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SOLUTIONS OF NONLINEAR ELLIPTIC PROBLEMS WITH LOWER ORDER TERMS

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Abstract. We give an existence result for strongly nonlinear elliptic equations of the form

$$-\mathrm{div}(a(x,u,\nabla u))+g(x,u,\nabla u)+H(x,\nabla u)=\mu \text{ in } \Omega,$$

where the right hand side belongs to $L^1(\Omega)+W^{-1,p'}(\Omega)$ and $-\mathrm{div}(a(x,u,\nabla u))$ is a Leray–Lions type operator with growth $|\nabla u|^{p-1}$ in ∇u. The critical growth condition on g is with respect to ∇u and no growth condition with respect to u, while the function $H(x,\nabla u)$ grows as $|\nabla u|^{p-1}$.

1. Introduction and preliminaries

In the present paper, we study the existence result for a class of nonlinear elliptic equations. The model problem is the following

$$\begin{cases} -\nabla(A(x)\nabla u)+B(x,u,\nabla u)=\mu & \text{in } \Omega,\\ u=0 & \text{in } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a regular open bounded set in $\mathbb{R}^N$ (N $\geq$1) and B involves the unknown u and its first derivatives. Precisely, B splits into terms which are linear with respect to u and ∇u and a nonlinear term as follows

$$B(x,u,\nabla u)=b(x)\nabla u+g(x,u,\nabla u).$$

Here, A and b are given functions defined on Ω with values in $\mathbb{R}^N\times\mathbb{R}^N$ and $\mathbb{R}^N$, respectively. In fact, we focus our attention on the following problem

$$\begin{cases} -\mathrm{div}(a(x,u,\nabla u))+g(x,u,\nabla u)+H(x,\nabla u)=f-\mathrm{div}\ F & \text{in } \Omega,\\ u=0 & \text{in } \partial\Omega, \end{cases} \tag{1.2}$$

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where $-\mathrm{div}(a(x,u,\nabla u))$ is a Leray–Lions operator acting from $W_0^{1,p}(\Omega)$ into its dual $W^{-1,p'}(\Omega)$, which is coercive and grows like $|\nabla u|^{p-1}$ with respect to ∇u, $p' := \frac{p}{p-1}$. Furthermore,The functions g and H are two the Carathéodory functions with suitable assumptions (see Assumption H(2)). Main difficulties in this work arise from the fact that we consider data which only belong to $L^1(\Omega) + W^{-1,p'}(\Omega)$, namely

$$f \in L^1(\Omega) \text{ and } F \in (L^{p'}(\Omega))^N. \tag{1.3}$$

Many physicals models lead to elliptic and parabolic problems. For instance, in [17] the authors study the modeling of an electronic device. The derived elliptic system coupled the temperature (denoted u) and the electronically potential (denoted Φ). The temperature equation is considered as an elliptic equation where the second member $f = |\nabla\Phi|^2$ belongs to $L^1(\Omega)$. In [18] a Fokker–Planck equation arising in populations dynamics is studied. Models of turbulent flows in oceanography and climatology also lead to such kind of problems (see [19] and the references therein).

In [20] the author studies the Navier-Stokes equations completed by an equation for the temperature ($u = T$). In this case, if we denote by v the velocity of the fluid, then the temperature equation reduces associate to (1.1). Note that for compressible flows the divergence of the velocity does not vanish, and the temperature equation can be considered with linear terms having the form $b(x)\nabla u$. These linear terms introduce new difficulties in the sense that the compactness results, do not apply directly to (1.1) which needs further technical investigations.

We are interested in existence results for weak solutions to (3.2). We have proved such an existence result, when $H \equiv 0$, there is a wide literature in which one can find existence results for problem (3.2). For instance, in the variational case (i.e. when $f \in W^{-1,p'}(\Omega)$), existence result can be found in [10] while if $f \in L^1(\Omega)$ initiated basic works were given in [15, 12, 25], also an existence result for (3.2) was proved in [11] (see the references therein). Related topics can be found in [24].

When H is not necessarily the null function, existence result for problem (3.2) was proved first in [16] in the case where g does not depend on the gradient and then in [23] using, in both works, the rearrangement techniques. for different approach used in the setting of Orlicz Sobolev space the reader can refer to [3]-[8]. See also [9] for related topics

The main features of (3.2) are both the fact that the operator has two lower order terms, which produce a lack of coercivity and the right-hand side which is a measure. The operator has no lower order terms (i.e. $H \equiv g \equiv 0$), in this case the difficulties in studying problem (3.2) are due only to the right-hand side belong to $L^1(\Omega) + W^{-1,p'}(\Omega)$. Simple examples (the Laplace operator in a ball, i.e. $p = 2$, $H \equiv g \equiv 0$, and second member the Dirac mass in the center) show that, in general, the solution of (3.2) does not belong to the space $W^{1,1}_{loc}(\Omega)$. Thus it is necessary to change the classical framework of Sobolev spaces in order to prove existence results. In the present paper we consider operators where both the lower order terms $H(x,\nabla u)$ appear without any coerciveness assumption on the operator.

Our aim in this paper is to investigate the existence of unbounded solutions to some the strongly nonlinear elliptic equations (3.2), in the case where the right-hand side belongs to $L^1(\Omega)+W^{-1,p'}(\Omega)$. The function $g(x,s,\xi)$ is assumed to have exactly the natural growth (i.e. of order p), but no growth assumption is imposed with respect to s to the function g which only satisfies the sign condition and the coercivity condition. The function $H(x,\xi)$, which induces a convection term, is assumed only to grow at most as $|\xi|^{p-1}$.

Now we state the Lemma is a slight modification of Gronwall's lemma (see [2]).

Lemma 1.1. *Given the function $\lambda,\gamma,\varphi,\rho$ defined on $[a,+\infty[$, suppose that $a\geq 0$, $\lambda\geq 0$, $\gamma\geq 0$ and that $\lambda\gamma,\lambda\varphi$ and $\lambda\rho$ belong to $L^1(a,+\infty)$. If for a.e. $t\geq 0$ we have*

$$\varphi(t)\leq \rho(t)+\gamma(t)\int_t^{+\infty}\lambda(\tau)\varphi(\tau)d\tau.$$

then for a.e. $t\geq 0$

$$\varphi(t)\leq \rho(t)+\gamma(t)\int_t^{+\infty}\rho(\tau)\lambda(\tau)\left(\int_t^{\tau}\lambda(r)\gamma(r)dr\right)d\tau.$$

We recall that, for $k>1$ and s in $\mathbb{R}$, the truncation is defined as

$$T_k(s)=\begin{cases} s & \text{if } |s|\leq k,\\ k\frac{s}{|s|} & \text{if } |s|>k.\end{cases}$$

2. Main results

Let us now give the precise hypotheses on the problem (3.2), we assume that the following assumptions:

Assumption H(1). Ω is a bounded open set of $\mathbb{R}^N$ ($N\geq 1$), Let $a:\Omega\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}^N$ be a Carathéodory function, such that

$$|a(x,s,\xi)|\leq \beta[k(x)+|s|^{p-1}+|\xi|^{p-1}], \tag{2.1}$$

for a.e. $(x)\in\Omega$, all $(s,\xi)\in\mathbb{R}\times\mathbb{R}^N$, some positive function $k(x)\in L^{p'}(\Omega)$ and $\beta>0$.

$$[a(x,s,\xi)-a(x,s,\eta)](\xi-\eta)>0 \text{ for all } (\xi,\eta)\in\mathbb{R}^N\times\mathbb{R}^N, \text{ with } \xi\neq\eta, \tag{2.2}$$

$$a(x,s,\xi)\xi\geq\alpha|\xi|^p, \tag{2.3}$$

where α is a strictly positive constant.

