4.11}$$

where $\lim_{n\to\infty} \overline{a}_{nk} = \overline{\alpha}_k$.

Proof. By combining Lemma 4.7 and Lemma 4.12, we deduce that the expression in (4.7) exists. We write $S = S_X$, for short. Then, we obtain by (4.5) and Lemma 4.1 that

$$\|L_A\|_\chi = \chi(AS) \tag{4.12}$$

which means $AS \in \mathcal{M}_c$, where is the class of all bounded subsets of c. Then, we are going to apply Lemma 4.10 to get an estimate for the value of $\chi(AS)$ in (4.12). For this, let $p_r : c \longrightarrow c$ be the projectors defined by (4.4). Then, we have for every $r \in \mathbb{N}$ that $(I - p_r)(z) = \sum_{n=r+1}^{\infty} (z_n - z)e^n$ and hence,

$$\|(I - p_r)(z)\|_{\ell_\infty} = \sup_{n>r} |z - \overline{z}| \tag{4.13}$$

for all $z \in c$. Thus, from (4.12) and Lemma 4.10 that

$$\frac{1}{2} \lim_{r\to\infty} \left(\sup_{x\in S} \|(I - p_r)(Ax)\|_{\ell_\infty} \right) \leq \|L_A\|_\chi \leq \lim_{r\to\infty} \left(\sup_{x\in S} \|(I - p_r)(Ax)\|_{\ell_\infty} \right) \tag{4.14}$$

Now, for every given $x \in X$ and $y \in Y$ be associated sequence defined by (3.1), where Y be the respective one of the space c_0 or ℓ_∞. Since $A \in (X, c)$, we have by Lemma 4.7 that $\overline{A} \in (Y, c)$ and $Ax = \overline{A}y$. Further, it follows from Lemma

4.12 that the limits $\overline{\alpha}_k = \lim_{n\to\infty} \overline{a}_{nk}$ exists for all k, $\overline{\alpha} = (\overline{\alpha}_k) \in \ell_1 = Y^\beta$ and $\lim_{n\to\infty}(\overline{A}y)_n = \sum_{k=0}^{\infty} \overline{\alpha}_k y_k$. Thus, we derive from (4.13) that

$$\begin{aligned} \|(I-p_r)(Ax)\|_{\ell_\infty} &= \|(I-p_r)(\overline{A}y)\|_{\ell_\infty} \\ &= \sup_{n>r} \left| \overline{A}_n(y) - \sum_{k=0}^{\infty} \overline{\alpha}_k y_k \right| \\ &= \sup_{n>r} \left| \sum_{k=0}^{\infty} (\overline{a}_{nk} - \overline{\alpha}_k) y_k \right| \end{aligned}$$

for $r \in \mathbb{N}$. Furthermore, since $x \in S = S_X$ if and only if $y \in S_Y$, we obtain by (4.1) and Lemma 4.1

$$\begin{aligned} \sup_{X\in S} \|(I-p_r)(Ax)\|_{\ell_\infty} &= \sup_{n>r} \left(\sup_{Y\in S_Y} \left| \sum (\overline{a}_{nk} - \overline{\alpha}_k) y_k \right| \right) \\ &= \sup_{n>r} \|\overline{A}_n - \overline{\alpha}\|_Y^* \\ &= \sup_{n>r} \|\overline{A}_n - \overline{\alpha}\|_{\ell_1} \end{aligned}$$

for all $r \in \mathbb{N}$. Thus, we get (4.10) and (4.11) from (4.14) and (4.6), respectively and this concludes the proof. □

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