Annals of Functional Analysis
Ann. Funct. Anal. 6 (2015), no. 2, 91–103
http://doi.org/10.15352/afa/06-2-9
ISSN: 2008-8752 (electronic)
http://projecteuclid.org/afa

TRUNCATED MOMENT PROBLEMS FOR J-SELF-ADJOINT, J-SKEW-SELF-ADJOINT AND J-UNITARY OPERATORS

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Communicated by S. R. Garcia

Abstract. In this paper we study truncated moment problems for J-self-adjoint, J-skew-self-adjoint and J-unitary operators. Conditions of the solvability are given. Some canonical solutions of the moment problems are constructed. As a by-product, some extension results for J-skew-symmetric and J-isometric operators are obtained.

1. Introduction and preliminaries

During the past decade an increasing interest was devoted to the investigations of operators related to a conjugation in a Hilbert space, see, e.g. [3], [4], [12], [8] and references therein. A conjugation J in a Hilbert space H is an *antilinear* operator on H such that $J^2x = x$, $x \in H$, and

$$(Jx, Jy)_H = (y, x)_H, \qquad x, y \in H.$$

The conjugation J generates the following bilinear form:

$$[x, y]_J := (x, Jy)_H, \qquad x, y \in H.$$

A linear operator A in H is said to be J-symmetric (J-skew-symmetric) if

$$[Ax, y]_J = [x, Ay]_J, \qquad x, y \in D(A),$$

or, respectively,

$$[Ax, y]_J = -[x, Ay]_J, \qquad x, y \in D(A).$$

A linear operator A in H is said to be J-isometric if

$$[Ax, Ay]_J = [x, y]_J, \qquad x, y \in D(A).$$

Date: Received: Jun 18, 2014; Accepted: Jul. 10, 2014.
2010 *Mathematics Subject Classification.* Primary 47A57; Secondary 44A60.
Key words and phrases. Moment problem, conjugation, extensions of operators.

A linear operator A in H is called J-self-adjoint (J-skew-self-adjoint, or J-unitary) if

$$A = JA^* J,$$

or

$$A = -JA^* J,$$

or

$$A^{-1} = JA^* J,$$

respectively. In this paper we shall study the following three problems:

- **Problem A.** Given a finite set of complex numbers $\{s_{n,m}\}_{n,m=0}^d$, $d \in \mathbb{N}$. Find a J-self-adjoint operator A in a Hilbert space H, and an element $x_0 \in H$, such that
$$[A^n x_0, A^m x_0]_J = s_{n,m}, \qquad n, m = 0, 1, \cdots, d. \tag{1.1}$$
- **Problem B.** Given a finite set of complex numbers $\{s_{n,m}\}_{n,m=0}^d$, $d \in \mathbb{N}$. Find a J-skew-self-adjoint operator A in a Hilbert space H, and an element $x_0 \in H$, such that relation (1.1) holds.
- **Problem C.** Given a finite set of complex numbers $\{s_{n,m}\}_{n,m=0}^d$, $d \in \mathbb{N}$. Find a J-unitary operator A in a Hilbert space H, and an element $x_0 \in H$, such that relation (1.1) holds.

The problem A / B / C is said to be **the truncated moment problem for J-self-adjoint / J-skew-self-adjoint / J-unitary operators**, respectively. These moment problems are analogs of the well-known truncated Hamburger and trigonometric moment problems, which are usually formulated in terms of prescribed integrals of powers with respect to an unknown positive Borel measure, see, e.g. [1], [2], [11] and references therein. However, the operator statements for such problems, close to our definitions of problems A-C, are also known, see [5, pp. 411,413].

A solution of the moment problem A / B / C is said to be **canonical** if $\mathrm{Lin}\{A^n x_0\}_{n=0}^d = H$. A solution of the moment problem A / B / C is said to be **almost canonical** if $\mathrm{Lin}\{A^n x_0,\ JA^n x_0\}_{n=0}^d = H$. Our aim is to present conditions of the solvability for the problems A, B, C and to describe some of their canonical solutions. As a by-product, we shall obtain some extension results for J-skew-symmetric and J-isometric operators. It should be said that under some conditions a description of J-skew-self-adjoint extensions of a J-skew-symmetric operator was presented earlier, see [9] and references therein. A description of J-unitary extensions of a J-isometric operator, under some conditions which are different from our assumptions in this paper, was given in [7, Lemma 6].

Notations. As usual, we denote by $\mathbb{R}, \mathbb{C}, \mathbb{N}, \mathbb{Z}, \mathbb{Z}_+$, the sets of real numbers, complex numbers, positive integers, integers and non-negative integers, respectively. Let $m, n \in \mathbb{N}$. The set of all complex matrices of size $(m \times n)$ we denote by $\mathbb{C}_{m\times n}$. The set of all complex non-negative Hermitian matrices of size $(n \times n)$ we denote by $\mathbb{C}_{n\times n}^{\geq}$. If $M \in \mathbb{C}_{m\times n}$ then M^T denotes the transpose of M, and M^* denotes the complex conjugate of M. The identity matrix from $\mathbb{C}_{n\times n}$ we denote by I_n. By $\mathrm{Ker}\, M$ we denote the kernel of M, i.e. all $x \in \mathbb{C}_{n\times 1}$ such that $Mx = 0$.