Assumption H(2). Furthermore, let $g(x,s,\xi):\Omega\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}$ and $H(x,\xi):\Omega\times\mathbb{R}^N\to\mathbb{R}$ are two Carathéodory functions which satisfy, for almost every $(x)\in\Omega$ and for all $s\in\mathbb{R},\ \xi\in\mathbb{R}^N$, the following conditions

$$|g(x,s,\xi)|\le L_1(|s|)(L_2(x)+|\xi|^p), \tag{2.4}$$

$$g(x,s,\xi)s\ge 0, \tag{2.5}$$

where $L_1:\mathbb{R}^+\to\mathbb{R}^+$ is a continuous increasing function, while $L_2(x)$ is positive and belongs to $L^1(\Omega)$.

$$\exists \quad \delta>0, \quad \nu'>0: \quad \text{for} \quad |s|\ge\delta, \quad |g(x,s,\xi)|\ge\nu'|\xi|^p, \tag{2.6}$$

$$|H(x,\xi)|\le b(x)|\xi|^{p-1}, \tag{2.7}$$

where $b(x)$ is positive and belongs to $L^r(\Omega)$ with $r>\ \max(N,p)$.

Assumption H(3). As far as the right-hand side of (3.2) is concerned, we assume that

$$f\in L^1(\Omega) \text{ and } F\in(L^{p'}(\Omega))^N. \tag{2.8}$$

We shall use the following definitions of weak solutions for problem (3.2) in the following sense:

Definition 2.1. A weak solution of (3.2) is a measurable function $u:\Omega\to\overline{\mathbb{R}}$, such that

$$\begin{cases} u\in W_0^{1,p}(\Omega) \\ \displaystyle\int_\Omega a(x,u,\nabla u)\cdot\nabla v\,dx+\int_\Omega g(x,u,\nabla u)v\,dx+\int_\Omega H(x,\nabla u)v\,dx \\ \qquad\qquad\displaystyle =\int_\Omega fv\,dx+\int_\Omega F\cdot\nabla v\,dx \\ \forall \quad v\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega). \end{cases} \tag{2.9}$$

Remark 2.2. Observe that, in (2.9), we can not replace $W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ by only $W_0^{1,p}(\Omega)$, since in general the two integrals $\displaystyle\int_\Omega g(x,u,\nabla u)v\,dx$ and $\displaystyle\int_\Omega fv\,dx$ may have no meaning.

We are interested in finding weak solutions of problem (3.2), i.e. solutions of (3.2) in the sense of distributions.

Existence result. Our main results are collected in the following theorems:

Theorem 2.3. *Assume that* (2.1)–(2.8) *hold true. Then the problem* (3.2) *has at least one weak solution* u.

Proof. The proof of theorem 2.3 is done in five steps.

Step 1: Approximate problem and a priori estimates. For $n > 0$, let us define the following approximation of g, H and , f. First, set

$$g_n(x,s,\xi) = \frac{g(x,s,\xi)}{1+\frac{1}{n}|g(x,s,\xi)|} \text{ and } H_n(x,\xi) = \frac{H(x,\xi)}{1+\frac{1}{n}|H(x,\xi)|}. \tag{2.10}$$

Note that $g_n(x,s,\xi)$ and $H_n(x,\xi)$ are satisfying the following conditions

$$|g_n(x,s,\xi)| \leq n \quad \text{and} \quad |H_n(x,\xi)| \leq n.$$

Let f_n is a regular functions such that f_n strongly converges to f in $L^1(\Omega)$ and $||f_n||_{L^1} \leq c_1$ for some constant c_1.

Let us now consider the approximate problem

$$-\text{div}(a(x,u_n,\nabla u_n)) + g_n(x,u_n,\nabla u_n) + H_n(x,\nabla u_n) = f_n - \text{div } F \text{ in } \Omega. \tag{2.11}$$

From the Leray–Lions existence theorem (cf. Theorem 2.1 and Remark 2.1 in chapter 2 of [22]), there exists at least one weak solution $u_n \in W_0^{1,\,p}(\Omega)$ of the approximate problem (2.11).

Now, we prove the solution u_n of problem (2.11) is bounded in $W_0^{1,\,p}(\Omega)$, we prove the following

Lemma 2.4. *Let $u_n \in W_0^{1,\,p}(\Omega)$ be a weak solution of (2.11). Then, the following estimates holds,*

$$||u_n||_{W_0^{1,\,p}(\Omega)} \leq D, \tag{2.12}$$

where D depend only on Ω, N, p, p', f, F and $||b||_{L^r(\Omega)}$.

Proof. To get (2.12), we divide the integral $\displaystyle\int_\Omega |\nabla u_n|^p\,dx$ in two parts and we prove the following estimates: for all $k \geq 0$

$$\int_{\{|u_n|\leq k\}} |\nabla u_n|^p\,dx \leq M_1 k, \tag{2.13}$$

and

$$\int_{\{|u_n|>k\}} |\nabla u_n|^p\,dx \quad \leq \quad M_2, \tag{2.14}$$

where M_1 and M_2 are positive constants. In what follows we will denote by M_i, $i = 3,4,\cdots$, some generic positive constants. For $\varepsilon > 0$ and $s \geq 0$, we define

$$\varphi_\varepsilon(r) = \begin{cases} \text{sign}(r) & \text{if } \ |r| > s+\varepsilon \\ \dfrac{\text{sign}(r)(|r|-s)}{\varepsilon} & \text{if } \ s < |r| \leq s+\varepsilon \\ 0 & \qquad \text{otherwise.} \end{cases}$$

We choose $v = \varphi_\varepsilon(u_n)$ as test function in (2.11), we have

$$\begin{aligned}
&\int_\Omega a(x,u_n,\nabla u_n)\nabla(\varphi_\varepsilon(u_n))\,dx\\
&+\int_\Omega g_n(x,u_n,\nabla u_n)\varphi_\varepsilon(u_n)\,dx+\int_\Omega H_n(x,\nabla u_n)\varphi_\varepsilon(u_n)\,dx\\
&=\int_\Omega f_n\varphi_\varepsilon(u_n)\,dx+\int_\Omega F\nabla(\varphi_\varepsilon(u_n))\,dx.
\end{aligned}$$

Using $g_n(x,u_n,\nabla u_n)\varphi_\varepsilon(u_n)\ge 0$, (2.7) and Hölder's inequality, we obtain

$$\begin{aligned}
&\frac{1}{\varepsilon}\int_{\{s<|u_n|\le s+\varepsilon\}} a(x,u_n,\nabla u_n)\nabla u_n\,dx\\
&\le\Big(\int_{\{s<|u_n|\le s+\varepsilon\}}|F|^{p'}\,dx\Big)^{\frac{1}{p'}}\Big(\int_{\{s<|u_n|\le s+\varepsilon\}}\Big(\frac{|\nabla u_n|}{\varepsilon}\Big)^p\,dx\,dt\Big)^{\frac1p}\\
&\quad+\int_{\{s<|u_n|\}} b(x)|\nabla u_n|^{p-1}dx+\int_{\{s<|u_n|\}}|f_n|\,dx.
\end{aligned}$$

Observe that,

$$\begin{aligned}
&\int_{\{s<|u_n|\}} b(x)|\nabla u_n|^{p-1}dx\\
&\qquad\le\int_s^{+\infty}\Big(\frac{-d}{d\sigma}\int_{\{\sigma<|u_n|\}}b^p\,dx\Big)^{\frac1p}\Big(\frac{-d}{d\sigma}\int_{\{\sigma<|u_n|\}}|\nabla u_n|^p\,dx\Big)^{\frac{1}{p'}}\,d\sigma.
\end{aligned}\tag{2.15}$$

