NOTE ON $(m, q)$-ISOMETRIES ON AN HYPERSPACE OF A NORMED SPACE

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Abstract. Given a normed space $X$ we consider the hyperspace $k(X)$ of all non-empty compact convex subsets of $X$ endowed with the Hausdorff distance. We prove that if $T : X \rightarrow X$ is an $(m, q)$-isometry, then it is possible that the map $k(T) : k(X) \rightarrow k(X)$, $k(T)C := TC$, is not an $(m, q)$-isometry. Moreover, if $k(X)$ is the Rådström space associated to the hyperspace $k(X)$, then $T : k(X) \rightarrow k(X)$ is an $(m, q)$-isometry if and only if $\overline{T} : k(X) \rightarrow k(X)$ is an $(m, q)$-isometry.

1. Introduction

Throughout this paper, $X$ is a real normed space and $\| \cdot \|$ its norm, $L(X)$ the class of all bounded linear operators $T : X \rightarrow X$, $m$ a positive integer and $q$ a positive real number, unless stated otherwise.

The notion of $(m, q)$-isometry in the setting of metric spaces was introduced in [3]: a map $T : E \rightarrow E$, on a metric space $E$ with distance $d$, is called an $(m, q)$-isometry if

$$\sum_{i=0}^{m} (-1)^{m-i} \binom{m}{i} d(T^i x, T^i y)^q = 0 \quad (x, y \in E).$$

(1.1)

An $(m, q)$-isometry is called strict whenever it is not an $(m - 1, q)$-isometry. Of course, the $(1, q)$-isometries are the isometries. This definition generalizes the concept of $m$-isometry firstly introduced on Hilbert spaces by J. Agler [1]. Some time after the notion of $(m, q)$-isometry on Banach spaces was defined by Bayart [2] and Sid Ahmed [7].

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In [4] it was introduced a notion of \( m \)-isometry on certain hyperspaces of a Banach space. In this paper we study \( (m, q) \)-isometries on the hyperspace \( k(X) \) of all non-empty convex compact subsets of a normed space \( X \). Given an operator \( T \in L(X) \) we consider the map \( k(T) : k(X) \to k(X) \), defined by \( k(T)C := TC \). It is possible that \( T \) is an \( (m, q) \)-isometry but \( k(T) \) is not an \( (m, q) \)-isometry. More precisely, we prove that any weighted shift operator \( S_w \in L(\ell_2) \) which is a \((2, 2)\)-isometry induces a map \( k(S_w) : k(\ell_2) \to k(\ell_2) \) which is not an \((2, 2)\)-isometry.

Using a construction by Råström we associate to \( k(X) \) the normed space \( \hat{k}(X) \), being \( k(X) \) a subspace of \( \hat{k}(X) \). We prove that \( T : k(X) \to k(X) \) is an \( (m, q) \)-isometry if and only if \( \hat{T} : \hat{k}(X) \to \hat{k}(X) \) is an \((m, q)\)-isometry.

2. The hyperspace \( k(X) \)

Given a real normed space \( X \), we consider the hyperspace
\[
k(X) := \{ C \subset X : \emptyset \neq C \text{ compact convex} \}.
\]
For \( C, D \in k(X) \) and \( \alpha \) scalar, we write \( C + D := \{ x + y : x \in C, y \in D \} \) and \( \alpha C := \{ \alpha x : x \in C \} \). Some properties of the class \( k(X) \) are given in the following proposition:

**Proposition 2.1.** For \( C, D, E \in k(X) \), \( \lambda, \mu \geq 0 \) and \( \alpha \) scalar,

1. \( C + D \in k(X) \)
2. \( (C + D) + E = C + (D + E) \) and \( C + D = D + C \)
3. \( C + E = D + E \implies C = D \)
4. \( \alpha C \in k(X) \)
5. \( \alpha(C + D) = \alpha C + \alpha D \) and \( (\lambda + \mu)C = \lambda C + \mu C \)

**Proof.** The property (3) is [6, Lemma 2]. The other properties are simple. \( \square \)

We introduce the norm of \( C \in k(X) \):
\[
\|C\| := \sup_{x \in C} \|x\|.
\]

**Proposition 2.2.** For \( C, D \in k(X) \) and \( \alpha \) scalar,

1. \( \|C\| = 0 \iff C = \{0\} \)
2. \( \|C + D\| \leq \|C\| + \|D\| \)
3. \( \|\alpha C\| = |\alpha|\|C\| \)

**Proof.** Routine. \( \square \)

The class \( k(X) \) is endowed with the Hausdorff distance \( h \): given \( C, D \in k(X) \), we put
\[
h(C, D) := \inf\{\varepsilon > 0 : C \subset D + \varepsilon B_X \text{ and } D \subset C + \varepsilon B_X \},
\]
where \( B_X \) is the unit closed ball of \( X \). In the next result we collect some basic facts about the distance \( h \).