By $\mathbb{C}^N$ we denote the finite-dimensional Hilbert space of complex column vectors of size N with the usual scalar product $(\vec{x},\vec{y})_{\mathbb{C}^N} = \sum_{j=0}^{N-1} x_j\overline{y_j}$, for $\vec{x},\vec{y}\in\mathbb{C}^N$, $\vec{x} = (x_0,x_1,\dots,x_{N-1})^T$, $\vec{y} = (y_0,y_1,\dots,y_{N-1})^T$, $x_j,y_j\in\mathbb{C}$; $N\in\mathbb{N}$.

If H is a Hilbert space then $(\cdot,\cdot)_H$ and $\|\cdot\|_H$ mean the scalar product and the norm in H, respectively. Indices may be omitted in obvious cases. For a linear operator A in H, we denote by $D(A)$ its domain, by $R(A)$ its range, and A^* means the adjoint operator if it exists. If A is invertible then A^{-1} means its inverse. $\overline{A}$ means the closure of the operator, if the operator is closable. If A is bounded then $\|A\|$ denotes its norm. For a set $M\subseteq H$ we denote by $\overline{M}$ the closure of M in the norm of H. For an arbitrary set of elements $\{x_n\}_{n\in I}$ in H, we denote by $\mathrm{Lin}\{x_n\}_{n\in I}$ the set of all linear combinations of elements x_n, and $\mathrm{span}\{x_n\}_{n\in I} := \overline{\mathrm{Lin}\{x_n\}_{n\in I}}$. Here I is an arbitrary set of indices. By E_H we denote the identity operator in H, i.e. $E_Hx = x$, $x\in H$. In obvious cases we may omit the index H. If H_1 is a subspace of H, then $P_{H_1} = P_{H_1}^H$ is an operator of the orthogonal projection on H_1 in H. By $[H_1,H_2]$ we denote a set of all bounded linear operators, which map a Hilbert space H_1 into a Hilbert space H_2.

2. A Hilbert space generated by a complex symmetric matrix.

We shall need the following theorem.

Theorem 2.1. *Let $\{s_{n,m}\}_{n,m=0}^d$ be a finite set of complex numbers, $d\in\mathbb{N}$. There exist a Hilbert space H, a conjugation J in H, and elements $\{x_n\}_{n=0}^d$ in H, such that*

$$[x_n,x_m]_J = s_{n,m},\qquad n,m = 0,1,\cdots,d;$$

if and only if the matrix $(s_{n,m})_{n,m=0}^d$ is complex symmetric.

If the last conditions are satisfied, the Hilbert space H may be chosen with $\dim H = d+1$.

Proof. The necessity follows directly from the property of the J-form: $[x,y]_J = [y,x]_J$.

Let us check the necessity. Suppose that the matrix $\Gamma := (s_{n,m})_{n,m=0}^d$ is complex symmetric. By a corollary from Takagi's factorization [6, Corollary 4.4.6], there exists a matrix $\Lambda = (a_{n,j})_{n,j=0}^d \in \mathbb{C}_{(d+1)\times(d+1)}$ such that

$$\Gamma = \Lambda\Lambda^T.$$

Set $H = \mathbb{C}^{d+1}$, $\vec{e}_n = (\delta_{n,0},\delta_{n,1},\dots,\delta_{n,d})^T$, $0\le n\le d$, and

$$J\sum_{k=0}^d \alpha_k\vec{e}_k = \sum_{k=0}^d \overline{\alpha_k}\vec{e}_k,\qquad \alpha_k\in\mathbb{C}.$$

Elements $\{x_n\}_{n=0}^d$ we define in the following way:

$$x_n = \sum_{j=0}^d a_{n,j}\vec{e}_j,\qquad 0\le j\le d.$$

Then

$$[x_n, x_m]_J = (x_n, Jx_m)_H = \sum_{j,k=0}^{d} a_{n,j}a_{m,k}\delta_{j,k} = \sum_{j=0}^{d} a_{n,j}a_{m,j} = s_{n,m},$$

for $0 \leq n, m \leq d$. □

Let $\Gamma = (s_{n,m})_{n,m=0}^d$ be a complex symmetric matrix. According to the proof of the last theorem, we see that the Gram matrix $G = ((x_n, x_m)_H)_{n,m=0}^d$ of the constructed sequence $\{x_n\}_{n=0}^d$ is equal to

$$G = \Lambda\Lambda^*.$$

If $\det\Gamma \neq 0$, then $\det\Lambda \neq 0$, $\det G \neq 0$, and $\{x_n\}_{n=0}^d$ form a linear basis in $\mathbb{C}^{d+1}$. In this case we have:

$$\mathrm{Lin}\{x_n\}_{n=0}^d = H. \tag{2.1}$$

Conversely, if relation (2.1) holds then $\det G \neq 0$ and therefore $\det\Gamma \neq 0$.

Notice that the matrix Λ can be constructed explicitly, since Takagi's factorization can be computed, see [6, p.205].