Because,

$$\begin{aligned}
\int_{\{s<|u_n|\}} b(x)|\nabla u_n|^{p-1}dx&=\int_s^{+\infty}\frac{-d}{d\sigma}\Big(\int_{\{\sigma<|u_n|\}}b(x)|\nabla u_n|^{p-1}\,dx\Big)\,d\sigma\\
&=\int_s^{+\infty}\lim_{\delta\to0}\frac1\delta\Big(\int_{\{\sigma<|u_n|\le\sigma+\delta\}}b(x)|\nabla u_n|^{p-1}\,dx\Big)\,d\sigma\\
&\le\int_s^{+\infty}\lim_{\delta\to0}\frac1\delta\Big(\int_{\{\sigma<|u_n|\le\sigma+\delta\}}b^p\,dx\Big)^{\frac1p}\Big(\int_{\{\sigma<|u_n|\le\sigma+\delta\}}|\nabla u_n|^p\,dx\Big)^{\frac{1}{p'}}\,d\sigma\\
&=\int_s^{+\infty}\Big(\lim_{\delta\to0}\frac1\delta\int_{\{\sigma<|u_n|\le\sigma+\delta\}}b^p\,dx\,dt\Big)^{\frac1p}\Big(\lim_{\delta\to0}\frac1\delta\int_{\{\sigma<|u_n|\le\sigma+\delta\}}|\nabla u_n|^p\,dx\Big)^{\frac{1}{p'}}\,d\sigma\\
&=\int_s^{+\infty}\Big(\frac{-d}{d\sigma}\int_{\{\sigma<|u_n|\}}b^p\,dx\Big)^{\frac1p}\Big(\frac{-d}{d\sigma}\int_{\{\sigma<|u_n|\}}|\nabla u_n|^p\,dx\Big)^{\frac{1}{p'}}\,d\sigma.
\end{aligned}$$

By (2.3) and (2.15), we deduce that

$$\begin{aligned}
&\frac1\varepsilon\int_{\{s<|u_n|\le s+\varepsilon\}}\alpha|\nabla u_n|^p\,dx\le\int_{\{s<|u_n|\}}|f_n|\,dx\\
&\quad+\Big(\frac1\varepsilon\int_{\{s<|u_n|\le s+\varepsilon\}}|F|^{p'}\,dx\Big)^{\frac{1}{p'}}\Big(\frac1\varepsilon\int_{\{s<|u_n|\le s+\varepsilon\}}|\nabla u_n|^p\,dx\Big)^{\frac1p}\\
&\qquad+\int_s^{+\infty}\Big(\frac{-d}{d\sigma}\int_{\{\sigma<|u_n|\}}b^p\,dx\Big)^{\frac1p}\Big(\frac{-d}{d\sigma}\int_{\{\sigma<|u_n|\}}|\nabla u_n|^p\,dx\Big)^{\frac{1}{p'}}\,d\sigma.
\end{aligned}$$

Letting ε go to zero, we obtain

$$\begin{aligned}\frac{-d}{ds}\int_{\{s<|u_n|\}}\alpha|\nabla u_n|^p\,dx \leq &\int_{\{s<|u_n|\}}|f_n|\,dx\\ &+\left(\frac{-d}{ds}\int_{\{s<|u_n|\}}|F|^{p'}\,dx\right)^{\frac{1}{p'}}\left(\frac{-d}{ds}\int_{\{s<|u_n|\}}|\nabla u_n|^p\,dx\right)^{\frac{1}{p}}\\ &\quad+\int_s^{+\infty}\left(\frac{-d}{d\sigma}\int_{\{\sigma<|u_n|\}}b^p\,dx\right)^{\frac{1}{p}}\left(\frac{-d}{d\sigma}\int_{\{\sigma<|u_n|\}}|\nabla u_n|^p\,dx\right)^{\frac{1}{p'}}d\sigma,\end{aligned} \tag{2.16}$$

where $\{s<|u_n|\}$ denotes the set $\{(x)\in\Omega, s<|u_n(x)|\}$ and $\mu(s)$ stands for the distribution function of u_n, that is $\mu(s)=|\{(x)\in\Omega,\ |u_n(x)|<s\}|$ for all $s\geq 0$.

Now, we recall the following inequality (see for example [21]), we have for almost every $s>0$

$$1\leq\left(NC_N^{\frac{1}{N}}\right)^{-1}(\mu(s))^{\frac{1}{N}-1}(-\mu'(s))^{\frac{1}{p'}}\left(-\frac{d}{ds}\int_{\{s<|u_n|\}}|\nabla u_n|^p dx\right)^{\frac{1}{p}}. \tag{2.17}$$

Using (2.17), we have

$$\begin{aligned}\frac{-d}{ds}\int_{\{s<|u_n|\}}\alpha|\nabla u_n|^p\,dx &= \alpha\left(\frac{-d}{ds}\int_{\{s<|u_n|\}}|\nabla u_n|^p\,dx\right)^{\frac{1}{p'}}\left(\frac{-d}{ds}\int_{\{s<|u_n|\}}|\nabla u_n|^p\,dx\right)^{\frac{1}{p}}\\ &\leq\left(\int_{\{s<|u_n|\}}|f_n|\,dx\right)\left(NC_N^{\frac{1}{N}}\right)^{-1}(\mu(s))^{\frac{1}{N}-1}(-\mu'(s))^{\frac{1}{p'}}\left(-\tfrac{d}{ds}\int_{\{s<|u_n|\}}|\nabla u_n|^p dx\right)^{\frac{1}{p}}\\ &\quad+\left(\frac{-d}{ds}\int_{\{s<|u_n|\}}|F|^{p'}\,dx\right)^{\frac{1}{p'}}\left(\frac{-d}{ds}\int_{\{s<|u_n|\}}|\nabla u_n|^p\,dx\right)^{\frac{1}{p}}\\ &\quad+\left(NC_N^{\frac{1}{N}}\right)^{-1}(\mu(s))^{\frac{1}{N}-1}(-\mu'(s))^{\frac{1}{p'}}\left(\frac{-d}{ds}\int_{\{s<|u_n|\}}|\nabla u_n|^p\,dx\right)^{\frac{1}{p}}\\ &\quad\times\int_s^{+\infty}\left(\frac{-d}{d\sigma}\int_{\{\sigma<|u_n|\}}b^p\,dx\right)^{\frac{1}{p}}\left(\frac{-d}{d\sigma}\int_{\{\sigma<|u_n|\}}|\nabla u_n|^p\,dx\right)^{\frac{1}{p'}}d\sigma,\end{aligned} \tag{2.18}$$

which implies that

$$\begin{aligned}&\alpha\left(\frac{-d}{ds}\int_{\{s<|u_n|\}}|\nabla u_n|^p\,dx\right)^{\frac{1}{p'}}\\ &\leq\left(NC_N^{\frac{1}{N}}\right)^{-1}(\mu(s))^{\frac{1}{N}-1}(-\mu'(s))^{\frac{1}{p'}}\left(\int_{\{s<|u_n|\}}|f_n|\,dx\right)+\left(\frac{-d}{ds}\int_{\{s<|u_n|\}}|F|^{p'}\,dx\right)^{\frac{1}{p'}}\\ &+\left(NC_N^{\frac{1}{N}}\right)^{-1}(\mu(s))^{\frac{1}{N}-1}(-\mu'(s))^{\frac{1}{p'}}\int_s^{+\infty}\left(\frac{-d}{d\sigma}\int_{\{\sigma<|u_n|\}}b^p\,dx\right)^{\frac{1}{p}}\left(\frac{-d}{d\sigma}\int_{\{\sigma<|u_n|\}}|\nabla u_n|^p\,dx\right)^{\frac{1}{p'}}d\sigma.\end{aligned} \tag{2.19}$$

Now, we consider three functions B, F_1 and ψ (see Lemma 2.2 of [1]) defined by

$$\int_{\{s<|u_n|\}}b^p(x)\,dx=\int_0^{\mu(s)}B^p(\sigma)d\sigma. \tag{2.20}$$

$$\int_{\{s<|u_n|\}}|F|^{p'}\,dx=\int_0^{\mu(s)}F_1^{p'}(\sigma)d\sigma \text{ and } \psi(s)=\int_{\{s<|u_n|\}}|f_n|\,dx. \tag{2.21}$$

