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# APPROXIMATION OF LOWER BOUND FOR MATRIX OPERATORS ON THE WEIGHTED SEQUENCE SPACE $C_p^r(w)(0<p<1)$

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ABSTRACT. Let $A=(a_{n,k})_{n,k\geq 0}$ be a non-negative matrix. We denote by $L_{\ell_p(w),C_q^r(w)}(A)$ the supremum of those $\ell$, satisfying the following inequality:

$$\left(\sum_{n=0}^{\infty} w_n \left(\frac{1}{(1+r)^n}\sum_{k=n}^{\infty}\frac{(1+r)^k}{1+k}\sum_{j=0}^{\infty} a_{k,j}x_j\right)^q\right)^{1/q} \geq \ell\left(\sum_{n=0}^{\infty} w_n x_n^p\right)^{1/p},$$

where $x\geq 0$, $x\in \ell_p(w)$, $0<r<1, 0<q\leq p<1$ and $(w_n)_{n=0}^{\infty}$ is a non-negative and non-increasing sequence of real numbers. In this paper, first we introduce the weighted sequence space $C_p^r(w)$ $(0<p<1)$ of non-absolute type which is a $p$-normed space and is isometrically isomorphic to the space $\ell_p(w)$. Then we focus on the evaluation of $L_{\ell_p(w),C_q^r(w)}(A^t)$ for a lower triangular matrix $A$, where $0<q\leq p<1$. A lower estimate is obtained. Moreover, in this paper a Hardy type formula is obtained for $L_{\ell_p,C_q^r}(H_\mu^\alpha)$ where $H_\mu^\alpha$ is the generalized Hausdorff matrix, $0<q\leq p\leq 1$ and $\alpha\geq 0$. A similar result is also established for the case in which $H_\mu^\alpha$ is replaced by $(H_\mu^\alpha)^t$.

## 1. Introduction and preliminaries

Let $p\in\mathbb{R}\backslash\{0\}$ and let $\ell_p(w)$ denote the space of all real sequences $x=(x_k)_{k=0}^{\infty}$ such that

$$\|x\|_{\ell_p(w)} := \left(\sum_{k=0}^{\infty} w_k|x_k|^p\right)^{\frac{1}{p}} < \infty,$$

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where $w = (w_n)_{n=0}^{\infty}$ is a non-increasing and non-negative sequence of real numbers. We write $x \geq 0$ if $x_k \geq 0$ for all $k$. We also write $x \uparrow$ for the case that $x_0 \leq x_1 \leq \cdots \leq x_n \leq \cdots$. The symbol $x \downarrow$ is defined in a similar way.

Let $X$ be a normed sequence space, $Y$ be the same as $X$ with a different norm and $A = (a_{n,k})_{n,k\geq 0}$ be an infinite matrix of real or complex numbers; Then, it is said that $A$ defines a matrix mapping from $X$ into $Y$, and we denote it by writing $A : X \to Y$, if for every sequence $x = (x_k)$ in $X$ the sequence $Ax = \{(Ax)_n\}$, the $A$-transform of $x$ is in $Y$, where

$$(Ax)_n = \sum_{k=0}^{\infty} a_{n,k} x_k, \ \ n = 0, 1, \dots .$$

For $p, q \in \mathbb{R}\backslash\{0\}$, the lower bound involved here is the number $L_{\ell_p(w),C_q^r(w)}(A)$ which is defined as the supremum of those $\ell$ obeying the following inequality:

$$\|Ax\|_{C_q^r(w)} \geq \ell \, \|x\|_{\ell_p(w)} \,,$$

where $x \geq 0$, $x \in \ell_p(w)$ and $A = (a_{n,k})_{n,k\geq 0}$ is a non-negative matrix. Here $0 < q \leq p \leq 1$ and

$$\|x\|_{C_q^r(w)} := \left( \sum_{n=0}^{\infty} w_n \left| \frac{1}{(1+r)^n} \sum_{k=n}^{\infty} \frac{(1+r)^k}{1+k} x_k \right|^q \right)^{\frac{1}{q}}.$$

If $w = (1, 1, 1, \dots)$, we use $L_{\ell_p,C_q^r}(A)$ instead of $L_{\ell_p(w),C_q^r(w)}(A)$.

The organization of this paper is given as follows: In Section 2, we introduce the weighted sequence space $C_p^r(w)(0 < p < 1)$ of non-absolute type and give an inclusion relation concerning with this space. We also show that $C_p^r(w)$ is a $p-$normed space which is isometrically isomorphic to the space $\ell_p(w)$.

In section 3, we consider the transpose of non-negative lower triangular matrices as operators from the sequence space $\ell_p(w)$ into the weighted sequence space $C_q^r(w)$ where $0 < q \leq p < 1$, and obtained a lower estimate for $L_{\ell_p(w),C_q^r(w)}(A^t)$(see Theorem 3.1). Then we apply our results to some famous classes of non-negative lower triangular matrices such as row stochastic matrices, weighted mean matrices and Nörlund matrices.

