On Marshall-Olkin Burr X family of distribution

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Abstract

Generalizing distributions is important for applied statisticians and recent literature has suggested several ways of extending well-known distributions. We propose a new class of distributions called the Marshall-Olkin Burr X family, which yields flexible shapes for its density such as symmetrical, left-skewed, right-skewed and reversed-J shaped, and can have increasing, decreasing,constant, bathtub and upside-down bathtub hazard rates shaped. Some of its structural properties including quantile and generating functions, ordinary and incomplete moments, and mean deviations are obtained. One special model of this family, the Marshall-Olkin-Burr-Lomax distribution, is investigated in details. We also derive the density of the order statistics. The model parameters are estimated by the maximum likelihood method. For illustrative purposes, three applications to real life data are presented.

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1 Introduction

Generalizing distributions is an old practice and has ever been considered as precious as other practical problems in applied probability and statistics. The modern era on lifetime models aims to study new classes in order to explain how the lifetime phenomenon arises in many fields like biology, medicine, public health, engineering, industry, communications, computer science, insurance, life-testing and many others. For example, the classical distributions such as exponential, Rayleigh, Weibull and gamma are very limited in their characteristics and are unable to show wide flexibility. In many practical situations, classical distributions do not provide adequate fits to real data. For example, if the data are asymmetric, the normal distribution can not be a good choice. So, several generators having one or more extra shape parameters have been proposed in the statistical literature to generate new models. Marshall and Olkin (1997) pioneered an ingenious general method of adding a shape parameter to a family of distributions. The new parameter gives more flexibility to the generated distribution. Let $\bar{G}(x) = 1 - G(x)$ be the baseline survival function (sf) of a random variable X. Let $g(x) = \frac{d}{dx}\, G(x)$ and $g(x)$ be the probability density function (pdf) and cumulative distribution function (cdf) of X. Then, the sf of the Marshall-Olkin (MO for short) family is defined by

$$\bar{F}_{MO}(x;\alpha) = \frac{\alpha\,\bar{G}(x)}{1-\bar{\alpha}\,\bar{G}(x)}, \quad \alpha > 0 \tag{1.1}$$

where $\bar{\alpha} = 1 - \alpha$ is an additional positive parameter. Clearly, $\alpha = 1$ leads to $\bar{G}(x,\alpha) = \bar{F}(x)$. The cdf and pdf of the MO family are, respectively, given by

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$$F_{MO}(x;\alpha) = \frac{G(x)}{1-\bar{\alpha}\,\bar{G}(x)}, \tag{1.2}$$

$$f_{MO}(x;\alpha) = \frac{\alpha\, g(x)}{\left\{1-\bar{\alpha}\,\bar{G}(x)\right\}^2}. \tag{1.3}$$

The Burr X (BX) (Burr, 1942) distribution has wide applications in agriculture, health, biology, actuarial sciences, lifetime and survival analysis. The BX model is also known as the generalized Rayleigh (Surles and Padgett, 2001) distribution. Its cdf is given by

$$\Pi(x;\lambda,\theta) = \left[1-e^{-(\lambda x)^2}\right]^{\theta}, \quad x>0 \tag{1.4}$$

where $\lambda > 0$ is a scale parameter and $\theta > 0$ is a shape parameter. Recently, Santos-Neto et al. (2014) used the adequate function $H_R(x) = -\log[1-R(x)]$ in $(0,\infty)$ for any baseline cdf $R(x)$ in order to define a new generator cdf based on a lifetime distribution. We assume that $R(x)$ depends on a parameter vector ξ. The corresponding pdf is $h_R(x) = \frac{d}{dx}H_R(x) = \frac{r(x)}{1-R(x)}$, where $r(x)$ is the baseline pdf.

Further, we propose the BX-R family of distributions using Santos-Neto et al.s approach and taking the BX cdf in (1.4) for the lifetime distribution. We can write

$$\Pi(x) = \left\{1-e^{-[\lambda H_R(x)]^2}\right\}^{\theta}. \tag{1.5}$$

Furthemore, the basic motivations for using the MOBX-G family in practice are the following:

- to make the kurtosis more flexible compared to the baseline model;
- to produce a skewness for symmetrical distributions;
- to construct heavy-tailed distributions that are not longer-tailed for modeling real data;
- to generate distributions with symmetric, left-skewed, right-skewed and reversed-J shaped;
- to define special models with all types of the hrf;
- to provide consistently better fits than other generated models under the same baseline distribution.