**Proposition 2.3.** For \( C, D, E \in k(X) \) and \( \alpha \) scalar,
(1) \( h \) is a metric on \( k(X) \); moreover, if \( X \) is a Banach space, then \( k(X) \) is complete.
(2) \( h(C + E, D + E) = h(C, D) \)
(3) \( h(\alpha C, \alpha D) = |\alpha| h(C, D) \)
(4) \( h(C, \{0\}) = \|C\| \)

Proof. The property (1) is well known and (4) is clear. In order to prove (2), notice that, for every \( \varepsilon > 0 \), we can write
\[
h(C + E, D + E) < \varepsilon \implies C + E \subset D + E + \varepsilon B_X \text{ and } D + E \subset C + E + \varepsilon B_X
\]
\[
\implies C \subset D + \varepsilon B_X \text{ and } D \subset C + \varepsilon B_X
\]
\[
\implies h(C, D) \leq \varepsilon,
\]
by Proposition 2.1 (3). Analogously, \( h(C, D) < \varepsilon \implies h(C + E, D + E) \leq \varepsilon \).
Therefore, (2) is true.

Now we prove (3). We have that the equality is obvious if \( \alpha = 0 \). Assume \( \alpha \neq 0 \). Then
\[
h(\alpha C, \alpha D) < \varepsilon \implies \alpha C \subset \alpha D + \varepsilon B_X \text{ and } \alpha D \subset \alpha C + \varepsilon B_X
\]
\[
\implies C \subset D + \alpha^{-1} \varepsilon B_X = D + |\alpha|^{-1} \varepsilon B_X \text{ and } D \subset C + \alpha^{-1} \varepsilon B_X = C + |\alpha|^{-1} \varepsilon B_X
\]
\[
\implies h(C, D) \leq |\alpha|^{-1} \varepsilon
\]
\[
\implies |\alpha| h(C, D) \leq \varepsilon.
\]
Analogously, \( |\alpha| h(C, D) < \varepsilon \implies h(\alpha C, \alpha D) \leq \varepsilon \). Consequently, (3) holds. \( \square \)

Observe that the property (2) in the above proposition depends on the fact that \( E \) is bounded and that both sets \( C + \varepsilon B_X \) and \( D + \varepsilon B_X \) are convex closed, since \( C \) and \( D \) are convex compact (see [6, Lemmas 2 and 3]).

It is obvious that we can identify \( X \) with \( \{\{x\} : x \in X\} \subset k(X) \). For \( x, y \in X \) and \( \alpha \) scalar we have that \( \{x\} + \{y\} = \{x + y\} \), \( \alpha \{x\} = \{\alpha x\} \) and \( h(\{x\}, \{y\}) = \|x - y\| \). Notice that, in general,
\[
h(C, D) \leq \|C - D\| \quad (C, D \in k(X))
\]
and it is possible that \( h(C, D) < \|C - D\| \). For example, \( h(C, C) = 0 < \|C - C\| \) whenever \( C \) is not a singleton.

3. Maps on \( k(X) \)

We say that a map \( T : k(X) \to k(X) \) is linear if, for \( C, D \in k(X) \) and \( \alpha \) scalar,
\[
T(C + D) = TC + TD \quad \text{and} \quad T(\alpha C) = \alpha TC.
\]
Given \( T : k(X) \to k(X) \) linear we define the norm of \( T \) by
\[
\|T\| = \sup_{\{0\} \neq C \in k(X)} \frac{\|TC\|}{\|C\|} = \sup_{C \in k(X), \|C\| = 1} \|TC\|.
\]
Hence, for every \( C \in k(X) \), we have that \( \|TC\| \leq \|T\| \|C\| \). We say that \( T \) is bounded if \( \|T\| < \infty \).
The following results are very similar to analogous facts about linear operators between normed spaces and we omit the proof.