Suppose that $\det\Gamma \neq 0$. If $\Lambda_0 \in \mathbb{C}_{d+1,l}$, $l \geq d+1$, is such that $\Lambda_0\Lambda_0^T = \Gamma$, then

$$\Lambda_0\Lambda_0^T = \Lambda\Lambda^T,$$

and therefore

$$\Lambda^{-1}\Lambda_0\Lambda_0^T(\Lambda^{-1})^T = I_{d+1}.$$

Thus $C := \Lambda^{-1}\Lambda_0 \in \mathbb{C}_{d+1,l}$ is such that $CC^T = I_{d+1}$. Consequently,

$$\Lambda_0 = \Lambda C, \tag{2.2}$$

where $C \in \mathbb{C}_{d+1,l}$, $l \geq d+1$, is such that $CC^T = I_{d+1}$. Conversely, any matrix of the form (2.2) satisfy the condition: $\Lambda_0\Lambda_0^T = \Gamma$.

3. Necessary conditions for the solvability of moment problems A, B, C.

Let one of the the moment problems A,B,C be given with a set $\{s_{n,m}\}_{n,m=0}^d$, $d \in \mathbb{N}$. Suppose that the moment problem has a solution: a J-self-adjoint (J-skew-self-adjoint, or a J-unitary) operator A in a Hilbert space H, and an element $x_0 \in H$, such that relation (1.1) holds. Set

$$x_n := A^n x_0, \qquad 0 \leq n \leq d.$$

Then

$$[x_n, x_m]_J = s_{n,m}, \qquad n, m = 0, 1, \cdots, d. \tag{3.1}$$

Set

$$\Gamma := (s_{n,m})_{n,m=0}^d,\ H_0 := \mathrm{Lin}\{x_n\}_{n=0}^{d-1},\ \mathbf{H} := \mathrm{Lin}\{x_n, Jx_n\}_{n=0}^d, \mathbf{J} = J|_{\mathbf{H}}.$$

Observe that $\mathbf{J}$ is a conjugation in a Hilbert space $\mathbf{H}$. By (3.1) and the property of the J-form $[x, y]_J = [y, x]_J$ we conclude that the matrix Γ is complex symmetric. Consider the following operator A_0 in $\mathbf{H}$ with the domain $D(A) = H_0$:

$$A_0 h = Ah, \qquad h \in H_0.$$

Observe that

$$A_0 \sum_{n=0}^{d-1} \alpha_n x_n = \sum_{n=0}^{d-1} \alpha_n x_{n+1}, \qquad \alpha_n \in \mathbb{C}.$$

Of course, the operator A_0 is $\mathbf{J}$-symmetric (respectively $\mathbf{J}$-skew-symmetric, or $\mathbf{J}$-isometric).
The operator A_0 is well-defined (as a restriction of A). Consequently, the equality

$$\sum_{n=0}^{d-1} \alpha_n x_n = 0, \qquad \alpha_n \in \mathbb{C}, \tag{3.2}$$

implies

$$\sum_{n=0}^{d-1} \alpha_n x_{n+1} = 0. \tag{3.3}$$

Firstly, suppose that $\mathbf{H} \neq \{0\}$. Let $\{f_j\}_{j=0}^{\rho}$, $0 \leq \rho \leq 2d+1$, be an orthonormal basis in $\mathbf{H}$ such that $\mathbf{J} f_j = f_j$, $0 \leq j \leq \rho$. Set

$$a_{n,j} = (x_n, f_j), \quad \Lambda = (a_{n,j})_{0 \leq n \leq d,\ 0 \leq j \leq \rho}.$$

Let Λ_1 (Λ_2) be a matrix which consists of the first (respectively last) d rows of the matrix Λ. Observe that (3.2) is equivalent to the following condition:

$$0 = \left(\sum_{n=0}^{d-1} \alpha_n x_n, f_j \right) = \sum_{n=0}^{d-1} \alpha_n a_{n,j}, \qquad 0 \leq j \leq \rho, \tag{3.4}$$

or, briefly

$$(\alpha_0, \alpha_1, \cdots, \alpha_{d-1}) \Lambda_1 = 0. \tag{3.5}$$

On the other hand, condition (3.3) is equivalent to the following condition:

$$0 = \left(\sum_{n=0}^{d-1} \alpha_n x_{n+1}, f_j \right) = \sum_{n=0}^{d-1} \alpha_n a_{n+1,j}, \qquad 0 \leq j \leq \rho, \tag{3.6}$$

or, briefly

$$(\alpha_0, \alpha_1, \cdots, \alpha_{d-1}) \Lambda_2 = 0. \tag{3.7}$$

Thus, we get

$$\operatorname{Ker} \Lambda_1^T \subseteq \operatorname{Ker} \Lambda_2^T. \tag{3.8}$$

Moreover, we may write:

$$s_{n,m} = [x_n, x_m]_J = (x_n, J x_m) = \sum_{j,k=0}^{\rho} a_{n,j} a_{m,k} \delta_{j,k}$$

$$= \sum_{j=0}^{\rho} a_{n,j} a_{m,j}, \qquad 0 \leq n, m \leq d.$$

Consequently, we obtain that

$$\Gamma = \Lambda \Lambda^T. \tag{3.9}$$

In the case $\mathbf{H} = \{0\}$ we get $x_n = 0$, $0 \leq n \leq d$, and therefore all $s_{n,m}$ are zeros. Then $\Gamma = 0$ and we may choose $\Lambda = \Gamma$, so that relations (3.8) and (3.9) holds.