We have

$$||B||_{L^p(\Omega)} \leq ||h||_{L^p(\Omega)} \tag{2.22}$$

$$||F_1||_{L^{p'}(\Omega)} \leq ||F||_{L^{p'}(\Omega)} \quad \text{and} \quad |\psi(s)| \leq ||f_n||_{L^1(\Omega)}.$$

From (2.19), (2.20) and (2.21) becomes

$$\begin{aligned}
&\alpha\left(\frac{-d}{ds}\int_{\{s<|u_n|\}}|\nabla u_n|^p dx\right)^{\frac{1}{p'}}\\
&\leq F_1(\mu(s))(-\mu'(s))^{\frac{1}{p'}}+(NC_N^{\frac{1}{N}})^{-1}(\mu(s))^{\frac{1}{N}-1}(-\mu'(s))^{\frac{1}{p'}}\psi(s)\\
&+(NC_N^{\frac{1}{N}})^{-1}(\mu(s))^{\frac{1}{N}-1}(-\mu'(s))^{\frac{1}{p'}}\int_s^{+\infty}B(\mu(\nu))(-\mu'(\nu))^{\frac{1}{p}}\left(-\frac{d}{d\nu}\int_{\{\nu<|u_n|\}}|\nabla u_n|^p dx\right)^{\frac{1}{p'}}d\nu.
\end{aligned}$$

From Lemma 1.1, we obtain

$$\begin{aligned}
&\alpha\left(\frac{-d}{ds}\int_{\{s<|u_n|\}}|\nabla u_n|^p\,dx\right)^{\frac{1}{p'}}\\
&\leq F_1(\mu(s))(-\mu'(s))^{\frac{1}{p'}}+(NC_N^{\frac{1}{N}})^{-1}(\mu(s))^{\frac{1}{N}-1}(-\mu'(s))^{\frac{1}{p'}}\psi(s)\\
&+(NC_N^{\frac{1}{N}})^{-1}(\mu(s))^{\frac{1}{N}-1}(-\mu'(s))^{\frac{1}{p'}}\int_s^{+\infty}\left[F_1(\mu(\sigma))+(NC_N^{\frac{1}{N}})^{-1}(\mu(\sigma))^{\frac{1}{N}-1}\psi(\sigma)\right]\\
&\quad\times B(\mu(\sigma))(-\mu'(\sigma))\,\exp\left(\int_s^{\sigma}(NC_N^{\frac{1}{N}})^{-1})B(\mu(r))(\mu(r))^{\frac{1}{N}-1}(-\mu'(r))dr\right)d\sigma.
\end{aligned} \tag{2.23}$$

Now, by a variable change and by Hölder inequality, we estimate the argument of the exponential function on the right hand side of (2.23)

$$\begin{aligned}
\int_s^{\sigma}B(\mu(r))(\mu(r))^{\frac{1}{N}-1}(-\mu'(r))dr &= \int_s^{\sigma}B(z)z^{\frac{1}{N}-1}dz\\
&\leq \int_0^{|\Omega|}B(z)z^{\frac{1}{N}-1}dz\\
&\leq ||B||_{L^r}\left(\int_0^{|\Omega|}z^{(\frac{1}{N}-1)r'}\right)^{\frac{1}{r'}}.
\end{aligned} \tag{2.24}$$

Raising to the power p' in (2.23) and we can write

$$\frac{-d}{ds}\int_{\{s<|u_n|\}}|\nabla u_n|^p\,dx \leq M_1. \tag{2.25}$$

where M_1 depend only on Ω, N, p, p', f, α and $||b||_{L^r(\Omega)}$, integrating between 0 and k, and then (2.13) is proved.

We now give the proof of (2.14), using $T_k(u_n)$ as test function in (2.11), gives

$$\begin{aligned}
&\int_\Omega a(x,u_n,\nabla u_n)\nabla T_k(u_n)\,dx\\
&\qquad+\int_\Omega\Big(g_n(x,u_n,\nabla u_n)+H_n(x,\nabla u_n)\Big)T_k(u_n)\,dx\\
&\qquad=\int_\Omega f_nT_k(u_n)\,dx+\int_\Omega F\nabla T_k(u_n)\,dx.
\end{aligned}$$

Using (2.7), we deduce that,

$$\int_{\{|u_n|\le k\}} a(x,u_n,\nabla u_n)\nabla u_n\,dx + \int_{\{|u_n|\le k\}} g_n(x,u_n,\nabla u_n)u_n\,dx + \int_{\{|u_n|>k\}} g_n(x,u_n,\nabla u_n)T_k(u_n)\,dx \le \int_\Omega f_nT_k(u_n)\,dx + \int_\Omega F\nabla T_k(u_n)\,dx + \int_\Omega b(x)|\nabla u_n|^{p-1}|T_k(u_n)|\,dx \tag{2.26}$$

and by using in the fact that $g_n(x,u_n,\nabla u_n)u_n \ge 0$ and (2.3), we have

$$\begin{aligned}
&\alpha\int_{\{|u_n|\le k\}}|\nabla u_n|^p\,dx + \int_{\{|u_n|>k\}} g(x,u_n,\nabla u_n)T_k(u_n)\,dx\\
&\quad\le k||f||_{L^1} + \int_{\{|u_n|\le k\}} F\nabla u_n\,dx + k\int_{\{|u_n|\le k\}} b(x)|\nabla u_n|^{p-1}\,dx\\
&\qquad + k\int_{\{|u_n|\ge k\}} b(x)|\nabla u_n|^{p-1}\,dx,
\end{aligned}$$

which implies that,

$$\begin{aligned}
&\int_{\{|u_n|>k\}} g(x,u_n,\nabla u_n)T_k(u_n)\,dx\\
&\quad\le k||f||_{L^1} + \int_{\{|u_n|\le k\}} F\nabla u_n\,dx + k\int_{\{|u_n|\le k\}} b(x)|\nabla u_n|^{p-1}\,dx\\
&\qquad + k\int_{\{|u_n|\ge k\}} h(x)|\nabla u_n|^{p-1}\,dx.
\end{aligned}$$

By Hölder inequality and (2.13), we obtain

$$\begin{aligned}
&\int_{\{|u_n|>k\}} g(x,u_n,\nabla u_n)T_k(u_n)\,dx\\
&\le k||f||_{L^1(\Omega)} + kM_1||F||_{(L^{p'}(\Omega))^N} + k^{1+\frac{1}{p'}}M_1||b||_{L^p(\Omega)} + k\int_{\{|u_n|>k\}} b(x)|\nabla u_n|^{p-1}\,dx.
\end{aligned}\tag{2.27}$$

From (2.6) and applying Young's inequality, we get for all $k>\delta$

$$\begin{aligned}
\nu' k\int_{\{|u_n|>k\}}|\nabla u_n|^p\,dx &\le k||f||_{L^1(\Omega)} + kM_1||F||_{(L^{p'}(\Omega))^N} + k^{1+\frac{1}{p'}}M_1||b||_{L^p(\Omega)}\\
&\quad + k\int_{\{|u_n|>k\}} b(x)|\nabla u_n|^{p-1}\,dx\\
&\le k||f||_{L^1(\Omega)} + kM_1||F||_{(L^{p'}(\Omega))^N} + k^{1+\frac{1}{p'}}M_1||b||_{L^p(\Omega)}\\
&\quad + M_6k||b||^p_{L^p} + \tfrac{1}{p'}\nu' k\int_{\{|u_n|>k\}}|\nabla u_n|^p\,dx.
\end{aligned}\tag{2.28}$$

Hence

$$\begin{aligned}
&(1-\tfrac{1}{p'})\int_{\{|u_n|>k\}}|\nabla u_n|^p\,dx\\
&\quad\le M_3||f||_{L^1(\Omega)} + M_4||F||_{(L^{p'}(\Omega))^N} + k^{\frac{1}{p'}}M_5||b||_{L^p(\Omega)} + M_7||b||^p_{L^p},
\end{aligned}\tag{2.29}$$

and Lemma 2.4 is proved. □

Step 2: Almost everywhere convergence of u_n. We prove that u_n converges to some function u locally in measure (and therefore, we can always assume that the convergence is a.e. after passing to a suitable subsequence). We will show that u_n is a Cauchy sequence in measure in any ball B_R.