**Proposition 3.1.** Let $T : k(X) \rightarrow k(X)$ a linear map. The following assertions are equivalent:

1. $T$ is uniformly continuous
2. $T$ is continuous
3. $T$ is continuous at $\{0\}$
4. There exists $M > 0$ such that, for every $C \in k(X)$, $\|TC\| \leq M\|C\|
5. $T$ is bounded

We denote by $L(k(X))$ the class of all bounded linear maps $T : k(X) \rightarrow k(X)$.

**Proposition 3.2.** For $T, S \in L(k(X))$ and scalar $\alpha$,

1. $T + S \in L(k(X))$ and $\|T + S\| \leq \|T\| + \|S\|
2. $\alpha T \in L(k(X))$ and $\|\alpha T\| = |\alpha|\|T\|
3. $TS \in L(k(X))$ and $\|TS\| \leq \|T\|\|S\|

**Proof.** Routine.

Given $T \in L(X)$ we define the map

$$k(T) : k(X) \rightarrow k(X), \quad k(T)C := TC.$$ 

Obviously, the restriction of $k(T)$ to $X$ is $T$: $k(T)\{x\} = T\{x\} = \{Tx\}$, for any $x \in X$.

**Proposition 3.3.** Let $T \in L(X)$. Then $k(T) \in L(k(X))$ and $\|k(T)\| = \|T\|$.

**Proof.** For $C \in k(X)$, we have that $\|TC\| \leq \|T\|\|C\|$, hence

$$\|k(T)\| = \sup_{\{0\} \neq C \in k(X)} \frac{\|k(T)C\|}{\|C\|} = \sup_{\{0\} \neq C \in k(X)} \frac{\|TC\|}{\|C\|} \leq \|T\|.$$ 

Moreover

$$\|T\| = \sup_{0 \neq x \in X} \frac{\|Tx\|}{\|x\|} \leq \sup_{\{0\} \neq C \in k(X)} \frac{\|TC\|}{\|C\|} = \|k(T)\|$$

and the proof is completed.

**Proposition 3.4.** Let $T \in L(X)$. Then $T$ is an isometry if and only if the map $k(T)$ is an isometry.

**Proof.** It is enough to observe that the equalities

$$\|k(T)C\| = \|C\| = \|TC\|$$

are equivalent to that both $k(T)$ and $T$ are isometries.

Our main interest is the study of $(m,q)$-isometries ($m \geq 1$ integer, $q > 0$ real) on the hyperspace $k(X)$. Recall that the general definition was given in (1.1). For $T : k(X) \rightarrow k(X)$ the condition (1.1) is equivalent to

$$\sum_{i=0}^{m} (-1)^{m-i} \binom{m}{i} h(T^iC, T^iD)^q = 0 \quad (C, D \in k(X)).$$ (3.1)
The equivalence given in Proposition 3.4 can not be extended to \((m, q)\)-isometries, although an implication is true.

**Proposition 3.5.** Let \(T \in L(X)\). If the map \(k(T)\) is an \((m, q)\)-isometry, then \(T\) is an \((m, q)\)-isometry.

**Proof.** It is enough to observe that any restriction of an \((m, q)\)-isometry to an invariant subset is also an \((m, q)\)-isometry and that \(T\) is the restriction of \(k(T)\) to \(X\) as explained before. \(\square\)

The converse of above proposition is false, as we show in the next example.

**Example 3.6.** Let \(S_w : \ell_2 \rightarrow \ell_2\) the weighted shift operator on \(\ell_2\) with weight sequence \(w = (w_n)_{n \geq 1} \in \ell_\infty\). That is, for \(x = (x_n)_{n \geq 1} \in \ell_2\),

\[
S_wx = S_w(x_1, x_2, x_3... ) = (0, w_1x_1, w_2x_2, w_3x_3...) .
\]

If \(S_w\) is a strict \((2, 2)\)-isometry, then \(k(S_w)\) is not a \((2, 2)\)-isometry.

**Proof.** We put \(\alpha := |w_1|^2\). Then, for \(n \geq 1\) [4, Remark 3.9(1)(b)]

\[
|w_n|^2 = \frac{\alpha n - (n - 1)}{\alpha (n - 1) - (n - 2)} ,
\]

hence

\[
|w_2|^2 = \frac{2\alpha - 1}{\alpha} \quad \text{and} \quad |w_4|^2 = \frac{3\alpha - 2}{2\alpha - 1} .
\]

We have that \(\alpha \neq 1\) since \(S_w\) is not an isometry, and \(\alpha > 1\) since \(S_w\) is a \((2, 2)\)-isometry ([4, Remark 3.9(1)(b)], [5, Corollary 2.3]).

