

FUNCTIONAL VERSIONS FOR SOME OPERATOR ENTROPY INEQUALITIES

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ABSTRACT. In this paper, some functional versions for operator entropies as well as for their related operator inequalities are investigated. The theoretical results obtained by our functional approach immediately imply those of operator version in a simple and nice way.

1. INTRODUCTION

Let A and B be two positive invertible operators on a Hilbert space H and $\lambda \in (0, 1]$. The relative operator entropy $S(A|B)$ and the Tsallis operator entropy $T_\lambda(A|B)$ were defined by, [2, 3, 11]

$$S(A|B) = A^{1/2} \log \left(A^{-1/2} B A^{-1/2} \right) A^{1/2},$$

$$T_\lambda(A|B) = \frac{A \sharp_\lambda B - A}{\lambda},$$

where

$$A \sharp_\lambda B = A^{1/2} \left(A^{-1/2} B A^{-1/2} \right)^\lambda A^{1/2}$$

is the power geometric operator mean of A and B . It is well known that $T_\lambda(A|B)$ extends $S(A|B)$ in the sense that the following equality

$$\lim_{\lambda \downarrow 0} T_\lambda(A|B) = S(A|B)$$

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holds for all positive invertible operators A and B . The following operator inequality

$$A - AB^{-1}A \leq S(A|B) \leq B - A \tag{1.1}$$

was shown in [2] and later extended for $T_\lambda(A|B)$ as well, [3]

$$A - AB^{-1}A \leq T_\lambda(A|B) \leq B - A. \tag{1.2}$$

Since then other extensions of $S(A|B)$ have been introduced and related operator inequalities have been investigated, see [4] and [7] for instance.

Otherwise, in two earlier papers [8] and [9] we extended $S(A|B)$ and $T_\lambda(A|B)$ from operators to convex functionals, respectively, as follows:

$$\begin{aligned} \mathcal{F}(f|g) &= \int_0^1 \frac{\left((1-t)f^* + tg^* \right)^* - f}{t} dt, \\ \mathcal{R}_\lambda(f|g) &= \frac{\mathcal{G}_\lambda(f, g) - f}{\lambda}, \end{aligned} \tag{1.3}$$

where, for $\lambda \in (0, 1)$,

$$\mathcal{G}_\lambda(f, g) = \frac{\sin(\lambda\pi)}{\pi} \int_0^1 \frac{t^{\lambda-1}}{(1-t)^\lambda} \left((1-t)f^* + tg^* \right)^* dt \tag{1.4}$$

is the power geometric functional mean, extension of $A\sharp_\lambda B$ from positive operators to convex functionals. Here, f^* denotes the conjugate of f (see Section 2 below).

The following

$$\mathcal{A}_\lambda(f, g) := (1 - \lambda)f + \lambda g \quad \text{and} \quad \mathcal{H}_\lambda(f, g) = \left((1 - \lambda)f^* + \lambda g^* \right)^*$$

have also been introduced (see [8]) as the extensions of the weighted arithmetic and harmonic means from operators to convex functionals, respectively. The three above functional means satisfy the following double functional inequality

$$\mathcal{H}_\lambda(f, g) \leq \mathcal{G}_\lambda(f, g) \leq \mathcal{A}_\lambda(f, g), \tag{1.5}$$

which extends the weighted arithmetic-geometric-harmonic operator mean inequality, namely

$$\left((1 - \lambda)A^{-1} + \lambda B^{-1} \right)^{-1} \leq A\sharp_\lambda B \leq (1 - \lambda)A + \lambda B.$$

The double functional inequality (1.5) allowed us to take

$$\mathcal{G}_1(f, g) = g \quad \text{and} \quad \mathcal{G}_0(f, g) := \lim_{\lambda \downarrow 0} \mathcal{G}_\lambda(f, g) = f.$$

For further detail about these functional means, see [8, 9] and the related references cited therein.