In Section 4, a Hardy type formula is obtained for $L_{\ell_p,C_q^r}(H_\mu^\alpha)$, where $H_\mu^\alpha$ is the generalized Hausdorff matrix, $0 < q \leq p \leq 1$ and $\alpha \geq 0$. A similar result is also established for the case in which $H_\mu^\alpha$ is replaced by $(H_\mu^\alpha)^t$. (see Theorems 4.3 and 4.7). In continue, we apply our results to some special generalized Hausdorff matrices such as generalized Gamma, generalized Hölder, generalized Cesàro and generalized Euler matrices. Our results provide some analogue to those given in [9] and [10].

## 2. The weighted sequence space $C_p^r(w)(0 < p < 1)$

The main purpose of the present section, following [1, 2], is to introduce the weighted sequence space $C_p^r(w)(0 < p < 1)$ of non-absolute type and is to derive an inclusion relation concerning with its. Moreover, we show that $C_p^r(w)$ is a $p-$normed space and is isometrically isomorphic to the space $\ell_p(w)$.

Let $0 < r < 1$ and $w = (w_n)$ be a non-increasing sequence of non-negative real numbers. The weighted sequence space $C_p^r(w)$ is defined as below:

$$C_p^r(w) = \left\{ (x_n) \in \mathbb{C} : \quad \sum_{n=0}^{\infty} w_n \left| \left( \frac{1}{1+r} \right)^n \sum_{k=n}^{\infty} \frac{(1+r)^k}{1+k} x_k \right|^p < \infty \right\},$$

where $0 < p < 1$. More precisely, $C_p^r(w)$ is the set of all sequences such that $C^r$-transforms of them are in the space $\ell_p(w)$, where $C^r$ denotes the matrix $C^r = (c_{n,k}^r)_{n,k\geq 0}$ defined by

$$c_{n,k}^r = \begin{cases} 0 & 0 \leq k < n, \\ \frac{(1+r)^{k-n}}{1+k} & k \geq n. \end{cases}$$

If the weighted sequence $w$ be $(1, 1, 1, ...)$, we use the notation $C_p^r$ instead of $C_p^r(w)$.

Now, we may begin with the following theorem which is essential in the text.

**Theorem 2.1.** *The set $C_p^r(w)$ becomes a linear space with the coordinatewise addition and scalar multiplication which is the p-normed space with the p-norm*

$$|||x||| := \|x\|_{C_p^r(w)}^p = \sum_{n=0}^{\infty} w_n \left| \frac{1}{(1+r)^n} \sum_{k=n}^{\infty} \frac{(1+r)^k}{1+k} x_k \right|^p.$$

*Proof.* This is a routine verification and so we omit the detail. □

One can easily check that the absolute property does not hold on the space $C_p^r(w)$, that is, $\|x\|_{C_p^r(w)} \neq \||x|\|_{C_p^r(w)}$ for at least one sequence in the space $C_p^r(w)$, and this says us that $C_p^r(w)$ is a sequence space of non-absolute type, where $|x| = (|x_k|)$ and $0 < p < 1$. Also, it is immediate by the well known inclusion $\ell_q(w) \subseteq \ell_p(w)$ that the inclusion $C_q^r(w) \subseteq C_p^r(w)$ holds whenever $0 < q \leq p < 1$.

**Theorem 2.2.** *The weighted sequence space $C_p^r(w)$ is isometrically isomorphic to the space $\ell_p(w)$, that is $C_p^r(w) \cong \ell_p(w)$.*

*Proof.* It is enough to show the existence of an isometric isomorphism between the spaces $C_p^r(w)$ and $\ell_p(w)$. Consider the transformation $\Lambda$ defined from $C_p^r(w)$ to $\ell_p(w)$ by $x \mapsto y = \Lambda x$, where $y = \{y_n\}$ is the $C^r$-transform of the sequence $x$, i.e.

$$y_n = \left( \frac{1}{1+r} \right)^n \sum_{k=n}^{\infty} \frac{(1+r)^k}{1+k} x_k; \quad n \in \mathbb{N} \cup \{0\}. \tag{2.1}$$

The linearity of $\Lambda$ is clear. Further, it is trivial that $x = 0$ whenever $\Lambda x = 0$ and hence $\Lambda$ is injective. Let $y \in \ell_p(w)$ for $0 < p < 1$ and define the sequence $x = \{x_n\}$ by

$$x_n = (n+1) \left[ y_n - (1+r)\, y_{n+1} \right]; \; n \in \mathbb{N} \cup \{0\}.$$