Let $k > 0$ large enough, we have

$$\begin{aligned} k\ \text{meas}(\{|u_n| > k\} \cap B_R) &= \int_{\{|u_n|>k\}\cap B_R} |T_k(u_n)| dx \\ &\leq \int_{B_R} |T_k(u_n)| dx \\ &\leq C \left(\int_\Omega |\nabla T_k(u_n)|^p dx \right)^{\frac{1}{p}} \\ &\leq c_1, \end{aligned} \tag{2.30}$$

which implies

$$\text{meas}(\{|u_n| > k\} \cap B_R) \leq \tfrac{c_1}{k^1}, \text{ for all } k > 1. \tag{2.31}$$

We have, for every $\delta > 0$,

$$\begin{aligned} \text{meas}(\{|u_m - u_n| > \delta\} \cap B_R) \leq\ & \text{meas}(\{|u_n| > k\} \cap B_R) \\ &+ \text{meas}(\{|u_m| > k\} \cap B_R) + \text{meas}(\{|T_k(u_n) - T_k(u_m)| > \delta\}). \end{aligned} \tag{2.32}$$

Since $T_k(u_n)$ is bounded in $W_0^{1,p}(\Omega)$, there exists some $v_k \in W_0^{1,p}(\Omega)$, such that

$$\begin{aligned} &T_k(u_n) \rightharpoonup v_k \text{ weakly in } W_0^{1,p}(\Omega), \\ &T_k(u_n) \to v_k \text{ strongly in } L^p(\Omega) \text{ and a.e. in } \Omega. \end{aligned}$$

Consequently, we can assume that $T_k(u_n)$ is a Cauchy sequence in measure in Ω.

Let $\varepsilon > 0$, then, by (2.31) and (2.32), there exists some $k(\varepsilon) > 0$ such that $\text{meas}(\{|u_n - u_m| > \delta\} \cap B_R) < \varepsilon$ for all $n, m \geq n_0(k(\varepsilon), \delta, R)$. This proves that (u_n) is a Cauchy sequence in measure in B_R, thus converges almost everywhere to some measurable function u. Then

$$\begin{aligned} &T_k(u_n) \rightharpoonup T_k(u) \text{ weakly in } W_0^{1,p}(\Omega), \\ &T_k(u_n) \to T_k(u) \text{ strongly in } L^p(\Omega) \text{ and a.e. in } \Omega, \end{aligned}$$

which implies, by using (2.1), for all $k > 0$ there exists a function $h_k \in (L^{p'}(\Omega))^N$, such that

$$a(x, T_k(u_n), \nabla T_k(u_n)) \rightharpoonup h_k \text{ weakly in } (L^{p'}(\Omega))^N. \tag{2.33}$$

Step 3: Strong convergence of truncations. Let $k > 0$, we consider the function $\phi(s) = se^{\lambda s^2}$, with $\lambda \geq (\frac{L_1(k)}{\alpha})^2$, we have the following inequality

$$\phi'(s) - \frac{L_1(k)}{\alpha} \mid \phi(s) \mid \geq \frac{1}{2}, \tag{2.34}$$

holds for all $s \in \mathbb{R}$. Here, we define $w_n = T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u))$ where $h > 2k > 0$, and the following function

$$v_n = \phi(w_n) \tag{2.35}$$

The use of v_n as test function in (2.11), we obtain

$$\begin{aligned}\int_\Omega a(x,u_n,\nabla u_n)\cdot\nabla w_n\phi'(w_n)\,dx+\int_\Omega(g_n(x,u_n,\nabla u_n)+H_n(x,\nabla u_n))\phi(w_n)\,dx\\=\int_\Omega f_n\phi(w_n)\,dx+\int_\Omega F\cdot\nabla\phi(w_n)\,dx.\end{aligned}\tag{2.36}$$

Note that, $\nabla w_n=0$ on the set where $\{|u_n|>h+4k\}$, therefore, setting $M=4k+h$, and denoting by $\alpha_h^1(n)$, $\alpha_h^2(n),\cdots$, various sequences of real numbers which converge to zero when n tends to infinity for any fixed value of h, we get by (2.36)

$$\begin{aligned}\int_\Omega a(x,T_M(u_n),\nabla T_M(u_n))\cdot\nabla w_n\phi'(w_n)\,dx+\int_\Omega g_n(x,u_n,\nabla u_n)\phi(w_n)\,dx\\\le\int_\Omega f_n\phi(w_n)\,dx+\int_\Omega F\cdot\nabla\phi(w_n)\,dx+\int_\Omega|H_n(x,\nabla u_n)\phi(w_n)|\,dx.\end{aligned}\tag{2.37}$$

Since

$$\left|\int_\Omega H_n(x,\nabla u_n)\phi(w_n)\,dx\right|\le||\nabla u_n||^{p-1}_{L^p(\Omega)}||b\phi(T_{2k}(u-T_h(u)))||_{L^p},\tag{2.38}$$

(where $b\phi(w_n)\to b\phi(T_{2k}(u-T_h(u)))$ in L^p, by Lebesque's dominated convergence theorem, because $\phi(w_n)$ is bounded).

$$\left|\int_\Omega H_n(x,\nabla u_n)\phi(w_n)\,dx\right|=M_9||b\phi(T_{2k}(u-T_h(u)))||_{L^p}+\alpha_h^3(n),\tag{2.39}$$

and since $g_n(x,u_n,\nabla u_n)\phi(w_n)\ge0$ on the subset $\{x\in\Omega:|u_n(x)|>k\}$, we deduce from (2.37) that

$$\begin{aligned}\int_\Omega a(x,T_M(u_n),\nabla T_M(u_n))\cdot\nabla w_n\phi'(w_n)\,dx+\int_{\{|u_n(x)|\le k\}}g_n(x,u_n,\nabla u_n)\phi(w_n)\,dx\\\le\int_\Omega f_n\phi(w_n)\,dx+\int_\Omega F\cdot\nabla\phi(w_n)\,dx+M_9||b\phi(T_{2k}(u-T_h(u)))||_{L^p}+\alpha_h^3(n).\end{aligned}\tag{2.40}$$

Splitting the first integral on the left hand side of (2.40) where $|u_n|\le k$ and $|u_n|>k$, we can write, by using (2.3)

$$\begin{aligned}&\int_\Omega a(x,T_M(u_n),\nabla T_M(u_n))\cdot\nabla w_n\phi'(w_n)dx\\&\ge\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\cdot(\nabla T_k(u_n)-\nabla T_k(u))\phi'(w_n)dx\\&-C_k\int_{\{|u_n|>k\}}|a(x,T_M(u_n),\nabla T_M(u_n))||\nabla T_k(u_n)|dx.\end{aligned}\tag{2.41}$$

where $C_k=\phi'(2k)$. Since, when n tends to infinity, we have $\nabla T_k(u)\chi_{\{|u_n|>k\}}$ tends to 0 strongly in $(L^p(\Omega))^N$ while, $(a(x,T_M(u_n),\nabla T_M(u_n)))_n$ is bounded in $(L^{p'}(\Omega))^N$ hence the last term in the previous inequality tends to zero for every

h fixed as n tends to infinity. Now, observe that

$$\begin{aligned}
&\int_\Omega a(x, T_k(u_n), \nabla T_k(u_n)) \cdot [\nabla T_k(u_n) - \nabla T_k(u)]\phi'(w_n)dx \\
&\quad = \int_\Omega [a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u))] \\
&\qquad \cdot [\nabla T_k(u_n) - \nabla T_k(u)]\phi'(w_n)dx \\
&\qquad + \int_\Omega a(x, T_k(u_n), \nabla T_k(u)) \cdot [\nabla T_k(u_n) - \nabla T_k(u)]\phi'(w_n)dx.
\end{aligned} \tag{2.42}$$