The previous extensions of $S(A|B)$ and $T_\lambda(A|B)$ from positive operators to convex functionals were investigated in the sense that the following connection-relationships

$$\mathcal{F}(f_A|f_B) = f_{S(A|B)}, \tag{1.6}$$

$$\mathcal{R}_\lambda(f_A|f_B) = f_{T_\lambda(A|B)}, \tag{1.7}$$

hold for all positive invertible operators A and B , respectively, see [8] and [9]. Here, f_A refers to the convex quadratic function generated by the positive operator A i.e. $f_A(x) = (1/2)\langle Ax, x \rangle$ for all $x \in H$.

The fundamental goal of the present paper is to extends the operator inequalities (1.1) and (1.2) for $\mathcal{F}(f|g)$ and $\mathcal{R}_\lambda(f|g)$, respectively. Our functional approach immediately implies, in a nice and fast way, the inequalities (1.1) and (1.2) whose proofs were differently stated in the literature, [2, 3, 6].

2. BASIC NOTIONS

We state here some basic notions needed later. For more details, we refer the reader to [1, 5, 10] for instance. Let H be a complex Hilbert space with its inner product $\langle \cdot, \cdot \rangle$ and the associate norm $\|\cdot\|$. Let $f : H \rightarrow \widetilde{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ be a functional (in short, we write $f \in \widetilde{\mathbb{R}}^H$).

- The (extended) space $\widetilde{\mathbb{R}}^H$ of such functionals is equipped with the following partial order (so-called the point-wise order)

$$f, g \in \widetilde{\mathbb{R}}^H, \quad f \leq g \iff f(x) \leq g(x) \text{ for all } x \in H,$$

where we extend the structure of \mathbb{R} to $\mathbb{R} \cup \{-\infty, \infty\}$ by setting

$$\forall s \in \mathbb{R} \cup \{-\infty, \infty\} \quad -\infty \leq s \leq \infty \quad \text{and} \quad s + \infty = \infty.$$

Since our involved functionals can take the value ∞ then the equality $f - f = 0$ does not always hold, since $\infty - \infty = \infty$. For the same reason, the two inequalities $f \leq g$ and $f - g \leq 0$ are not always equivalent.

- The effective domain $\text{dom } f$ of f is given by

$$\text{dom } f = \{x \in H, \quad f(x) < \infty\},$$

and its conjugate f^* is defined through

$$\forall x^* \in H \quad f^*(x^*) = \sup_{x \in H} \{Re\langle x^*, x \rangle - f(x)\} = \sup_{x \in \text{dom } f} \{Re\langle x^*, x \rangle - f(x)\}. \quad (2.1)$$

As well-known, $f^{**} := (f^*)^* \leq f$, f^* is always convex lower semi-continuous and, $f \leq g$ implies $g^* \leq f^*$, for all functionals $f, g \in \widetilde{\mathbb{R}}^H$. Further, the conjugate map $f \mapsto f^*$ is point-wise convex in the sense that

$$\forall f, g \in \widetilde{\mathbb{R}}^H \quad \forall t \in (0, 1) \quad \left((1-t)f + tg \right)^* \leq (1-t)f^* + tg^*. \quad (2.2)$$

- The sub-differential $\partial f(x)$ of f at $x \in \text{dom } f$ is the (possibly empty) subset of H defined by

$$x^* \in \partial f(x) \iff \forall y \in H \quad f(y) \geq f(x) + Re\langle y - x, x^* \rangle,$$

and it is well-known that

$$x^* \in \partial f(x) \iff Re\langle x^*, x \rangle = f^*(x^*) + f(x). \quad (2.3)$$

As usual, the symbol $\Gamma_\circ(H)$ denotes the cone of all convex lower semi-continuous functions $f \in \widetilde{\mathbb{R}}^H$ not identically equal to ∞ . With this, we have $f^{**} = f$ and only if $f \in \Gamma_\circ(H)$. Moreover, if $f \in \Gamma_\circ(H)$ then, $x^* \in \partial f(x)$ if and only if $x \in \text{int}(\text{dom } f)$. Denoting by $\text{int}(\text{dom } f)$ the topological interior of $\text{dom } f$, we recall

that if $f \in \Gamma_\circ(H)$ and $\text{int}(\text{dom } f)$ is nonempty then, for all $x \in \text{int}(\text{dom } f)$, f is continuous at x and $\partial f(x)$ is nonempty.