Then, we have

$$\begin{aligned}
|||x||| &= \sum_{n=0}^{\infty} w_n \left| \tfrac{1}{(1+r)^n} \sum_{k=n}^{\infty} \tfrac{(1+r)^k}{1+k} x_k \right|^p \\
&= \sum_{n=0}^{\infty} w_n \left| \tfrac{1}{(1+r)^n} \sum_{k=n}^{\infty} \tfrac{(1+r)^k}{1+k} (k+1) \left[ y_k - (1+r)\, y_{k+1} \right] \right|^p \\
&= \sum_{n=0}^{\infty} w_n \left| \tfrac{1}{(1+r)^n} \left( \sum_{k=n}^{\infty} (1+r)^k y_k - \sum_{k=n}^{\infty} (1+r)^{k+1} y_{k+1} \right) \right|^p \\
&= \sum_{n=0}^{\infty} w_n |y_n|^p = \ \|y\|_{\ell_p(w)}^p .
\end{aligned}$$

Thus, we have that $x \in C_p^r(w)$ and consequently $\Lambda$ is surjective and $p-$norm preserving. Hence, $\Lambda$ is a linear bijection which says us that the spaces $C_p^r(w)$ and $\ell_p(w)$ are linearly isomorphic, as desired. $\square$

The following lemma has an essential role in the rest of this paper.

**Lemma 2.3.** *([8], Corollary 4.3.3). Let $x \in \ell_p(w), x \geq 0$ and let $w$ be a non-increasing and non-negative sequence of real numbers. Then*

$$\sum_{n=0}^{\infty} w_n \left( \sum_{k=n}^{\infty} \frac{x_k}{k+1} \right)^p \geq p^p \left( \sum_{n=0}^{\infty} w_n x_n^p \right) \qquad (0 < p \leq 1) . \tag{2.2}$$

*Here the constant $p^p$ is the best possible.*

Inequality (2.2) which is the weighted version of Copson inequality [6], can be rewritten as $L_{\ell_p(w),\ell_p(w)}(C) = p$, where $C = (c_{n,k})_{n,k\geq 0}$ is the Copson matrix defined by

$$c_{n,k} = \begin{cases} 0 & 0 \leq k < n, \\ \frac{1}{k+1} & \quad k \geq n. \end{cases}$$

We conclude this section by giving a sequence of the points of the space $C_p^r(w)$ which forms a basis(Schauder basis) for that space, where $0 < p < 1$. Because of the isomorphism $\Lambda$, defined in the proof of Theorem 2.2, is onto the inverse image of the basis $\{e_k\}_{k=0}^{\infty}$ of the space $\ell_p$ is the basis of the new space $C_p^r(w)$. Therefore, we have the following:

**Theorem 2.4.** *Let $0 < p < 1$. Define the sequence $b^{(n)} = \left\{ b_k^{(n)} \right\}_{k=0}^{\infty}$ of elements of the weighted sequence space $C_p^r(w)$ by*

$$b_k^{(n)} = \begin{cases} -k\,(1+r) & n = k-1, \\ k+1 & n = k, \\ 0 & 0.w. \end{cases}$$

*for every fixed $n \in \mathbb{N} \cup \{0\}$ . Then the sequence $\left\{b^{(n)}\right\}_{n=0}^{\infty}$ is a basis for the space $C_p^r(w)$, and any $x \in C_p^r(w)$ has a unique representation of the form*

$$x = \sum_{n=0}^{\infty} (C^r x)_n b^{(n)}.$$

*Proof.* This is a routine verification and so we omit the detail. □

In the rest of this paper we consider matrix operators from the space $\ell_p(w)$ into the weighted sequence space $C_p^r(w)$ and try to calculate their lower bounds.

## 3. Lower bound for the transpose of lower triangular matrices

Let $A = (a_{n,k})_{n,k\geq 0}$ be a non-negative infinite matrix and define $T : \ell_p(w) \longrightarrow C_p^r(w)$ represented by $[T]_{\mathcal{A},\mathcal{B}} = A$, where $\mathcal{A}$ and $\mathcal{B}$ are the standard bases of $\ell_p(w)$ and $C_p^r(w)$, respectively. In this section, we focus on the evaluation of $L_{\ell_p(w),C_q^r(w)}\left(A^t\right)$, where $0 < q \leq p < 1$ and $A$ is a non-negative lower triangular matrix. Our result gives a lower estimate for this value in terms of the constant $M$ which is defined by Chen and Wang in [4], as:

$$a_{n,k} \leq M a_{n,j}, \qquad (0 < k \leq j \leq n). \tag{3.1}$$

Here $M \geq 1$. We shall assume that $M$ is the smallest value appeared in (3.1). If (3.1) fails, we set $M = \infty$. In continue, we apply our result to the weighted mean matrices, $\left(A_{w'}^{WM}\right) = (a_{n,k})_{n,k\geq 0}$, and the Nörlund matrices, $\left(A_{w'}^{NM}\right) = (b_{n,k})_{n,k\geq 0}$, for some cases, where the weighted mean matrices and the Nörlund matrices are defined as below:

$$a_{n,k} = \begin{cases} \frac{w'_k}{W'_n} & 0 \leq k \leq n \\ 0 & \text{otherwise,} \end{cases}$$

and

$$b_{n,k} = \begin{cases} \frac{w'_{n-k}}{W'_n} & 0 \leq k \leq n \\ 0 & \text{otherwise.} \end{cases}$$

Here $W'_n = \sum_{k=0}^{n} w'_k$ and $w' = (w'_n)$ is a non negative sequence with $w'_0 > 0$. The details are given below.