Observe that

$$[A_0 x_n, x_m]_{\mathbf{J}} = [x_{n+1}, x_m]_{\mathbf{J}} = s_{n+1,m},\ [x_n, A_0 x_m]_{\mathbf{J}} = s_{n,m+1},$$

$$[A_0 x_n, A_0 x_m]_{\mathbf{J}} = s_{n+1,m+1},\ [x_n, x_m]_{\mathbf{J}} = s_{n,m}, \qquad 0 \leq n, m \leq d-1.$$

In the case of the moment problem A / B / C, we conclude that

$$s_{n+1,m} = s_{n,m+1}, \qquad 0 \leq n, m \leq d-1; \tag{3.10}$$

or

$$s_{n+1,m} = -s_{n,m+1}, \qquad 0 \leq n, m \leq d-1; \tag{3.11}$$

or

$$s_{n+1,m+1} = s_{n,m}, \qquad 0 \leq n, m \leq d-1, \tag{3.12}$$

respectively.

From the preceding considerations we obtain the following result.

Theorem 3.1. *Let one of the moment problems A,B,C be given with a set $\{s_{n,m}\}_{n,m=0}^d$, $d \in \mathbb{N}$. If the moment problem A (B or C) has a solution, then the matrix $\Gamma := (s_{n,m})_{n,m=0}^d$ is complex symmetric, relation (3.10) holds (respectively relation (3.11) holds, or relation (3.12) holds), and there exists a representation of the matrix Γ of the form $\Gamma = \Lambda\Lambda^T$, where $\Lambda \in \mathbb{C}_{d+1,\rho+1}$, $0 \leq \rho \leq 2d+1$, such that $\operatorname{Ker} \Lambda_1^T \subseteq \operatorname{Ker} \Lambda_2^T$. Here Λ_1 (Λ_2) is a matrix which consists of the first (respectively last) d rows of the matrix Λ.*

4. Sufficient conditions for the solvability of moment problems A, B, C.

Let one of the moment problems A, B, C be given with a set $\{s_{n,m}\}_{n,m=0}^d$, $d \in \mathbb{N}$. Suppose that the matrix $\Gamma := (s_{n,m})_{n,m=0}^d$ is complex symmetric, relation (3.10) holds (respectively relation (3.11) holds, or relation (3.12) holds), and there exists a representation of the matrix Γ of the form

$$\Gamma = \Lambda\Lambda^T, \tag{4.1}$$

where $\Lambda \in \mathbb{C}_{d+1,\rho+1}$, $0 \leq \rho \leq 2d+1$, such that $\operatorname{Ker} \Lambda_1^T \subseteq \operatorname{Ker} \Lambda_2^T$. Here Λ_1 (Λ_2) is a matrix which consists of the first (respectively last) d rows of the matrix Λ. Let $\Lambda = (a_{n,j})_{0 \leq n \leq d,\ 0 \leq j \leq \rho}$, $a_{n,j} \in \mathbb{C}$.
Set $H = \mathbb{C}^{\rho+1}$, $\vec{e}_n = (\delta_{n,0}, \delta_{n,1}, \dots, \delta_{n,\rho})^T$, $0 \leq n \leq \rho$, and

$$J \sum_{k=0}^{\rho} \alpha_k \vec{e}_k = \sum_{k=0}^{\rho} \overline{\alpha_k} \vec{e}_k, \qquad \alpha_k \in \mathbb{C}.$$

Then J is a conjugation in H. Elements $\{x_n\}_{n=0}^d$ we define in the following way:

$$x_n = \sum_{j=0}^{\rho} a_{n,j} \vec{e}_j, \qquad 0 \leq n \leq d.$$

Then

$$[x_n, x_m]_J = (x_n, J x_m)_H = \sum_{j,k=0}^{\rho} a_{n,j} a_{m,k} \delta_{j,k} = \sum_{j=0}^{\rho} a_{n,j} a_{m,j} = s_{n,m},$$

$$0 \le n, m \le d. \tag{4.2}$$

Define the following operator:

$$A_0 \sum_{n=0}^{d-1} \alpha_n x_n = \sum_{n=0}^{d-1} \alpha_n x_{n+1}, \qquad \alpha_n \in \mathbb{C}, \tag{4.3}$$

with the domain $D(A_0) = \mathrm{Lin}\{x_n\}_{n=0}^{d-1}$. Let us check that it is well-defined. The latter fact means that relation (3.2) implies (3.3). Suppose that (3.2) holds. Then relations (3.4) and (3.5) hold, with $f_j = \vec{e}_j$. By our assumptions we conclude that relations (3.7) and (3.6) hold, with $f_j = \vec{e}_j$. Then relation (3.3) holds. Thus, A_0 is well-defined.

Choose arbitrary $h = \sum_{k=0}^{d-1} \alpha_k x_k$, $g = \sum_{j=0}^{d-1} \beta_j x_j$, $\alpha_k, \beta_j \in \mathbb{C}$, from $D(A_0)$. Then

$$[A_0 h, g]_J = \sum_{k,j=0}^{d-1} \alpha_k \beta_j [x_{k+1}, x_j]_J = \sum_{k,j=0}^{d-1} \alpha_k \beta_j s_{k+1,j},$$

$$[h, A_0 g]_J = \sum_{k,j=0}^{d-1} \alpha_k \beta_j [x_k, x_{j+1}]_J = \sum_{k,j=0}^{d-1} \alpha_k \beta_j s_{k,j+1}.$$

In the case of the moment problem A (B) by relation (3.10) (respectively by relation (3.11)) we conclude that A_0 is J-symmetric (respectively J-skew-symmetric). In the case of the moment problem C, by a similar argument we obtain that A_0 is J-isometric.