By the continuity of the Nymetskii operator, we have for all $i = 1, \cdots, N$.

$$a_i(x, T_k(u_n), \nabla T_k(u))\phi'(w_n) \to a_i(x, T_k(u), \nabla T_k(u))\phi'(T_{2k}(u - T_k(u)))$$

strongly in $L^{p'}(\Omega)$, and since $\frac{\partial(T_k(u_n))}{\partial x_i} \rightharpoonup \frac{\partial(T_k(u))}{\partial x_i}$ weakly in $L^p(\Omega)$, the second term of the right hand side of (2.42) tends to zero as n tends to infinity. So that (2.41) yields

$$\begin{aligned}
&\int_\Omega a(x, T_M(u_n), \nabla T_M(u)) \cdot [\nabla T_k(u_n) - \nabla T_k(u)]\phi'(w_n)dx \\
&\qquad \geq \int_\Omega [a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u))] \\
&\qquad \cdot [\nabla T_k(u_n) - \nabla T_k(u)]\phi'(w_n)dx + \alpha_h^5(n).
\end{aligned} \tag{2.43}$$

For the second term of the left hand side of (2.40), we can estimate as follows

$$\begin{aligned}
&\int_{\{|u_n| \leq k\}} g(x, u_n, \nabla u_n)\phi(w_n)dx \\
&\quad \leq \int_{\{|u_n| \leq k\}} L_1(k)\Big(L_2(x) + |\nabla T_k(u_n)|^p\Big)|\phi(w_n)|dx \\
&\quad \leq L_1(k)\int_\Omega L_2(x)|\phi(w_n)|dx \\
&\qquad + \frac{L_1(k)}{\alpha}\int_\Omega a(x, T_k(u_n), \nabla T_k(u_n)) \cdot \nabla T_k(u_n)|\phi(w_n)|dx.
\end{aligned} \tag{2.44}$$

Note that, we have

$$\begin{aligned}
&\int_\Omega a(x, T_k(u_n), \nabla T_k(u_n)) \cdot \nabla T_k(u_n)|\phi(w_n)|dx \\
&= \int_\Omega [a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u))] \cdot [\nabla T_k(u_n) - \nabla T_k(u)]|\phi(w_n)|dx \\
&\quad + \int_\Omega a(x, T_k(u_n), \nabla T_k(u_n)) \cdot \nabla T_k(u)|\phi(w_n)|dx \\
&\quad + \int_\Omega a(x, T_k(u_n), \nabla T_k(u)) \cdot [\nabla T_k(u_n) - \nabla T_k(u)]|\phi(w_n)|dx.
\end{aligned} \tag{2.45}$$

By Lebesgue's Theorem, we have

$$\nabla T_k(u)|\phi(w_n)| \to \nabla T_k(u)|\phi(T_{2k}(u - T_h(u)))| \text{ strongly in } (L^p(\Omega))^N$$

Moreover, in view of (2.33) the second term of the right hand side of (2.45) tends to

$$\int_\Omega h_k.\nabla T_k(u)|\phi(T_{2k}(u-T_h(u)))|\,dx.$$

The third term of the right hand side of (2.45) tends to 0 since for all $i=1,\cdots,N$.

$$a_i(x,T_k(u_n),\nabla T_k(u))\phi(w_n)\to a_i(x,T_k(u),\nabla T_k(u))\phi(T_{2k}(u-T_k(u)))$$

strongly in $L^{p'}(\Omega)$, while

$$\frac{\partial(T_k(u_n))}{\partial x_i}\rightharpoonup\frac{\partial(T_k(u))}{\partial x_i}\text{ weakly in }L^p(\Omega).$$

From (2.44) and (2.45), we obtain

$$\begin{aligned}\Big|\int_{\{|u_n|\le k\}} & g(x,u_n,\nabla u_n)\phi(w_n)dx\Big|\\ &\le\int_\Omega\Big[a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u))\Big]\\ &\qquad\cdot\Big[\nabla T_k(u_n)-\nabla T_k(u)\Big]|\phi(w_n)|dx\\ &\quad+\int_\Omega h_k.\nabla T_k(u)|\phi(T_{2k}(u-T_h(u)))|\,dx\\ &\quad+L_1(k)\int_\Omega L_2(x)|\phi(w_n)|dx+\alpha_h^{10}(n),\end{aligned}\tag{2.46}$$

Now, by the strongly convergence of f_n and the fact that

$$w_n\rightharpoonup T_{2k}(u-T_k(u))\text{ weakly in }W_0^{1,p}(\Omega)\text{ and weakly}_*\text{ in }L^\infty(\Omega),\tag{2.47}$$

and by combining (2.43) and (2.46), we conclude that

$$\begin{aligned}&\int_\Omega\Big[a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u))\Big]\\ &\quad\cdot\Big[\nabla T_k(u_n)-\nabla T_k(u)\Big](\phi'(w_n)-\tfrac{L_1(k)}{\alpha}|\phi(w_n)|)dx\\ &\le L_1(k)\int_\Omega L_2(x)|\phi(T_{2k}(u-T_h(u)))|dx+\int_\Omega f\phi(T_{2k}(u-T_h(u)))|dx\\ &\quad+\int_\Omega F\cdot\nabla T_{2k}(u-T_h(u))\phi'(T_{2k}(u-T_h(u)))|dx\\ &\quad+\int_\Omega h_k.\nabla T_k(u)|\phi(T_{2k}(u-T_h(u)))|\,dx+M_9||b\phi(T_{2k}(u-T_h(u)))||_{L^p}+\alpha_h^{11}(n),\end{aligned}\tag{2.48}$$

which and (2.34), implies that

$$\begin{aligned}
&\int_\Omega \Big[a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u))\Big]\\
&\quad\cdot\Big[\nabla T_k(u_n) - \nabla T_k(u)\Big]\,dx\\
&\le 2L_1(k)\int_\Omega L_2(x)|\phi(T_{2k}(u-T_h(u)))|dx + 2\int_\Omega f\phi(T_{2k}(u-T_h(u)))|dx\\
&\ +2\int_\Omega F\cdot\nabla T_{2k}(u-T_h(u))\phi'(T_{2k}(u-T_h(u)))|dx\\
&\ +2\int_\Omega h_k.\nabla T_k(u)|\phi(T_{2k}(u-T_h(u)))|\,dx + 2M_9||b\phi(T_{2k}(u-T_h(u)))||_{L^p} + \alpha_h^{12}(n),
\end{aligned} \tag{2.49}$$

hence, passing to the limit over n, we obtain

$$\begin{aligned}
\limsup_{n\to+\infty}&\int_\Omega \Big[a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u))\Big]\\
&\cdot\Big[\nabla T_k(u_n) - \nabla T_k(u)\Big]\,dx\\
&\le 2L_1(k)\int_\Omega L_2(x)|\phi(T_{2k}(u-T_h(u)))|dx + 2\int_\Omega f\phi(T_{2k}(u-T_h(u)))|dx\\
&\ +2M_9||b\phi(T_{2k}(u-T_h(u)))||_{L^p}\\
&\ 2\int_\Omega h_k.\nabla T_k(u)|\phi(T_{2k}(u-T_h(u)))|\,dx\\
&\ +2\int_\Omega F\cdot\nabla T_{2k}(u-T_h(u))\phi'(T_{2k}(u-T_h(u)))|dx + \alpha_h^{13}(n).
\end{aligned} \tag{2.50}$$

It remains to show, for our purposes, that the all terms on the right hand side of (2.50) converge to zero as h goes to infinity. The only difficulty that exists is in the last term. For the other terms it suffices to apply Lebesgue's Theorem.