**Theorem 3.1.** *Let $0 < q \leq p < 1$ and $A = (a_{n,k})_{n,k\geq 0}$ be lower triangular matrix with non-negative entries. Then*

$$L_{\ell_p(w),C_q^r(w)}\left(A^t\right) \geq q^2 M^{q-1} \left(\inf_{j\geq 0} \sum_{k=0}^{j} a_{j,k}\right).$$

*Here $M$ is defined by (3.1).*

*Proof.* Let $x \geq 0$ with $||x||_{\ell_p(w)} = 1$. Since $q - 1 < 0$, it follows from Hölder's inequality, Lemma 3.13 of [3] and Fubini's theorem with the monotonicity of the weighted sequence $w$, that

$$
\begin{aligned}
\|A^t x\|^q_{C^r_q(w)} &= \sum_{k=0}^{\infty} w_k \left( \frac{1}{(1+r)^k} \sum_{n=k}^{\infty} \frac{(1+r)^n}{1+n} \sum_{i=n}^{\infty} a_{i,n} x_i \right)^q \\
&\overset{0<r<1}{\geq} \sum_{k=0}^{\infty} w_k \left( \sum_{n=k}^{\infty} \frac{1}{1+n} \sum_{i=n}^{\infty} a_{i,n} x_i \right)^q \\
&\overset{Lem.\ 2.3}{\geq} q^q \sum_{k=0}^{\infty} w_k \left( \sum_{n=k}^{\infty} a_{n,k} x_n \right)^q \\
&\geq q^{q+1} \sum_{k=0}^{\infty} w_k \sum_{j=k}^{\infty} a_{j,k} x_j \left( \sum_{n=j}^{\infty} a_{n,k} x_n \right)^{q-1} \\
&\geq q^{q+1} M^{q-1} \sum_{k=0}^{\infty} w_k \sum_{j=k}^{\infty} a_{j,k} x_j \left( \sum_{n=j}^{\infty} a_{n,j} x_n \right)^{q-1} \\
&\geq q^{q+1} M^{q-1} \sum_{j=0}^{\infty} w_j x_j \left( \sum_{n=j}^{\infty} a_{n,j} x_n \right)^{q-1} \left( \sum_{k=0}^{j} a_{j,k} \right) \\
&\geq q^{q+1} M^{q-1} \left( \inf_{j\geq 0} \sum_{k=0}^{j} a_{j,k} \right) \|x\|_{\ell_q(w)} \left\|A^t x\right\|^{q-1}_{\ell_q(w)} . \qquad (3.2)
\end{aligned}
$$

where the last inequality is based on Hölder's inequality. On the other hand, since $q-1<0$, we have

$$
\left\|A^t x\right\|^{q-1}_{\ell_q(w)} \geq \left(\frac{1}{q}\right)^{q-1} \left\|A^t x\right\|^{q-1}_{C^r_q(w)}
$$

Inserting this estimate into the corresponding term in (3.2) gives

$$
\left\|A^t x\right\|_{C^r_q(w)} \geq q^2 M^{q-1} \left( \inf_{j\geq 0} \sum_{k=0}^{j} a_{j,k} \right) \|x\|_{\ell_q(w)}.
$$

Also, since $0<q\leq p<1$, we have $\|x\|_{\ell_q(w)} \geq \|x\|_{\ell_p(w)} = 1$. Therefore

$$
\left\|A^t x\right\|_{C^r_q(w)} \geq q^2 M^{q-1} \left( \inf_{j\geq 0} \sum_{k=0}^{j} a_{j,k} \right).
$$

This leads us to the lower estimate in Theorem 3.1 and completes the proof. □

In the following we state some application of Theorem 3.1. First, consider the non-negative matrix $A$ for which $a_{n,k} \leq a_{n,k+1} (0 \leq k < n)$. Then Equation (3.1) with $M=1$, is satisfied. Applying Theorem 3.1, we get the following results.

**Corollary 3.2.** *Let $0 < q \le p < 1$, and $A$ be a lower triangular matrix with non-negative entries. If $a_{n,k} \le a_{n,k+1}$ for $0 \le k < n$, then*

$$L_{\ell_p(w),C_q^r(w)}\left(A^t\right) \ge q^2 \left(\inf_{j\ge 0} \sum_{k=0}^{j} a_{j,k}\right). \tag{3.3}$$

For row stochastic matrix(A non-negative square matrix for which the sum of all rows are 1), (3.3) takes the form

$$L_{\ell_p(w),C_q^r(w)}\left(A^t\right) \ge q^2.$$

We also have the following corollaries for the Nörlund matrices and the weighted mean matrices.