Theorem 4.1. *Let one of the moment problems A, B, C be given with a set $\{s_{n,m}\}_{n,m=0}^d$, $d \in \mathbb{N}$. Suppose that the matrix $\Gamma := (s_{n,m})_{n,m=0}^d$ is complex symmetric, relation (3.10) holds (respectively relation (3.11) holds, or relation (3.12) holds), and there exists a representation of the matrix Γ of the form $\Gamma = \Lambda\Lambda^T$, where $\Lambda \in \mathbb{C}_{d+1,\rho+1}$, $0 \le \rho \le 2d+1$, such that $\mathrm{Ker}\,\Lambda_1^T \subseteq \mathrm{Ker}\,\Lambda_2^T$. Here Λ_1 (Λ_2) is a matrix which consists of the first (respectively last) d rows of the matrix Λ. Suppose that for the corresponding J-symmetric (respectively J-skew-symmetric, or J-isometric) operator A_0 from (4.3) there exists a J-self-adjoint (respectively J-skew-self-adjoint, or J-unitary) extension in a possibly larger Hilbert space (with an extension of J). Then the moment problem has a solution.*

Proof. Let $A \supseteq A_0$ be a J-self-adjoint (J-skew-self-adjoint, or J-unitary) extension of the J-symmetric (respectively J-skew-symmetric, or J-isometric) operator A_0 from (4.3). By the induction argument we conclude that

$$A^n x_0 = x_n, \qquad 0 \le n \le d. \tag{4.4}$$

By (4.2) and (4.4) we obtain that relation (1.1) holds. □

Thus, similar to the case of classical moment problems, we arrive to a problem of an extension of the corresponding operator.

In the case $\det\Gamma \ne 0$, the above sufficient conditions of the solvability may be simplified.

Corollary 4.2. *Let one of the moment problems A, B, C be given with a set* $\{s_{n,m}\}_{n,m=0}^{d}$, $d \in \mathbb{N}$. *Suppose that the matrix* $\Gamma := (s_{n,m})_{n,m=0}^{d}$ *is complex symmetric, relation* (3.10) *holds (respectively relation* (3.11) *holds, or relation* (3.12) *holds), and* $\det \Gamma \neq 0$. *Suppose that for the corresponding J-symmetric (respectively J-skew-symmetric, or J-isometric) operator* A_0 *from* (4.3) *(acting in* $\mathbb{C}^\rho$, $\rho = d+1$, *with* Λ *in* (4.1)*, provided by the corollary from Takagi's factorization) there exists a J-self-adjoint (respectively J-skew-self-adjoint, or J-unitary) extension in a possibly larger Hilbert space (with an extension of J). Then the moment problem has a solution.*

Proof. By the above-mentioned corollary from Takagi's factorization ([6, Corollary 4.4.6]), there exists a matrix $\Lambda \in \mathbb{C}_{(d+1)\times(d+1)}$ such that

$$\Gamma = \Lambda\Lambda^T.$$

It is clear that $\det \Lambda \neq 0$. Then $\operatorname{Ker} \Lambda_1^T \subseteq \operatorname{Ker} \Lambda_2^T$, where Λ_1 (Λ_2) is a matrix which consists of the first (respectively last) d rows of the matrix Λ. In fact, if $(\alpha_0, \alpha_1, \cdots, \alpha_{d-1})\Lambda_1 = 0$, $\alpha_j \in \mathbb{C}$, then $(\alpha_0, \alpha_1, \cdots, \alpha_{d-1}, 0)\Lambda = 0$. Therefore all α_j are zeros.
It remains to apply Theorem 4.1 to complete the proof. □

Let H_1, H_2 be some Hilbert spaces. An operator J, which maps H_1 into H_2 is said to be anti-isometric, if ([10])

$$(Jx, Jy)_{H_2} = (y, x)_{H_1}, \qquad x, y \in H_1.$$

An operator $A \in [H_1, H_2]$ is said to be J-self-adjoint (J-skew-self-adjoint) if $B = JB^*J$ (respectively $B = -JB^*J$).

Theorem 4.3. *(*[10, Theorem 1]*) Let B be a bounded closed J-symmetric operator in a Hilbert space H, with the domain* $H_1 := D(B)$ *and* $H_2 := H \ominus D(B) \neq \{0\}$. *Let* B^* *be the adjoint to B, viewed as an operator from* $[H_1, H]$. *The following formula:*

$$\widehat{B} = BP_{H_1}^H + (JB^*J + S)P_{H_2}^H \tag{4.5}$$

establishes a one-to-one correspondence between a set of all bounded J-self-adjoint extensions on the whole H of B, and a set of all J-self-adjoint operators $S \in [H_2, JH_2]$.

The following analog of Theorem 4.3 holds.

Theorem 4.4. *Let B be a bounded closed J-skew-symmetric operator in a Hilbert space H, with the domain* $H_1 := D(B)$ *and* $H_2 := H \ominus D(B) \neq \{0\}$. *Let* B^* *be the adjoint to B, viewed as an operator from* $[H_1, H]$. *The following formula:*

$$\widehat{B} = BP_{H_1}^H + (-JB^*J + S)P_{H_2}^H \tag{4.6}$$

establishes a one-to-one correspondence between a set of all bounded J-skew-self-adjoint extensions on the whole H of B, and a set of all J-skew-self-adjoint operators $S \in [H_2, JH_2]$.