The rest of the paper is organized as follows. In section 2, the probability density function (pdf), cumulative distribution function (cdf), hazard rate function (hrf) and quantile function (qf) are given. In section 3, mixture representation of the density is given and the asymptotic and shapes of pdf and hrf are given. In section 4, some mathematical properties are discussed. In section 5, we gave an expression to calculate stochastic ordering and reliability (stress-strength). In section 6, the expression of ith order statistics is given and estimation of parameters is discussed for complete samples as well as for censored samples. In section 7, four special models are considered and one of them is discussed in detail. In section 8, simulation is carried out and application on real life data is given to see the goodness of fit of the proposed family.

2 The new family

Inserting Eq. (1.5) in Eq. (1.2), the cdf of the new Marshall-Olkin Burr X (MOBX) family is defined by

$$F(x;\alpha,\lambda,\theta,\xi)=\frac{\left\{1-e^{-[\lambda H_R(x)]^2}\right\}^{\theta}}{1-\bar{\alpha}\left\{1-\left[1-e^{-\{\lambda H_R(x)\}^2}\right]^{\theta}\right\}}. \tag{2.1}$$

The pdf and hrf corresponding to Eq. (2.1) are, respectively, given by

$$f(x;\alpha,\lambda,\theta,\xi)=\frac{2\lambda^2\theta h_R(x)\,H_R(x)\,e^{-[\lambda H_R(x)]^2}\,\left\{1-e^{-[\lambda H_R(x)]^2}\right\}^{\theta-1}}{\left\{1-\bar{\alpha}\left[1-\left\{1-e^{-[\lambda H_R(x)]^2}\right\}^{\theta}\right]\right\}^2}, \tag{2.2}$$

and

$$\gamma(x;\alpha,\lambda,\theta,\xi)=\frac{2\lambda^2\theta h_R(x)\,H_R(x)\,e^{-[\lambda H_R(x)]^2}\,\left\{1-e^{-[\lambda H_R(x)]^2}\right\}^{\theta-1}}{\left\{1-\left[1-e^{-\{\lambda H_R(x)\}^2}\right]^{\theta}\right\}\left\{1-\bar{\alpha}\left[1-\left\{1-e^{-[\lambda H_R(x)]^2}\right\}^{\theta}\right]\right\}}. \tag{2.3}$$

Henceforth, we denote by $X\sim MOBX(\lambda,\theta,\alpha)$ a random variable having the family density (2.2), where the baseline parameter vector ξ in $R(x)$ and $r(x)$ is omitted.
The quantile function (qf) of X is determined by inverting (2.1). It can be expressed in terms of the baseline qf $Q_R(u)=R^{-1}(u)$ as

$$Q_X(u)=R^{-1}\left\{1-\exp\left[-\left\{-\frac{1}{\lambda^2}\ln\left[1-\left(\frac{u\alpha}{1-\bar{\alpha}u}\right)^{\frac{1}{\theta}}\right]\right\}^{\frac{1}{2}}\right]\right\}, \tag{2.4}$$

where and $X=Q(u)$ follows the MOBX family.

where $U\sim u(0,1)$. Then, $Q_X(U)$ follows the MOBX family.
Generating new families from existing distributions makes sense when there are important properties in the new family and some characteristics of flexibility in applications. This is the case in the present family. We shall omit the dependence on the parameters in $F(.), f(.)$, etc.

3 Asymptotic and shapes

Let $a=\inf\{x|G(x)>0\}$, the asymptotics of equations (2.2), (2.1) and (2.3) as $x\to a$ are given by

$$F(x)\sim\frac{[\lambda\,H_R(x)]^{2\,\theta}}{\alpha}\qquad \text{as}\quad \text{x}\to\text{a},$$
$$f(x)\sim\frac{2\,\theta\,\lambda^{2\theta}\,h_R(x)\,H_R(x)^{2\,\theta-1}}{\alpha}\qquad \text{as}\quad \text{x}\to\text{a},$$
$$h(x)\sim\frac{2\,\theta\,\lambda^{2\theta}\,h_R(x)\,H_R(x)^{2\,\theta-1}}{\alpha}\qquad \text{as}\quad \text{x}\to\text{a}.$$

The asymptotics of equations (2.2), (2.1) and (2.3) as $x \to \infty$ are given by

$$
\begin{aligned}
&F(x) \sim \alpha\theta\, \mathrm{e}^{-[\lambda\, \mathrm{H_R(x)}]^2} \qquad \text{as} \quad \mathrm{x} \to \infty,\\
&f(x) \sim \alpha\theta\, \lambda^2\, h_R(x) \mathrm{e}^{-[\lambda\, \mathrm{H_R(x)}]^2} \qquad \text{as} \quad \mathrm{x} \to \infty,\\
&h(x) \sim \alpha\theta\, \lambda^2\, h_R(x) \qquad \text{as} \quad \mathrm{x} \to \infty.
\end{aligned}
$$