We deal with this term. Let us observe that, if we take $\phi(T_{2k}(u_n - T_h(u_n)))$ as test function in (2.11), use (2.3) and (2.7), we obtain

$$\begin{aligned}
&\alpha\int_{\{h\le|u_n|\le h+2k\}} |\nabla u_n|^p\phi'(T_{2k}(u_n-T_h(u_n)))\,dx\\
&\qquad\qquad +\int_\Omega g_n(x,u_n,\nabla u_n)\phi(T_{2k}(u_n-T_h(u_n)))\,dx\\
&\le \int_\Omega f_n\phi(T_{2k}(u_n-T_h(u_n)))\,dx + \int_{\{h\le|u_n|\le h+2k\}} F\cdot\nabla u_n\phi'(T_{2k}(u_n-T_h(u_n)))\,dx\\
&\qquad\qquad +\int_\Omega b(x)|\nabla u_n|^{p-1}|\phi(T_{2k}(u_n-T_h(u_n)))|\,dx,
\end{aligned} \tag{2.51}$$

thanks to the sign condition (2.5), (2.6) and Young inequality, we get

$$\begin{aligned}
\alpha\int_{\{h\le|u_n|\le h+2k\}} &|\nabla u_n|^p\phi'(T_{2k}(u_n-T_h(u_n)))\,dx\\
&+\nu'\int_\Omega|\nabla u_n|^p|\phi(T_{2k}(u_n-T_h(u_n)))|\,dx\\
\le&\int_\Omega f_n\phi(T_{2k}(u_n-T_h(u_n)))\,dx\\
&+\int_{\{h\le|u_n|\le h+2k\}} F\cdot\nabla u_n\phi'(T_{2k}(u_n-T_h(u_n)))\,dx\\
&+C(p,p',N,\nu')\int_\Omega|b(x)|^p|\phi(T_{2k}(u_n-T_h(u_n)))|\,dx\\
&+\nu'\int_\Omega|\nabla u_n|^p|\phi(T_{2k}(u_n-T_h(u_n)))|\,dx.
\end{aligned}\tag{2.52}$$

Hence

$$\begin{aligned}
\alpha\int_{\{h\le|u_n|\le h+2k\}} &|\nabla u_n|^p\phi'(T_{2k}(u_n-T_h(u_n)))\,dx\\
\le&\int_\Omega f_n\phi(T_{2k}(u_n-T_h(u_n)))\,dx\\
&+\int_{\{h\le|u_n|\le h+2k\}} F\cdot\nabla u_n\phi'(T_{2k}(u_n-T_h(u_n)))\,dx\\
&+C(p,p',N,\nu')\int_\Omega|b(x)|^p|\phi(T_{2k}(u_n-T_h(u_n)))|\,dx.
\end{aligned}\tag{2.53}$$

Using the Young inequality, we have

$$\begin{aligned}
\tfrac{\alpha}{2}\int_{\{h\le|u_n|\le h+2k\}} &|\nabla u_n|^p\phi'(T_{2k}(u_n-T_h(u_n)))\,dx\\
\le&\int_\Omega f_n\phi(T_{2k}(u_n-T_h(u_n)))\,dx+C_k\int_{\{h\le|u_n|\}}|F|^{p'}\,dx\\
&+C(p,p',N,\nu')\int_\Omega|b(x)|^p|\phi(T_{2k}(u_n-T_h(u_n)))|\,dx,
\end{aligned}\tag{2.54}$$

so that, since $\phi'\ge 1$, we have

$$\int_\Omega|\nabla T_{2k}(u-T_h(u))|^p\,dx \;\le\; \int_\Omega|\nabla T_{2k}(u-T_h(u))|^p\phi'(T_{2k}(u-T_h(u)))\,dx,\tag{2.55}$$

again because the norm is lower semi-continuity, we get

$$\begin{aligned}
\int_\Omega&|\nabla T_{2k}(u-T_h(u))|^p\phi'(T_{2k}(u-T_h(u)))\,dx\\
&\le C_k\int_\Omega|\nabla T_{2k}(u-T_h(u))|^p\,dx\\
&\le C_k\liminf_{n\to+\infty}\int_\Omega|\nabla T_{2k}(u_n-T_h(u_n))|^p\,dx\\
&\le C_k\liminf_{n\to+\infty}\int_\Omega|\nabla T_{2k}(u_n-T_h(u_n))|^p\,dx\phi'(T_{2k}(u_n-T_h(u_n))).
\end{aligned}\tag{2.56}$$

Consequently, in view of (2.54) and (2.56), we obtain

$$\begin{aligned}\int_\Omega &|\nabla T_{2k}(u-T_h(u))|^p \phi'(T_{2k}(u-T_h(u)))\,dx \\ &\le C_k \liminf_{n\to+\infty} \int_{\{h\le|u_n|\}} |F|^{p'}\,dx \\ &\quad + \liminf_{n\to+\infty} \int_\Omega f_n \phi(T_{2k}(u_n-T_h(u_n)))\,dx \\ &\quad + C(p,p',N,\nu') \liminf_{n\to+\infty} \int_\Omega |b(x)|^p |\phi(T_{2k}(u_n-T_h(u_n)))|\,dx.\end{aligned} \tag{2.57}$$

Finally, the strong convergence in $L^1(\Omega)$ of f_n, and $b \in L^r(\Omega)$, we have, as first n and then h tend to infinity,

$$\limsup_{h\to+\infty} \int_{\{h\le|u|\le h+2k\}} |\nabla u|^p \phi'(T_{2k}(u-T_h(u)))\,dx = 0,$$

hence

$$\limsup_{h\to+\infty} \int_\Omega F\cdot\nabla T_{2k}(u-T_h(u))\phi'(T_{2k}(u-T_h(u)))\,dx = 0.$$

Therefore by (2.50), letting h go to infinity, we conclude,

$$\begin{aligned}\lim_{n\to+\infty} \int_\Omega &[a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u))]\cdot[\nabla T_k(u_n)-\nabla T_k(u)]dx \\ &= 0.\end{aligned} \tag{2.58}$$

Then, Lemma 5 of [14] implies,

$$T_k(u_n) \to T_k(u) \text{ strongly in } W_0^{1,p}(\Omega). \tag{2.59}$$

Step 4: Equi-integrability of H_n and g_n. We shall now prove that $H_n(x,\nabla u_n)$ converges to $H(x,\nabla u)$ and $g_n(x,u_n,\nabla u_n)$ converges to $g(x,u,\nabla u)$ strongly in $L^1(\Omega)$ by using Vitali's theorem. Since $H_n(x,\nabla u_n) \to H(x,\nabla u)$ a.e. Ω and $g_n(x,u_n,\nabla u_n) \to g(x,u,\nabla u)$ a.e. Ω, thanks to (2.4) and (2.7), it suffices to prove that $H_n(x,\nabla u_n)$ and $g_n(x,u_n,\nabla u_n)$ are uniformly equi-integrable in Ω. We will now prove that $H_n(x,\nabla u_n)$ is uniformly equi-integrable, we use Hölder's inequality and (2.12), we have for any measurable subset $E \subset \Omega$:

$$\begin{aligned}\int_E |H_n(x,\nabla u_n)|\,dx &\le \Big(\int_E b^p(x)\,dx\Big)^{\frac{1}{p}} \Big(\int_\Omega |\nabla u_n|^p\Big)^{\frac{1}{p'}} \\ &\le C\Big(\int_E b^p(x)\,dx\Big)^{\frac{1}{p}}\end{aligned} \tag{2.60}$$

which is small uniformly in n when the measure of E is small.