Proof. The proof is similar to the proof of the Theorem 4.3, given in [10]. Since B is J-skew symmetric, it follows that

$$P^H_{JH_1} B \subseteq -JB^* J, \tag{4.7}$$

where B^* is understood for B, which is viewed as an operator from $[H_1, H]$. The following property holds:

$$P^H_F J = JP^H_{JF}, \tag{4.8}$$

where F is an arbitrary subspace of H. Observe that

$$\widehat{B}^* = B^* - P^H_{H_2} JBP^H_{H_1} J + S^* P^H_{JH_2}. \tag{4.9}$$

By (4.9),(4.7) and (4.8) we obtain that $J\widehat{B}^* J = -\widehat{B}$.

On the other hand, let $\widetilde{B}$ be an arbitrary bounded J-skew-self-adjoint extension of B on the whole H. Let $\widehat{B}_0$ be the operator $\widehat{B}$ from (4.6) with $S = 0$. Since $(\widetilde{B} - \widehat{B}_0)H_1 = \{0\}$, then $R\left((\widetilde{B} - \widehat{B}_0)^*\right) \subseteq H_2$. We have $R\left(\widetilde{B} - \widehat{B}_0\right) = R\left(J(\widetilde{B} - \widehat{B}_0)^* J\right) \subseteq JH_2$. We set $\mathcal{S} = \left(\widetilde{B} - \widehat{B}_0\right)|_{H_2} \in [H_2, JH_2]$. Observe that $\mathcal{S}$ is J-skew-self-adjoint, and $\widetilde{B} = \widehat{B}_0 + \mathcal{S}P^H_{H_2}$ has the required form. □

Notice that $S = 0$ is J-self-adjoint and J-skew-self-adjoint. Therefore, by Theorems 3.1, 4.1, 4.3, 4.4 and Corollary 4.2 we obtain the following two theorems.

Theorem 4.5. *Let the moment problem A (B) be given with a set* $\{s_{n,m}\}^d_{n,m=0}$, $d \in \mathbb{N}$. *The moment problem A (B) has a solution if and only if the matrix* $\Gamma := (s_{n,m})^d_{n,m=0}$ *is complex symmetric, relation* (3.10) *(respectively* (3.11)*) holds, and there exists a representation of the matrix* Γ *of the form* $\Gamma = \Lambda\Lambda^T$, *where* $\Lambda \in \mathbb{C}_{d+1,\rho+1}$, $0 \leq \rho \leq 2d+1$, *such that* $\operatorname{Ker}\Lambda_1^T \subseteq \operatorname{Ker}\Lambda_2^T$. *Here* Λ_1 (Λ_2) *is a matrix which consists of the first (respectively last) d rows of the matrix* Λ.

Theorem 4.6. *Let the moment problem A (B) be given with a set* $\{s_{n,m}\}^d_{n,m=0}$, $d \in \mathbb{N}$. *Suppose that the matrix* $\Gamma := (s_{n,m})^d_{n,m=0}$ *is complex symmetric, relation* (3.10) *(respectively* (3.11)*) holds, and* $\det\Gamma \neq 0$. *Then the moment problem has a solution.*

Moreover, if conditions of Theorem 4.6 are satisfied, then canonical solutions in a Hilbert space H, constructed as in Corollary 4.2, can be obtained by formula (4.5).

On the other hand, suppose that we only know that conditions of Theorem 4.5 are satisfied. In this case, we may proceed as at the beginning of Section 3 and construct a $\mathbf{J}$-symmetric ($\mathbf{J}$-skew-symmetric) operator A_0 in a finite-dimensional Hilbert space $\mathbf{H}$. By Theorem 4.3 (Theorem 4.4) extending (if necessary) this operator to a $\mathbf{J}$-self-adjoint (respectively $\mathbf{J}$-skew-self-adjoint) operator in $\mathbf{H}$, we obtain an almost canonical solution of the moment problem A (respectively B). Thus, *if the moment problem A (B) is solvable, then it always has an almost canonical solution.*

5. Extensions of J-isometric operators. Applications to the moment problem C.

Let V be a bounded closed J-isometric operator in a Hilbert space H, with the domain $H_0 := D(V)$ and $H_1 := H \ominus D(V) \neq \{0\}$. Let $W \supseteq V$ be a bounded operator, defined on the whole H. Choose arbitrary $h, g \in H$, $h = h_0 + h_1$, $g = g_0 + g_1$, $g_j, h_j \in H_j$, $j = 1, 2$. We may write:

$$[Wh, Wg]_J = [Wh_0 + Wh_1, Wg_0 + Wg_1]_J$$

$$= [h_0, g_0]_J + [Vh_0, Wg_1]_J + [Wh_1, Vg_0]_J + [Wh_1, Wg_1]_J;$$

$$[h, g]_J = [h_0 + h_1, g_0 + g_1]_J = [h_0, g_0]_J + [h_0, g_1]_J + [h_1, g_0]_J + [h_1, g_1]_J.$$

Thus, W is J-isometric if and only if

$$[Vh_0, Wg_1]_J + [Wh_1, Vg_0]_J + [Wh_1, Wg_1]_J = [h_0, g_1]_J + [h_1, g_0]_J + [h_1, g_1]_J.$$