- The directional derivative of f at x in the direction y is defined by

$$f'(x, y) = \lim_{t \downarrow 0} \frac{f(x + ty) - f(x)}{t},$$

provided that this limit exists. In this case, if moreover the map $y \mapsto f'(x, y)$ is linear then we say that f is Gateaux-differentiable (in short G-differentiable) at x and we write $f'(x, y) = \nabla f(x)(y)$, where $\nabla f(x)$ is called the (operator) G-gradient of f at x . With this, if f is convex and G-differentiable at x then $\partial f(x) = \{\nabla f(x)\}$. It is well-known that every differentiable function is G-differentiable.

The next example will be of interest later.

Example 2.1. Let A and B be as above, f_A and f_B their generated quadratic functions, respectively. The following assertions hold:

- (1) $f_A \pm f_B = f_{A \pm B}$ and, $f_A = (\geq \leq) f_B$ if and only if $A = (\geq \leq) B$.
- (2) For all $x \in H$, $f_A^*(x) = (1/2)\langle A^{-1}x, x \rangle$. We can then write $f_A^* = f_{A^{-1}}$.
- (3) Since A is self-adjoint (resp. positive) then f_A is continuous (resp. convex). Since f_A is quadratic continuous then f_A is (infinitely) differentiable and so f_A is G-differentiable. To compute explicitly $\partial f_A(x)$ we use (2.3) for writing

$$x^* \in \partial f_A(x) \iff \text{Re}\langle x^*, x \rangle = f_A(x) + f_A^*(x^*) = \frac{1}{2}\langle Ax, x \rangle + \frac{1}{2}\langle A^{-1}x^*, x^* \rangle.$$

This, with the fact that A is self-adjoint and so $\langle Ax, x \rangle = \|A^{1/2}x\|^2$, yields

$$\|A^{1/2}x\|^2 - 2\text{Re}\langle x^*, x \rangle + \|A^{-1/2}x^*\|^2 = 0,$$

or equivalently $\|A^{1/2}x - A^{-1/2}x^*\|^2 = 0$. We then deduce $A^{1/2}x - A^{-1/2}x^* = 0$ and so $x^* = Ax$. In summary, $\partial f_A(x) = \{Ax\}$.

3. MAIN RESULTS

Before stating our first main result, we start with the next needed lemmas.

Lemma 3.1. *Let $\lambda \in (0, 1)$. Then we have*

$$\int_0^1 \frac{t^\lambda}{(1-t)^\lambda} dt = \frac{\lambda\pi}{\sin(\lambda\pi)}. \tag{3.1}$$

Proof. If Γ and B denote the standard special functions Gamma and Beta, respectively, then we can write

$$\int_0^1 \frac{t^\lambda}{(1-t)^\lambda} dt = B(1+\lambda, 1-\lambda) = \frac{\Gamma(1+\lambda)\Gamma(1-\lambda)}{\Gamma(2)} = \lambda\Gamma(\lambda)\Gamma(1-\lambda) = \frac{\lambda\pi}{\sin(\lambda\pi)}.$$

□

Lemma 3.2. [9]. *Let $f, g \in \widetilde{\mathbb{R}}^H$. Then we have*

$$\lim_{\lambda \downarrow 0} \mathcal{R}_\lambda(f|g) = \mathcal{F}(f|g),$$

where the limit is taken for the point-wise convergence.

Now, our first main result may be recited as follows.

Theorem 3.3. *Let $f \in \Gamma_{\circ}(H)$ be such that $\text{int}(\text{dom } f)$ is nonempty. Then, for each $\lambda \in (0, 1]$, the following inequalities*

$$\sup_{x^* \in \partial f(x)} (f^* - g^*)(x^*) \leq \mathcal{R}_{\lambda}(f|g)(x) \leq (g - f)(x) \quad (3.2)$$

$$\sup_{x^* \in \partial f(x)} (f^* - g^*)(x^*) \leq \mathcal{F}(f|g)(x) \leq (g - f)(x) \quad (3.3)$$

hold true for all $x \in \text{int}(\text{dom } f)$.