**Corollary 3.3.** *Let $0 < q \le p < 1$. If $w'_0 > 0$ and $w'_n \uparrow$, then*

$$L_{\ell_p(w),C_q^r(w)}\left(\left(A_{w'}^{WM}\right)^t\right) \ge q^2. \tag{3.4}$$

**Corollary 3.4.** *Let $0 < q \le p < 1$. If $w'_0 > 0$ and $w'_n \downarrow$, then*

$$L_{\ell_p(w),C_q^r(w)}\left(\left(A_{w'}^{NM}\right)^t\right) \ge q^2.$$

Consider the weighted mean matrix $\left(A_{w'}^{WM}\right)$, associated with the sequence $w' = (w'_n)$, where $w' \uparrow$ and $\inf\limits_{k} \frac{w'_k}{w'_0+\ldots+w'_k} > q$. For this matrix, by the same argument as in ([10], p. 2415), one can prove that the exact value of $L_{\ell_p(w),C_q^r(w)}\left(\left(A_{w'}^{WM}\right)^t\right)$ is greater than the lower estimate in (3.4). In fact

$$L_{\ell_p(w),C_q^r(w)}\left(\left(A_{w'}^{WM}\right)^t\right) > q^2.$$

Also, for the Nörlund matrix $\left(A_{w'}^{NM}\right)$, associated with the sequence $w' = (w'_n)$ with $w' \downarrow$ and $\inf\limits_{k} \frac{w'_0}{w'_0+\ldots+w'_k} > q$, we have

$$L_{\ell_p(w),C_q^r(w)}\left(\left(A_{w'}^{NM}\right)^t\right) > q^2.$$

Next, consider the weighted mean matrix $\left(A_{w'\ell}^{WM}\right)$ associated with the sequence $W' = (w'_n)_{n=0}^{\infty}$, where $l = 0, 1, 2, \cdots$, $w'_0 = w'_1 = \cdots = w'_l = 1$ and $w'_n = \frac{1}{2}$ for $n > l$. For this matrix Equation (3.1) with $M = 2$, is satisfied. By Theorem 3.1 we have

$$L_{\ell_p(w),C_q^r(w)}\left(\left(A_{w'\ell}^{WM}\right)^t\right) > q^2 2^{q-1}.$$

Applying Theorem 3.1 to a row stochastic matrix $A$ with $a_{n,k} \ge a_{n,k+1}(0 \le k < n)$, we have

$$L_{\ell_p(w),C_q^r(w)}\left(A^t\right) \ge q^2 M^{q-1}.$$

The cases $M = (\frac{\alpha}{w'_0})$ and $M = (\frac{w'_0}{\alpha})$ of the above statement give the following analogue of ([10], Corollaries 2.6 and 2.7), respectively.

**Corollary 3.5.** *Let $0 < q \leq p < 1$. If $w'_n \downarrow \alpha$ for $\alpha \geq 0$, then*

$$L_{\ell_p(w),C_q^r(w)}\left(\left(A_{w'}^{WM}\right)^t\right) \geq q^2\left(\frac{w'_0}{\alpha}\right)^{q-1}.$$

**Corollary 3.6.** *Let $0 < q \leq p < 1$. If $w'_0 > 0$, $w'_n \uparrow \alpha$, then*

$$L_{\ell_p(w),C_q^r(w)}\left(\left(A_{w'}^{NM}\right)^t\right) \geq q^2\left(\frac{\alpha}{w'_0}\right)^{q-1}.$$

## 4. Generalized Hausdorff matrices

Let $\alpha \geq 0$ and $d\mu$ is a Borel probability measure on [0,1]. The generalized Hausdorff matrix associated with $d\mu$, $H_\mu^\alpha = (h_{n,k}^\alpha)_{n,k\geq 0}$ is defined by

$$h_{n,k}^\alpha = \begin{cases} \begin{pmatrix} n+\alpha \\ n-k \end{pmatrix} \int_0^1 \theta^{k+\alpha}(1-\theta)^{n-k} d\mu(\theta) & n \geq k, \\ 0 & n < k. \end{cases}$$

Clearly $h_{n,k}^\alpha = \binom{n+\alpha}{n-k} \Delta^{n-k}\mu_k$ for $n \geq k \geq 0$, where

$$\mu_k = \int_0^1 \theta^{k+\alpha} d\mu(\theta) \qquad (k = 0, 1, 2, ...),$$

and $\Delta\mu_k = \mu_k - \mu_{k+1}$.
The generalized Hausdorff matrix contains some famous classes of matrices. These classes are as follows:

(1) The choice $d\mu(\theta) = \beta(1-\theta)^{\beta-1}d\theta$ gives rise the generalized Cesàro matrix of order $\beta$;
(2) The choice $d\mu(\theta) = |\log\theta|^{\beta-1}/\Gamma(\beta)d\theta$ gives the generalized Hölder matrix of order $\beta$;
(3) The choice $d\mu(\theta) = \beta\theta^{\beta-1}d\theta$ gives the generalized Gamma matrix of order $\beta$.
(4) The choice $d\mu(\theta) =$ point evaluation at $\theta = \beta$ gives rise to the generalized Euler matrix of order $\beta$;

The generalized Cesàro, generalized Hölder and generalized Gamma matrices have non-negative entries whenever $\beta > 0$, and also does the Euler matrices when $0 < \beta \leq 1$.