Let \((e_n)_{n \geq 1}\) be the canonical basis of \(\ell_2\). Take \(x = e_1\) and \(y = \lambda e_2\), such that \(\lambda\) is a scalar with

\[
1 < |\lambda|^2 < \frac{\alpha^2}{2\alpha - 1} .
\]

We obtain

\[
\|x\|^2 = 1 , \; \|S_wx\|^2 = \alpha , \; \|S_w^2x\|^2 = 2\alpha - 1 ,
\]

\[
\|y\|^2 = |\lambda|^2 , \; \|S_wy\|^2 = |\lambda|^2 \frac{2\alpha - 1}{\alpha} , \; \|S_w^2y\|^2 = |\lambda|^2 \frac{3\alpha - 2}{\alpha} .
\]

Consider the segment

\[
C = [x, y] := \{tx + (1 - t)y : 0 \leq t \leq 1\} \in k(\ell_2) .
\]

Then

\[
\|C\|^2 = \sup_{0 \leq t \leq 1} \|tx + (1 - t)y\|^2
\]

\[
= \sup_{0 \leq t \leq 1} \|(t, (1 - t)\lambda, 0, 0, 0...)\|^2
\]

\[
= \sup_{0 \leq t \leq 1} (t^2 + (1 - t)^2|\lambda|^2)
\]

\[
= |\lambda|^2 ,
\]
since $1 < |\lambda|^2$. Moreover,
\[
\|S_w C\|^2 = \sup_{0 \leq t \leq 1} \|(0, w_1 t, w_2(1-t)\lambda, 0, 0, 0...\|^2 \\
= \sup_{0 \leq t \leq 1} (|w_1|^2 t^2 + |w_2|^2(1-t)^2|\lambda|^2) \\
= \sup_{0 \leq t \leq 1} (\alpha t^2 + \frac{2\alpha - 1}{\alpha}(1-t)^2|\lambda|^2) \\
= \alpha
\]
and
\[
\|S_w^2 C\|^2 = \sup_{0 \leq t \leq 1} \|(0, 0, w_1 w_2 t, w_2 w_3(1-t)\lambda, 0, 0, 0...\|^2 \\
= \sup_{0 \leq t \leq 1} (|w_1 w_2|^2 t^2 + |w_2 w_3|^2(1-t)^2|\lambda|^2) \\
= \sup_{0 \leq t \leq 1} ((2\alpha - 1)t^2 + \frac{3\alpha - 2}{\alpha}(1-t)^2|\lambda|^2) \\
= 2\alpha - 1.
\]
We have that
\[
h(k(S_w)^2C, k(S_w)^2\{0\})^2 - 2h(k(S_w)C, k(S_w)\{0\})^2 + h(C, \{0\})^2 = \\
= \|k(S_w)^2C\|^2 - 2\|k(S_w)C\|^2 + \|C\|^2 = 2\alpha - 1 - 2\alpha + |\lambda|^2 = |\lambda|^2 - 1 \neq 0,
\]
because of $1 < |\lambda|^2$. By (3.1) we obtain that $S_w$ is not a $(2,2)$-isometry. \hfill \Box

4. The Rådström space $\hat{k}(X)$

Rådström [6] proved that $k(X)$ endowed with the Hausdorff distance can be isometrically embedded in a normed space $\hat{k}(X)$ in such a way that addition in $\hat{k}(X)$ induces addition in $k(X)$ and multiplication by scalars in $\hat{k}(X)$ induces multiplication by scalars in $k(X)$.

Now we give a description of the Rådström space associated to the hyperspace $k(X)$ (see [6]). On $k(X) \times k(X)$ we consider the equivalence relation $(C, D) \sim (E, F) \iff C + F = D + E$, where $C, D, E, F \in k(X)$. The class of $(C, D)$ is denoted by $[C, D]$.

The quotient space
\[
\hat{k}(X) := \frac{k(X) \times k(X)}{\sim}
\]
is a normed space with the following: for $C, D, E, F \in k(X)$ and $\lambda \geq 0$ scalar,
\[
\|[C, D]\| = h(C, D), [C, D] + [E, F] = [C + E, D + F], \\
\lambda[C, D] = [\lambda C, \lambda D], (-\lambda)[C, D] = [\lambda D, \lambda C], .
\]
From this, the distance between two classes of $\hat{k}(X)$ is given by
\[
\hat{h}([C, D], [E, F]) = \|[C, D] - [E, F]\| = \|[C + F, D + E]\| = h(C + F, D + E).
\]
Moreover the map $\psi : k(X) \to \hat{k}(X)$ defined by $\psi C := [C, \{0\}]$, is an isometric embedding of $k(X)$ into $\hat{k}(X)$; in fact, we have that $\psi(C + D) = \psi(C) + \psi(D)$, $\psi(\lambda C) = \lambda \psi(C)$ and $\|\psi(C)\| = \|C\|.$