Proof. First, (3.3) follows from (3.2) with Lemma 3.2. We now will show (3.2). From (1.3) and (1.4), with (3.1), it is easy to see that

$$\mathcal{R}_{\lambda}(f|g) = \frac{\sin \lambda \pi}{\lambda \pi} \int_0^1 \frac{t^{\lambda}}{(1-t)^{\lambda}} \frac{\mathcal{H}_t(f, g) - f}{t} dt. \quad (3.4)$$

Otherwise, (2.2) implies that

$$\mathcal{H}_t(f, g) := \left((1-t)f^* + tg^* \right)^* \leq (1-t)f^{**} + tg^{**} \leq (1-t)f + tg,$$

which, when substituted in (3.4) with a help of (3.1), gives the right inequality of (3.2). We now prove the left inequality of (3.2). By (2.1), for all $x, x^* \in H$ we can write

$$\mathcal{H}_t(f, g)(x) := \left((1-t)f^* + tg^* \right)^*(x) \geq \text{Re}\langle x, x^* \rangle - \left((1-t)f^* + tg^* \right)(x^*).$$

If moreover $x \in \text{int}(\text{dom } f) \subset \text{dom } f$ then this latter inequality is equivalent to the next one

$$\mathcal{H}_t(f, g)(x) - f(x) \geq \text{Re}\langle x, x^* \rangle - f^*(x^*) - f(x) + t(f^* - g^*)(x^*).$$

We can take $x^* \in \partial f(x)$ since $\partial f(x) \neq \emptyset$ by the assumption, so that according to (2.3) the last inequality becomes

$$\frac{\mathcal{H}_t(f, g)(x) - f(x)}{t} \geq (f^* - g^*)(x^*).$$

Substituting this in (3.4), with a help of (3.1), we then proved that the next inequality

$$\mathcal{R}_{\lambda}(f|g)(x) \geq (f^* - g^*)(x^*)$$

holds true for all $x \in \text{int}(\text{dom } f)$ and $x^* \in \partial f(x)$. It follows that the set

$$\left\{ (f^* - g^*)(x^*), \quad x^* \in \partial f(x) \right\}$$

is nonempty and upper bounded, for the point-wise order, by $\mathcal{R}_{\lambda}(f|g)(x)$. That is,

$$\forall x \in \text{int}(\text{dom } f) \quad \mathcal{R}_{\lambda}(f|g)(x) \geq \sup_{x^* \in \partial f(x)} (f^* - g^*)(x^*),$$

and so the left side of (3.2) is obtained, so completes the proof. \square

The above theorem has many interesting consequences. First, under a convenient hypothesis on f , it can be recited in the following form.

Corollary 3.4. *Let $f \in \Gamma_\circ(H)$ be G -differentiable at x . Then, for every $\lambda \in (0, 1]$, we have*

$$f^*(\nabla f(x)) - g^*(\nabla f(x)) \leq \mathcal{R}_\lambda(f/g)(x) \leq (g - f)(x). \quad (3.5)$$

$$f^*(\nabla f(x)) - g^*(\nabla f(x)) \leq \mathcal{F}(f/g)(x) \leq (g - f)(x). \quad (3.6)$$

Proof. As already pointed in the above section, since $f \in \Gamma_\circ(H)$ and f is G -differentiable (then continuous) at x then $\text{int}(\text{dom } f) \neq \emptyset$ and $\partial f(x) = \{\nabla f(x)\}$. Substituting this in the left side of (3.2) and (3.3) we obtain (3.5) and (3.6), respectively. \square

From the above corollary we find again, under a functional angle, the next known result giving upper and lower bounds of the Tsallis relative operator entropy, [2, 3, 6].