In this section we will consider the generalized Hausdorff matrix as an operator from $\ell_p$ into the sequence space $C_q^r$ where $0 < q \leq p \leq 1$. Afterwards we focus on the evaluation of $L_{\ell_p,C_q^r}\left(H_\mu^\alpha\right)$. A Hardy type formula is exhibited as a lower estimate. A similar result is also established for the case in which $H_\mu^\alpha$ is replaced by $\left(H_\mu^\alpha\right)^t$. As a consequence, we apply our results to the generalized Gamma, generalized Hölder and generalized Cesàro matrices which were recently considered in [5] on the $\ell_p$ spaces and in [7] and [10] on the block weighted sequence space $\ell_p(w,F)$ and on the sequential *weak* $\ell_p$, respectively.

First, we state and prove the following statement about below boundedness of lower triangular matrix operators which has essential role in this section.

**Lemma 4.1.** *Let $0 < q \leq p < 1$ and let $A = (a_{n,k})_{n,k\geq 0}$ be a lower triangular matrix with non-negative entries. If*

$$\sup_{n\geq 0}\sum_{k=0}^{n} a_{n,k} = R, \qquad \textit{and} \qquad \inf_{k\geq 0}\sum_{n=k}^{\infty} a_{n,k} = C > 0,$$

*then*

$$L_{\ell_p,C_q^r}(A) \geq qC^{\frac{1}{q}}R^{\frac{q-1}{q}}.$$

*Proof.* Suppose that $x$ is a non-negative sequence. By the same reason as we have seen in ([10], Lemma 3.1), we have

$$\sum_{k=0}^{n} a_{n,k}x_k^q \leq R^{1-q}\left(\sum_{k=0}^{n} a_{n,k}x_k\right)^q.$$

Since $A$ is a non-negative lower triangular matrix, using Lemma 2.3, we have

$$\begin{aligned}
R^{1-q}\sum_{n=0}^{\infty}&\left(\tfrac{1}{(1+r)^n}\sum_{k=n}^{\infty}\tfrac{(1+r)^k}{1+k}\sum_{j=0}^{k} a_{k,j}x_j\right)^q\\
&\overset{0<r<1}{\geq} R^{1-q}\sum_{n=0}^{\infty}\left(\sum_{k=n}^{\infty}\tfrac{1}{1+k}\sum_{j=0}^{k} a_{k,j}x_j\right)^q\\
&\geq R^{1-q}q^q\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_{n,k}x_k\right)^q\\
&\geq q^q\sum_{n=0}^{\infty}\sum_{k=0}^{n} a_{n,k}x_k^q = q^q\sum_{k=0}^{\infty} x_k^q\left(\sum_{n=k}^{\infty} a_{n,k}\right)\\
&\geq q^qC\sum_{k=0}^{\infty} x_k^q \geq q^qC\left\|x\right\|_{\ell_p}^q,
\end{aligned}$$

which implies $\|Ax\|_{C_q^r} \geq qC^{\frac{1}{q}}R^{\frac{q-1}{q}}\|x\|_{\ell_p}$, and this leads us to the desired inequality. ☐

Using the above lemma enables us to calculate the lower bound of the generalized Euler matrix which is essential in the rest of this section.

**Lemma 4.2.** *Let $\alpha \geq 0$ and $E^\alpha(\beta) = \left(e^\alpha_{n,k}(\beta)\right)_{n,k\geq 0}$ be the generalized Euler matrix of order $\beta$ where $0 < \beta \leq 1$. Then*

$$L_{\ell_p,C_q^r}(E^\alpha(\beta)) \geq q\beta^{-\frac{1}{q}} \qquad (0 < q \leq p < 1),$$

$$L_{\ell_1,C_q^r}(E^\alpha(\beta)) \geq \tfrac{q}{\beta} \qquad (0 < q \leq 1).$$

*Proof.* Let $0 < \beta \leq 1$. The column sums of $E^\alpha(\beta)$ are all equal to $\beta^{-1}$. Also, as we have seen in ([10], Lemma 3.2), $\sup_i\sum_{j=0}^{\infty} e^\alpha_{i,j}(\beta) = 1$. Thus, for $0 < p < 1$, applying Lemma 4.1 to case that $R = 1$ and $C = \beta^{-1}$, we deduce that