The shapes of the density and hazard functions can be described analytically. The critical points of the MOBX density function are the roots of the equation:

$$
\begin{aligned}
&\frac{r'(x)}{r(x)} + \frac{r(x)}{1-R(x)} - \frac{r(x)\,[1-R(x)]^{-1}}{\{\log[1-R(x)]\}} - 2\frac{r(x)\left\{\lambda^2\,[-\log\{1-R(x)\}]\right\}}{1-R(x)}\\
+ \quad &(\theta-1)\frac{2\lambda^2 r(x)\,\{-\log[1-R(x)]\}\, e^{-\lambda^2\{-\log[1-R(x)]\}^2}}{[1-R(x)]\left\{1-e^{-\lambda^2[-\log\{1-R(x)\}]^2}\right\}}\\
- \quad &\frac{2\bar{\alpha}\lambda^2\theta r(x) e^{-\lambda^2\{-\log[1-R(x)]\}^2}\left\{1-e^{-\lambda^2[-\log\{1-R(x)\}]^2}\right\}^{\theta-1}}{[1-R(x)]\left\{1-\bar{\alpha}\left[1-\left\{1-e^{-\lambda^2[-\log\{1-R(x)\}]^2}\right\}^{\theta}\right]\right\}} = 0
\end{aligned}
$$

This equation may have more than one root. The critical points of $\gamma(x)$ are obtained from the equation:

$$
\begin{aligned}
&\frac{r'(x)}{r(x)} + \frac{r(x)}{1-R(x)} - \frac{r(x)\,[1-R(x)]^{-1}}{\{\log[1-R(x)]\}} - 2\frac{r(x)\left\{\lambda^2\,[-\log\{1-R(x)\}]\right\}}{1-R(x)}\\
+ \quad &(\theta-1)\frac{2\lambda^2 r(x)\,\{-\log[1-R(x)]\}\, e^{-\lambda^2\{-\log[1-R(x)]\}^2}}{[1-R(x)]\left\{1-e^{-\lambda^2[-\log\{1-R(x)\}]^2}\right\}}\\
+ \quad &2\lambda^2 r(x)\,\{-\log[1-R(x)]\}\, e^{-\lambda^2\{-\log[1-R(x)]\}^2}\,\frac{\left\{1-e^{-\lambda^2[-\log\{1-R(x)\}]^2}\right\}^{\theta-1}}{1-\left\{1-e^{-\lambda^2[-\log\{1-R(x)\}]^2}\right\}^{\theta}}\\
- \quad &\frac{\bar{\alpha}\lambda^2\theta r(x)\, e^{-\lambda^2\{-\log[1-R(x)]\}^2}\left\{1-e^{-\lambda^2[-\log\{1-R(x)\}]^2}\right\}^{\theta-1}}{[1-R(x)]\left\{1-\bar{\alpha}\left[1-\left\{1-e^{-\lambda^2[-\log\{1-R(x)\}]^2}\right\}^{\theta}\right]\right\}} = 0
\end{aligned}
$$

By using most symbolic computation software platforms, we can examine these two equations to determine the local maximums and minimums and inflexion points.

3.1 Mixture representation

In this section, we present useful linear representations for Eqs. (2.2) and (2.1). Using the generalized binomial theorem

$$
(1-z)^{-k} = \sum_{i=0}^{\infty} \binom{k+i-1}{i} z^i,
$$

and log-power expansion, we get

$$[\log\,(1+z)]^{a}=a\sum_{k=0}^{\infty}\left(\begin{array}{c}k-a\\k\end{array}\right)\sum_{j=0}^{k}\frac{(-1)^{j}}{a-j}\left(\begin{array}{c}k\\j\end{array}\right)P_{j,k}z^{k},$$

where $c_k=\frac{(-1)^k}{k+1}, p_{j,0}=1$, and $P_{j,k}$ is determined recursively by

$$P_{j,k}=\frac{1}{k}\sum_{m=1}^{k}(jm-k+m)c_{m}P_{j,k-m}.$$

$$p_{j,0}=1 \text{ and } c_k=\frac{(-1)^k}{k+1}$$

(”http://functions.wolfram.com/ElementaryFunctions/Log/06/01/04/”)
Now Equation (2.2) becomes

$$f(x)=\sum_{u,k=0}^{\infty}a_{u+k}\,h_{u+k}(x). \tag{3.1}$$

where

$$a_{u+k}=\sum_{j=0}^{\infty}\frac{w_{\theta(i+1)-1}}{(u+k+1)}\left[2\lambda^{2}\theta\sum_{l=0}^{\infty}\left(\begin{array}{c}\theta(i+1)-1\\l\end{array}\right)(-1)^{l}\sum_{m=0}^{\infty}\frac{(-1)^{m}\,(l+1)^{m}\lambda^{2m}}{m!}V_{u+k}\right]$$

and $$w_{\theta(i+1)-1}=\sum_{i=0}^{j}\left(\begin{array}{c}j\\i\end{array}\right)(j+1)\,\bar{\alpha}^{j}\,(-1)^{i}$$