Corollary 3.5. *For all positive invertible operators A and B and each $\lambda \in (0, 1]$ one has*

$$A - AB^{-1}A \leq T_\lambda(A/B) \leq B - A \quad \text{and} \quad A - AB^{-1}A \leq S(A/B) \leq B - A$$

Proof. Take $f = f_A$ and $g = f_B$ where A and B are as above. Then, by Example 2.1, $\nabla f(x) = \{Ax\}$ for all $x \in H$, and so

$$f^*(\nabla f(x)) = f_{A^{-1}}(Ax) = f_A(x) \quad \text{and} \quad g^*(\nabla f(x)) = f_{B^{-1}}(Ax) = f_{AB^{-1}A}(x).$$

Substituting these in (3.5) and (3.6), with (1.6), (1.7) and Example 2.1 again, we obtain the desired operator inequalities. \square

Now, we state another consequence of the above theorem which, as far as we know, appears to us to be new.

Corollary 3.6. *Let f be convex and G -differentiable. Then we have*

$$\inf_{g \in \mathbb{R}^H} (g + g^* \circ \nabla f) = \min_{g \in \mathbb{R}^H} (g + g^* \circ \nabla f) = f + f^* \circ \nabla f,$$

where the "inf" is taken for the point-wise order.

Proof. If f is convex and G -differentiable then (3.5) implies that

$$\forall x \in H \quad f^*(\nabla f(x)) - g^*(\nabla f(x)) \leq (g - f)(x),$$

or equivalently

$$\forall x \in H \quad (f + f^* \circ \nabla f)(x) \leq (g + g^* \circ \nabla f)(x),$$

from which the desired result follows. \square

Theorem 3.3 can be formulated in another equivalent version recited in the following remark.

Remark 3.7. With the same hypothesis as in Theorem 3.3, the following inequalities

$$\begin{aligned} (f^* - g^*)(x^*) &\leq \inf_{x \in \partial f^*(x^*)} \mathcal{R}_\lambda(f|g)(x) \leq \inf_{x \in \partial f^*(x^*)} (g - f)(x) \\ (f^* - g^*)(x^*) &\leq \inf_{x \in \partial f^*(x^*)} \mathcal{F}(f|g)(x) \leq \inf_{x \in \partial f^*(x^*)} (g - f)(x) \end{aligned}$$

hold for all $x^* \in \text{int}(\text{dom } f^*)$.

Finally, we end this section by stating the following example.

Example 3.8. Let $1 < p, q < \infty$ be two real numbers and take

$$\forall x \in H \quad f(x) = \frac{1}{p}\|x\|^p, \quad g(x) = \frac{1}{q}\|x\|^q.$$

It is well known that the conjugates of f and g are given by, [1]

$$\forall x \in H \quad f^*(x) = \frac{1}{p^*}\|x\|^{p^*}, \quad g^*(x) = \frac{1}{q^*}\|x\|^{q^*},$$

where $1 < p^*, q^* < \infty$ are the conjugates numbers of p and q , respectively. Since f is convex and G-differentiable then

$$\partial f(x) = \{\nabla f(x)\} = \{\|x\|^{p-2}x\} \text{ for all } x \neq 0, \quad \partial f(0) = \{0\}.$$

Substituting these in the double inequality of Corollary 3.4 we obtain (after all computation and reduction)

$$\begin{aligned} \frac{1}{p^*}\|\cdot\|^p - \frac{1}{q^*}\|\cdot\|^{q^*(p-1)} &\leq \mathcal{R}_\lambda\left(\frac{1}{p}\|\cdot\|^p \mid \frac{1}{q}\|\cdot\|^q\right) \leq \frac{1}{q}\|\cdot\|^q - \frac{1}{p}\|\cdot\|^p. \\ \frac{1}{p^*}\|\cdot\|^p - \frac{1}{q^*}\|\cdot\|^{q^*(p-1)} &\leq \mathcal{F}\left(\frac{1}{p}\|\cdot\|^p \mid \frac{1}{q}\|\cdot\|^q\right) \leq \frac{1}{q}\|\cdot\|^q - \frac{1}{p}\|\cdot\|^p. \end{aligned}$$

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