$L_{\ell_p,C_q^r}\left(E^{\alpha}(\beta)\right) \geq q\beta^{-\frac{1}{q}}$. For $p=1$, using Hölder's inequality and Fubini's theorem, we have

$$\begin{aligned}
\|E^{\alpha}(\beta)x\|_{C_q^r} &= \left\{\sum_{k=0}^{\infty}\left(\frac{1}{(1+r)^k}\sum_{n=k}^{\infty}\frac{(1+r)^n}{1+n}\sum_{j=0}^{\infty}e_{n,j}^{\alpha}(\beta)\,x_j\right)^q\right\}^{1/q}\\
&\overset{0<r<1}{\geq} \left\{\sum_{k=0}^{\infty}\left(\sum_{n=k}^{\infty}\frac{1}{1+n}\sum_{j=0}^{\infty}e_{n,j}^{\alpha}(\beta)\,x_j\right)^q\right\}^{1/q}\\
&\overset{Lem.\ 2.3}{\geq} q\left\{\sum_{k=0}^{\infty}\left(\sum_{n=0}^{\infty}e_{k,n}^{\alpha}(\beta)\,x_n\right)^q\right\}^{1/q}\\
&\geq q\sum_{k=0}^{\infty}\left(\sum_{n=0}^{\infty}e_{k,n}^{\alpha}(\beta)\,x_n\right)\\
&= q\sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty}e_{k,n}^{\alpha}(\beta)\right)x_n = q\beta^{-1}\|x\|_{\ell_1},
\end{aligned}$$

which gives the desired inequality and completes the proof. □

For $x \geq 0$, we have $H_{\mu}^{\alpha}x = \int_0^1 E^{\alpha}(\beta)x\,d\mu(\beta)$. Hence Lemma 4.2 enables us to estimate the value of $L_{\ell_p,C_q^r}(H_{\mu}^{\alpha})$. The details are given below.

**Theorem 4.3.** *For $\alpha \geq 0$ we have*

$$L_{\ell_p,C_q^r}\left(H_{\mu}^{\alpha}\right) \geq q\int_{(0,1]}\beta^{-1/q}d\mu(\beta), \qquad (0<q\leq p<1). \tag{4.1}$$

*and*

$$L_{\ell_1,C_q^r}\left(H_{\mu}^{\alpha}\right) \geq q\int_{(0,1]}\frac{d\mu(\beta)}{\beta}, \qquad (0<q\leq 1). \tag{4.2}$$

*Proof.* Consider (4.1). Let $x \geq 0$ with $\|x\|_{\ell_p}=1$. Applying Minkowski's inequality and Lemma 4.2, we have

$$\begin{aligned}
\left\|H_{\mu}^{\alpha}x\right\|_{C_q^r} &= \left\|\int_{(0,1]}E^{\alpha}(\beta)x\,d\mu(\beta)\right\|_{C_q^r} \geq \int_{(0,1]}\|E^{\alpha}(\beta)x\|_{C_q^r}d\mu(\beta)\\
&\geq \left(q\int_{(0,1]}\beta^{-1/q}d\mu(\beta)\right)\|x\|_{\ell_p}\\
&= q\int_{(0,1]}\beta^{-1/q}d\mu(\beta).
\end{aligned}$$

This leads us to (4.1). The proof of (4.2) is similar. □

In the next corollary we apply Theorem 4.3 to some special cases of generalized Hausdorff matrices such as generalized Gamma, generalized Hölder and generalized Cesàro matrices. Its proof can be easily adopted from the one of Corollaries 3.4, 3.5 and 3.6 of [10].

**Corollary 4.4.** *Let $\beta, \alpha > 0$ and $0 < q \leq p \leq 1$. Then*

(1) $L_{\ell_p, C_q^r}(\Gamma^\alpha(\beta)) = \infty, \qquad (\beta \leq \frac{1}{q})$.

(2) $L_{\ell_p, C_q^r}(\Gamma^\alpha(\beta)) \geq \frac{q\beta}{\beta - \frac{1}{q}}, \qquad (\beta \geq \frac{1}{q}, p \neq 1)$.

(3) $L_{\ell_1, C_q^r}(\Gamma^\alpha(\beta)) \geq \frac{\beta}{\beta - \frac{1}{q}}, \qquad (\beta \geq \frac{1}{q})$.

(4) $L_{\ell_p, C_q^r}(C^\alpha(\beta)) = \infty$.

(5) $L_{\ell_p, C_q^r}(H^\alpha(\beta)) = \infty$.

In the rest of our paper we consider the transpose of the generalized Hausdorff matrix, $\left(H_\mu^\alpha\right)^t$, as an operator from $\ell_p$ into the sequence space $C_q^r$ where $0 < q \leq p \leq 1$. First, we state and prove the following statement about below boundedness of lower triangular matrices with non-negative entries which has essential role in rest of this section.