In the case $h_1 = 0$, we get

$$[Vh_0, Wg_1]_J = [h_0, g_1]_J, \qquad h_0 \in H_0,\ g_1 \in H_1. \tag{5.1}$$

In the case $g_1 = 0$, we get

$$[Wh_1, Vg_0]_J = [h_1, g_0]_J, \qquad h_1 \in H_1,\ g_0 \in H_0. \tag{5.2}$$

Therefore

$$[Wh_1, Wg_1]_J = [h_1, g_1]_J, , \qquad h_1, g_1 \in H_1.$$

It is clear that conditions (5.1) and (5.2) are equivalent. Consequently, *the operator W is J-isometric iff relation (5.1) holds and the operator $W|_{H_1}$ is J-isometric.*

Assume that V *has a bounded inverse.* Denote by $(V^{-1})^*$ the adjoint to V^{-1}, viewed as an operator from $[R(V), H_0]$. Rewrite condition (5.1) in the following form:

$$(Vh_0, JWg_1) = (V^{-1}Vh_0, P^H_{H_0} Jg_1) = (Vh_0, (V^{-1})^* P^H_{H_0} Jg_1),$$

where $h_0 \in H_0$, $g_1 \in H_1$. Then

$$(Vh_0, JWg_1 - (V^{-1})^* P^H_{H_0} Jg_1) = 0, \qquad h_0 \in H_0,\ g_1 \in H_1;$$

and therefore $Wg_1 - J(V^{-1})^* P^H_{H_0} Jg_1 \in H \ominus JR(V)$, $\forall g_1 \in H_1$. Then

$$P^H_{JR(V)} W|_{H_1} = J(V^{-1})^* P^H_{H_0} J|_{H_1}.$$

From our considerations we obtain the following theorem.

Theorem 5.1. *Let V be a bounded closed J-isometric operator in a Hilbert space H, with the domain $H_0 := D(V)$ and $H_1 := H \ominus D(V) \neq \{0\}$. Let V have a bounded inverse. The operator V can be extended to a bounded J-isometric operator, defined on the whole H, if and only if there exists a bounded J-isometric operator W_1 in H, with the domain H_1, such that*

$$P^H_{JR(V)} W_1 = J(V^{-1})^* P^H_{H_0} J|_{H_1}. \tag{5.3}$$

Here by $(V^{-1})^*$ the adjoint to V^{-1}, viewed as an operator from $[R(V), H_0]$, is understood. If such an operator W_1 exists then the following formula:

$$W(h_0 + h_1) = Vh_0 + W_1 h_1, \qquad h_0 \in H_0,\ h_1 \in H_1, \tag{5.4}$$

establishes a one-to-one correspondence between a set of bounded J-isometric extensions $W \supseteq V$, $D(W) = H$, and a set of all bounded J-isometric operators W_1 in H, with the domain H_1, such that relation (5.3) holds.

In general, it is not easy to construct an operator W_1 with the properties, described in Theorem 5.1. However, in the case of a finite-dimensional H and $\dim H_1 = 1$, a full explicit answer on a question of the possibility of the extension can be given.

Theorem 5.2. *Let V be an invertible J-isometric operator in a finite-dimensional Hilbert space H, with the domain $H_0 := D(V)$ and $H_1 := H \ominus D(V) \neq \{0\}$, $\dim H_1 = 1$. Let u be a non-zero element in H_1, and v be a non-zero element in $H \ominus JR(V)$. The operator V can be extended to a J-isometric operator, defined on the whole H, if and only if there exists a solution of the following quadratic equation with respect to an unknown $\lambda \in \mathbb{C}$:*

$$[v, v]_J \lambda^2 + 2\left(v, (V^{-1})^* P^H_{H_0} Ju\right)_H \lambda$$

$$+\overline{\left[(V^{-1})^* P^H_{H_0} Ju, (V^{-1})^* P^H_{H_0} Ju\right]_J} - [u, u]_J = 0. \tag{5.5}$$

Here by $(V^{-1})^*$ the adjoint to V^{-1}, viewed as an operator from $[R(V), H_0]$, is understood. If equation (5.5) is solvable then the following formula:

$$W(h_0 + \beta u) = V h_0 + \beta\lambda v + \beta J (V^{-1})^* P^H_{H_0} Ju, \qquad h_0 \in H_0,\ \beta \in \mathbb{C}, \tag{5.6}$$

establishes a one-to-one correspondence between a set of solutions λ of equation (5.5) and a set of all J-isometric operators $W \supseteq V$, $D(W) = H$. All such operators W are J-unitary operators.

Remark 5.3. It is clear that equation (5.5) has no solutions if and only if

$$[v, v]_J = 0, \quad \left(v, (V^{-1})^* P^H_{H_0} Ju\right)_H = 0,$$

and

$$\overline{\left[(V^{-1})^* P^H_{H_0} Ju, (V^{-1})^* P^H_{H_0} Ju\right]_J} \neq [u, u]_J.$$

However, we do not know, whether such a case can ever happen.

Proof. (Theorem 5.2.)