Given a map $\mathcal{T} : k(X) \to k(X)$, we define

$$\hat{T} : k(X) \to \hat{k}(X), \quad \hat{T}[C, D] := [\mathcal{T}C, \mathcal{T}D].$$

Notice that the restriction of $\hat{T}$ to $k(X)$ is $\mathcal{T}$.

**Proposition 4.1.** Let $\mathcal{T} : k(X) \to k(X)$ a linear map. Then

1. $\hat{T}$ is linear
2. $\mathcal{T}$ bounded $\implies \hat{T}$ bounded and $\|\hat{T}\| = \|\mathcal{T}\|.$

**Proof.** (1) Straightforward.

(2) As $\mathcal{T}$ is restriction of $\hat{T}$, we have that $\|\mathcal{T}\| \leq \|\hat{T}\|$. Now we show $\|\mathcal{T}\| \geq \|\hat{T}\|$. For this purpose, first we prove

$$h(\mathcal{T}C, \mathcal{T}D) \leq \|\mathcal{T}\| h(C, D) \quad (C, D \in k(X)). \quad (4.1)$$

Fix $C, D \in k(X)$. Let $\varepsilon > h(C, D)$. Then $C \subset D + \varepsilon B_X$ and $D \subset C + \varepsilon B_X$. Hence $\mathcal{T}C \subset \mathcal{T}D + \varepsilon \tilde{T}B_X$ and $\mathcal{T}D \subset \mathcal{T}C + \varepsilon \tilde{T}B_X$, where

$$\tilde{T}B_X := \bigcup_{b \in B_X} \mathcal{T}\{b\}.$$

(Observable that $\tilde{T}B_X$ is not always defined because of $B_X \notin k(X)$ if $X$ is infinite-dimensional). Notice that from $\mathcal{T}\{b\} \subset \|\mathcal{T}\||b||B_X \subset \|\mathcal{T}\|B_X$, we obtain $\tilde{T}B_X \subset \|\mathcal{T}\|B_X$ and consequently $\mathcal{T}C \subset \mathcal{T}D + \varepsilon \|\mathcal{T}\|B_X$ and $\mathcal{T}D \subset \mathcal{T}C + \varepsilon \|\mathcal{T}\|B_X$. Therefore $h(\mathcal{T}C, \mathcal{T}D) \leq \varepsilon \|\mathcal{T}\|$. Hence (4.1) follows. From this

$$\|\hat{T}\| = \sup_{\|\hat{T}[C, D]\| \leq 1} \|\hat{T}[C, D]\|$$

$$= \sup_{\|\mathcal{T}\| h(C, D) \leq 1} \||\mathcal{T}\| h(C, D)\|$$

$$\leq \sup_{\|\mathcal{T}\| h(C, D) \leq 1} \|\mathcal{T}\| h(C, D)$$

$$= \|\mathcal{T}\|. \quad \Box$$

**Proposition 4.2.** Let $\mathcal{T} \in L(k(X))$. The following assertions are equivalent:

1. $\mathcal{T}$ is a strict $(m, q)$-isometry
2. $\hat{T}$ is a strict $(m, q)$-isometry

**Proof.** For $C, D \in k(X)$ and $1 \leq k \leq m$, we have the following equalities

$$\|\hat{T}^k[C, D]\| = \|\mathcal{T}^k\mathcal{T}^k D\| = h(\mathcal{T}^k C, \mathcal{T}^k D).$$
Consequently, \( \mathcal{T} \) is an \((m, q)\)-isometry, that is it verifies (3.1), if and only if \( \hat{\mathcal{T}} \) verifies
\[
\sum_{i=0}^{m} (-1)^{m-i} \binom{m}{i} \| \hat{\mathcal{T}}^i[C,D] \|^q = 0 \quad (C, D \in k(X)) ;
\]
that is, \( \hat{\mathcal{T}} \) is an \((m, q)\)-isometry. From this, it is obvious that \( \mathcal{T} \) is a strict \((m, q)\)-isometry if and only if \( \hat{\mathcal{T}} \) is also a strict \((m, q)\)-isometry. \( \square \)

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