To prove the uniform equi-integrability of $g_n(x,u_n,\nabla u_n)$. For any measurable subset $E\subset\Omega$ and $m\geq 0$,

$$\begin{aligned}
&\int_E |g(x,u_n,\nabla u_n)|\,dx\\
&\quad=\int_{E\cap\{|u_n|\leq m\}}|g(x,u_n,\nabla u_n)|\,dx+\int_{E\cap\{|u_n|>m\}}|g(x,u_n,\nabla u_n)|\,dx\\
&\quad\leq L_1(m)\int_{E\cap\{|u_n|\leq m\}}\Big(L_2(x)+|\nabla u_n|^p\Big)\,dx+\int_{E\cap\{|u_n|>m\}}|g(x,u_n,\nabla u_n)|\,dx\\
&\quad\leq L_1(m)\int_{E\cap\{|u_n|\leq m\}}\Big(L_2(x)+|\nabla T_m(u_n)|^p\Big)\,dx\\
&\qquad+\int_{E\cap\{|u_n|>m\}}|g(x,u_n,\nabla u_n)|\,dx\\
&\quad=K_1+K_2.
\end{aligned} \tag{2.61}$$

For fixed m, we get

$$K_1 \;\leq\; L_1(m)\int_E\Big(L_2(x)+|\nabla T_m(u_n)|^p\Big)\,dx,$$

which is thus small uniformly in n for m fixed when the measure of E is small (recall that $T_m(u_n)$ tends to $T_m(u)$ strongly in $W_0^{1,p}(\Omega)$). We now discuss the behavior of the second integral of the right hand side of (2.61), let ψ_m be a function such that

$$\begin{cases}\psi_m(s)=0 & \text{if}\quad |s|\leq m-1,\\ \psi_m(s)=\operatorname{sign}(s) & \text{if}\quad |s|\geq m,\\ \psi_m'(s)=1 & \text{if}\quad m-1<|s|<m.\end{cases} \tag{2.62}$$

We chooses for $m>1$, $\psi_m(u_n)$ as a test function in (2.11), we obtain

$$\begin{aligned}
&\int_\Omega a(x,u_n,\nabla u_n)\nabla u_n\psi_m'(u_n)\,dx\\
&+\int_\Omega g_n(x,u_n,\nabla u_n)\psi_m(u_n)\,dx+\int_\Omega H_n(x,\nabla u_n)\psi_m(u_n)\,dx\\
&\quad=\int_\Omega f_n\psi_m(u_n)\,dx+\int_\Omega F\nabla u_n\psi_m'(u_n)\,dx.
\end{aligned}$$

Using (2.3) and Hölder's inequality

$$\begin{aligned}
&\int_{\{m-1\leq|u_n|\}}|g_n(x,u_n,\nabla u_n)|\,dx\leq\int_E|H_n(x,\nabla u_n)|\,dx\\
&+\int_{\{m-1\leq|u_n|\}}|f|\,dx+\|F\|_{L^{p'}(\Omega)}\Big(\int_{\{m-1\leq|u_n|\leq m\}}|\nabla u_n|^p\,dx\Big)^{\frac{1}{p}},
\end{aligned}$$

and by (2.12), we have

$$\lim_{m\to\infty}\sup_{n\in\mathbb{N}}\int_{\{|u_n|>m-1\}}|g_n(x,u_n,\nabla u_n)|\,dx=0.$$

Thus we proved that the second term of the right hand side of (2.61) is also small, uniformly in n and in E when m is sufficiently large, which shows that

$g_n(x,u_n,\nabla u_n)$ and $H_n(x,\nabla u_n)$ are uniformly equi-integrable in Ω as required, we conclude that

$$\begin{array}{lll} H_n(x,\nabla u_n) \to H(x,\nabla u) & \text{strongly in} & L^1(\Omega), \\ g_n(x,u_n,\nabla u_n) \to g(x,u,\nabla u) & \text{strongly in} & L^1(\Omega). \end{array} \tag{2.63}$$

Step 5: Passing to the limit. We take $T_k(u_n-v)$ as test function in (2.11), with $v\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$, we can write

$$\begin{aligned} \int_\Omega a(x,T_{k+||v||_\infty}(u_n),\nabla T_{k+||v||_\infty}(u_n))\cdot\nabla T_k(u_n-v)dx \\ +\int_\Omega (g(x,u_n,\nabla u_n)+H(x,\nabla u_n))T_k(u_n-v)dx \\ =\int_\Omega f_nT_k(u_n-v)dx+\int_\Omega F\cdot\nabla T_k(u_n-v)dx, \end{aligned} \tag{2.64}$$

By Fatou's lemma and in fact that

$$a(x,T_{k+||v||_\infty}(u_n),\nabla T_{k+||v||_\infty}(u_n))\rightharpoonup a(x,T_{k+||v||_\infty}(u),\nabla T_{k+||v||_\infty}(u))$$

weakly in $(L^{p'}(\Omega))^N$. It is easily see that

$$\begin{aligned} &\int_\Omega a(x,T_{k+||v||_\infty}(u),\nabla T_{k+||v||_\infty}(u))\cdot\nabla T_k(u-v)dx \\ &\quad\le \liminf_{n\to\infty}\int_\Omega a(x,T_{k+||v||_\infty}(u_n),\nabla T_{k+||v||_\infty}(u_n))\cdot\nabla T_k(u_n-v)dx. \end{aligned} \tag{2.65}$$

For the second term of the right hand side of (2.64), we have

$$\int_\Omega F\cdot\nabla T_k(u_n-v)dx\to\int_\Omega F\cdot\nabla T_k(u-v)dx \quad\text{as}\quad n\to+\infty, \tag{2.66}$$

since $\nabla T_k(u_n-v)\rightharpoonup\nabla T_k(u-v)$ weakly in $(L^p(\Omega))^N$. On the other hand, we have

$$\int_\Omega f_nT_k(u_n-v)dx\to\int_\Omega fT_k(u-v)dx \text{ as } n\to+\infty. \tag{2.67}$$

Tanks to (2.63) and (2.65)-(2.67), we can pass to the limit in (2.64), and we obtain that u is a solution of the problem (3.2). This completes the proof of Theorem 2.3. ☐

Remark 2.5. The condition (2.4) can be replaced by the weaker one

$$|g(x,s,\xi)|\le L_2(x)+L_1(|s|)|\xi|^p, \tag{2.68}$$

where $L_1:\mathbb{R}^+\to\mathbb{R}^+$ is a continuous increasing function and $L_2(x)\in L^1(\Omega)$.

Remark 2.6. Note that the solution given by Theorem 2.3 belongs to the energy space $W_0^{1,p}(\Omega)$. This better regularity is due the assumption (2.6). Indeed, if (2.6) is not satisfied we cannot expect that the solution belongs to (2.6). One can see a counterexample in [13].

3. Example

Let us consider

$$a(x,s,\xi)=|\xi|^{p-2}\xi, \quad g(x,s,\xi)=\frac{s^2+1}{s^2+3}\mathrm{sign}(s)|\xi|^p, \quad H(x,\xi)=h(x)|\xi|^{p-1}, \tag{3.1}$$

where $h(x)\in L^r(\Omega)$ with $r>\max\ (N,\ p)$, with $x\in\Omega$. It is easy to show that the $a(x,s,\xi)$ are Carathéodory functions satisfying the growth condition (2.1), the coercivity (2.3) and the monotonicity condition (2.2).

While the Carathéodory function $g(x,s,\xi)$ satisfies the condition (2.4) indeed

$$|g(x,s,\xi)|\le \frac{s^2+1}{s^2+3}|\xi|^p\equiv L_1(s)|\xi|^p$$

where $L_1(s)=\frac{s^2+1}{s^2+3}$ is clearly bounded continuous increasing function in $\mathbb{R}^+$.

Note that $g(x,s,\xi)$ satisfying the sign condition (2.5) and the coercivity condition (2.6), and the function $H(x,\xi)$ satisfy the condition (2.7).

Finally, the hypotheses of Theorem 2.3 are satisfied. Therefore, for all $f\in L^1(\Omega)$ and $F\in(L^{p'}(\Omega))^N$, the following problem

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\nabla u)+\frac{u^2+1}{u^2+3}\ \mathrm{sign}(u)|\xi|^p+h(x)|\nabla u|^{p-1}=f-\ \mathrm{div}\ F & \text{in } \Omega,\\ u=0 & \text{in } \partial\Omega, \end{cases} \tag{3.2}$$

has at least one solution in the sense of Definition 2.9.

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