By Theorem 5.1, the given operator V can be extended to a J-isometric operator, defined on the whole H, if and only if there exists a J-isometric operator W_1 in H, with the domain H_1, such that relation (5.3) holds. Let u be a non-zero element in H_1, and v be a non-zero element in $H \ominus JR(V)$. An arbitrary linear operator W_1 in H, with the domain H_1, such that relation (5.3) holds, has the following form:

$$W_1(\beta u) = \beta\lambda v + \beta J (V^{-1})^* P^H_{H_0} Ju, \qquad \beta \in \mathbb{C}, \tag{5.7}$$

where $\lambda \in \mathbb{C}$. On the other hand, an operator W_1 of the form (5.7) with an arbitrary complex parameter λ is a linear operator in H, with the domain H_1, such that relation (5.3) holds.

It is easy to check that an operator W_1 of the form (5.7) is J-isometric if and only if relation (5.5) holds. The statement about the correspondence (5.6) follows from the formulas (5.4) and (5.7). Let us check the last statement of the theorem. An arbitrary J-isometric operator $W \supseteq V$, $D(W) = H$, is invertible [12, p. 18]. Moreover, we have $W^{-1} \subseteq JW^*J$. Since H is finite-dimensional, then $WH = H$. Therefore $W^{-1} = JW^*J$. $\square$

Example 5.4. (J-isometric operator which has no J-unitary extensions) Let $H = \mathbb{C}^2$, $\vec{e}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\vec{e}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and

$$J(\alpha\vec{e}_0 + \beta\vec{e}_1) = \overline{\alpha}\vec{e}_1 + \overline{\beta}\vec{e}_0, \qquad \alpha, \beta \in \mathbb{C}.$$

Observe that J is a conjugation in H. Consider the following operator V:

$$V\alpha\vec{e}_0 = 0, \qquad \alpha \in \mathbb{C},$$

with the domain $D(V) = \mathrm{Lin}\{\vec{e}_0\}$. The operator V is J-isometric. Since it is not invertible, it has no J-unitary extensions.

Example 5.5. (J-isometric operator which has a unique J-unitary extension inside the original Hilbert space) Let H, $\vec{e}_0$, $\vec{e}_1$, and J be the same as in Example 5.4. Consider the following operator V:

$$V\alpha\vec{e}_0 = \alpha\vec{e}_0, \qquad \alpha \in \mathbb{C},$$

with the domain $D(V) = \mathrm{Lin}\{\vec{e}_0\} =: H_0$. The operator V is J-isometric. Observe that $H \ominus JR(V) = H_0$. Set $u = \vec{e}_1$, $v = \vec{e}_0$. Then $[u,u]_J = [v,v]_J = 0$, and

$$(V^{-1})^* P^H_{H_0} Ju = \vec{e}_0.$$

Equation (5.5) takes the following form: $2\lambda = 0$. Therefore $W = E_H$ is the unique J-unitary extension inside H.

Example 5.6. (J-isometric operator which has exactly two J-unitary extensions inside the original Hilbert space) Let H, $\vec{e}_0$, $\vec{e}_1$ be the same as in Example 5.4. Set

$$J(\alpha\vec{e}_0 + \beta\vec{e}_1) = \overline{\alpha}\vec{e}_0 + \overline{\beta}\vec{e}_1, \qquad \alpha, \beta \in \mathbb{C}.$$

Consider the following operator V:

$$V\alpha\vec{e}_0 = \alpha\vec{e}_0, \qquad \alpha \in \mathbb{C},$$

with the domain $D(V) = \mathrm{Lin}\{\vec{e}_0\} =: H_0$. The operator V is J-isometric. Observe that $H \ominus JR(V) = \mathrm{Lin}\{\vec{e}_1\} =: H_1$. Set $u = v = \vec{e}_1$. Then $[u,u]_J = [v,v]_J = 1$, and

$$(V^{-1})^* P^H_{H_0} Ju = 0.$$

Equation (5.5) takes the following form: $\lambda^2 - 1 = 0$. Therefore we have two possible J-unitary extensions of V inside H.

We can apply Theorem 5.2 to obtain some sufficient results for the solvability of the moment problem C.

Theorem 5.7. *Let the moment problem C be given with a set* $\{s_{n,m}\}_{n,m=0}^d$, $d \in \mathbb{N}$. *Suppose that the matrix* $\Gamma := (s_{n,m})_{n,m=0}^d$ *is complex symmetric, relation* (3.12) *holds, and* $\det\Gamma \neq 0$. *Consider the corresponding* J*-isometric operator* A_0 *from* (4.3) *(acting in* $\mathbb{C}^\rho$, $\rho = d+1$, *with* Λ *in* (4.1)*), provided by the corollary* *from Takagi's factorization). Let u be a non-zero element in $H \ominus D(A_0)$, and v be a non-zero element in $H \ominus JR(A_0)$; $H_0 := D(A_0)$. Suppose that equation (5.5) with respect to an unknown $\lambda \in \mathbb{C}$, where $V = A_0$, has a solution. Then the moment problem has a solution.*

Proof. Observe that $\det \Lambda \neq 0$. The Gram matrix $G = ((x_n, x_m)_H)_{n,m=0}^d$ is equal to $\Lambda\Lambda^*$. Then $\det G \neq 0$, and $\{x_n\}_{n=0}^\infty$ are linearly independent. Thus, we have $\dim(H \ominus D(A_0)) = 1$, and A_0 is invertible. Applying Corollary 4.2 and Theorem 5.2 we complete the proof. □

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