**Lemma 4.5.** *Let $0 < q \leq p \leq 1$ and let $A = (a_{n,k})_{n,k\geq 0}$ be a upper triangular matrix with non-negative entries. If*

$$\sup_{n\geq 0} \sum_{k=0}^{\infty} a_{n,k} = R > 0, \qquad \text{and} \qquad \inf_{k\geq 0} \sum_{n=0}^{k} a_{n,k} = C,$$

*then*

$$L_{\ell_p, C_q^r}(A) \geq q C^{1/q} R^{\frac{q-1}{q}}.$$

*Proof.* Suppose that $x$ is a non-negative sequence. By the same reason as we have seen in ([10], Lemma 3.7), we have

$$\sum_{k=n}^{\infty} a_{n,k} x_k^q \leq R^{1-q} \left( \sum_{k=n}^{\infty} a_{n,k} x_k \right)^q.$$

Since $A$ is a non-negative upper triangular matrix, using Lemma 2.3, we have

$$R^{1-q} \sum_{n=0}^{\infty} \left( \frac{1}{(1+r)^n} \sum_{k=n}^{\infty} \frac{(1+r)^k}{1+k} \sum_{j=k}^{\infty} a_{k,j} x_j \right)^q$$

$$\overset{0<r<1}{\geq} \quad R^{1-q} \sum_{n=0}^{\infty} \left( \sum_{k=n}^{\infty} \frac{1}{1+k} \sum_{j=k}^{\infty} a_{k,j} x_j \right)^q$$

$$
\begin{aligned}
&\geq \quad R^{1-q}q^q \sum_{n=0}^{\infty}\left(\sum_{k=n}^{\infty} a_{n,k}x_k\right)^q \\
&\geq \quad q^q \sum_{n=0}^{\infty}\sum_{k=n}^{\infty} a_{n,k}x_k^q \;=\; q^q \sum_{k=0}^{\infty} x_k^q \left(\sum_{n=0}^{k} a_{n,k}\right) \\
&\geq \quad q^q C \sum_{k=0}^{\infty} x_k^q \;\geq\; q^q C \left\|x\right\|_{\ell_p}^q ,
\end{aligned}
$$

which implies $\|Ax\|_{C_q^r} \geq qC^{1/q}R^{\frac{q-1}{q}}\|x\|_{\ell_p}$, and this leads us to the desired inequality. ☐

**Corollary 4.6.** *Let $0 < q \leq p \leq 1$ and $E^\alpha(\beta)$ be the generalized Euler matrix of order $\beta$ with $0 < \beta \leq 1$ and $\alpha \geq 0$. Then*

$$L_{\ell_p,C_q^r}\left((E^\alpha(\beta))^t\right) \geq q\beta^{\frac{1-q}{q}}.$$

*Proof.* The proof is similar to that of Lemma 4.2. The only difference is that we use Lemma 4.5 instead of Lemma 4.1. ☐

Now, we come to the evaluation of $L_{\ell_p,C_q^r}\left(\left(H_\mu^\alpha\right)^t\right)$.

**Theorem 4.7.** *Let $0 < q \leq p \leq 1$. Then for $\alpha \geq 0$ we have*

$$L_{\ell_p,C_q^r}\left(\left(H_\mu^\alpha\right)^t\right) \geq q \int_{(0,1]} \beta^{\frac{1-q}{q}} d\mu(\beta). \tag{4.3}$$

*Proof.* Let $x \geq 0$ and $||x||_{\ell_p} = 1$. Since $\left(H_\mu^\alpha\right)^t x = \int_0^1 (E^\alpha(\beta))^t x d\mu(\beta)$, applying Minkowski's inequality and Corollary 4.6, we have

$$
\begin{aligned}
\left\|\left(H_\mu^\alpha\right)^t x\right\|_{C_q^r} &= \left\|\int_0^1 (E^\alpha(\beta))^t x d\mu(\beta)\right\|_{C_q^r} \\
&\geq \int_0^1 \left\|(E^\alpha(\beta))^t x\right\|_{C_q^r} d\mu(\beta) \\
&\geq \left(q \int_{(0,1]} \beta^{\frac{1-q}{q}} d\mu(\beta)\right) \|x\|_{\ell_p} \\
&= q \int_{(0,1]} \beta^{\frac{1-q}{q}} d\mu(\beta).
\end{aligned}
$$

This leads us to (4.3) and completes the proof. ☐

Applying Theorem 4.7 to the special cases of generalized Hausdorff matrices we have the following corollary.

**Corollary 4.8.** *Let $0 < q \leq p \leq 1$. Then for $\beta > 0$, we have*

(1) $L_{\ell_p,C_q^r}\left((C^\alpha(\beta))^t\right) \geq q\frac{\Gamma(\beta+1)\Gamma(\frac{1}{q})}{\Gamma(\beta+\frac{1}{q})}$.

(2) $L_{\ell_p,C_q^r}\left((H^\alpha(\beta))^t\right) \geq \frac{q}{\Gamma(\beta)}\int_0^1 \theta^{\frac{1-q}{q}}|\log\theta|^{\beta-1}d\theta.$

(3) $L_{\ell_p,C_q^r}\left((\Gamma^\alpha(\beta))^t\right) \geq \frac{\beta q^2}{\beta q-q+1}.$

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