, $$V_{u+k}=(-1)^{a}a\sum_{k=0}^{\infty}\left(\begin{array}{c}k-a\\k\end{array}\right)\sum_{q=0}^{\infty}\frac{(-1)^{q}}{a-q}\left(\begin{array}{c}k\\q\end{array}\right)P_{q,k}\,(-1)^{k}$$

Similarly we can write the MOBX family cdf as a mixture of exp-G densities as

$$F(x)=\sum_{k=0}^{\infty}a_{u+k+1}\,H_{u+k+1}(x), \tag{3.2}$$

where $h_{m+1}(x)=(m+1)\,R^{m}(x)\,r(x)$ is the exp-R density with power parameter $u+k+1$. Equations (3.1) and (3.2) are linear combinations of the cdfs and pdfs of the exp-R distribution. These linear combination require to be computed numerically in software such as MAPLE, MATHEMATICA and Ox. Eq. (3.1) is important to derive some mathematical properties of the EG distribution from those exp-R properties. The mathematical properties of the exp-R model were studied by several authors, e.g. Mudholkar et al.(1995), Gupta et al.(1998), Gupta and Kundu (1999) and Nadarajah and Kotz (2006).

4 Mathematical properties

In this section, we obtain ordinary and incomplete moments, mean deviations about the mean and the median and moment generating function (mgf) of X using Eq. (3.1) from those quantities of the exp-R model. From now on, let $Y_{u+k+1}\sim$ exp-R with power parameter $u+k+1$.

4.1 Moment

Based on the linear representation (3.1), we can write

$$\mu'_r = \sum_{u,k=0}^{\infty} a_{u+k}\, E(Y_{u+k}^r), \tag{4.1}$$

where $E(Y_{u+k}^r) = (u+k+1) \int_{-\infty}^{\infty} x^r\, R^{u+k}(x)\, r(x)\, dx = (u+k+1) \int_0^1 Q_R(x)^n\, u^k\, du$. Explicit expressions for moments of several exponentiated distributions were derived by several authors. They can be used to obtain μ'_r. Here, we give only one example by taking the baseline Lomax distribution with shape parameter $b > 0$ and scale parameter $a > 0$, and cdf $R(x) = 1 - \left(1 + \frac{x}{a}\right)^{-b}$. The rth moment of the MOBX-Lomax distribution becomes

$$\mu'_r = \sum_{u,k=0}^{\infty} a_{u+k}\, (u+k+1)\, b \sum_{n=0}^{\infty} \binom{u+k}{n} (-1)^n\, a^r\, \mathrm{B}\left(r+1, b(n+1) - r\right)$$

The central moments (μ_s) and cumulants (K_s) of X are determined from (4.1) as $\mu_s = \sum_{n=0}^{p} \binom{s}{n} (-1)^n\, \mu'_1\, \mu'_{s-n}$ and $K_s = \mu'_s - \sum_{n=1}^{s-1} \binom{s-1}{n-1} K_n\, \mu'_{s-n}$ respectively, where $K_1 = \mu'_1$. Thus, $K_2 = \mu'_2 - (\mu'_1)^2$, $K_3 = \mu'_3 - 3\mu'_2\mu'_1 + 2(\mu'_1)^3$, $K_4 = \mu'_4 - 4\mu'_3\mu'_1 - 3(\mu'_2)^2 + 12\mu'_2\,(\mu'_1)^2 - 6(\mu'_1)^4$, etc. The skewness $\gamma_1 = \frac{K_3}{K_2^{\frac{3}{2}}}$ and kurtosis $\gamma_2 = \frac{K_4}{K_2^2}$ can be calculated from the third and fourth standardized cumulants.

Based on the linear representation (3.1), we can write moment generating function is

$$M_X(t) = \sum_{u,k=0}^{\infty} a_{u+k}\, M_{u+k}(t), \tag{4.2}$$

where $M_{u+k}(t) = \int_0^{\infty} e^{t\,x}\, h_{u+k}(x)\, dx$

The incomplete moments play an important role to obtain Lorenz and Bonferroni curves, mean deviations, Zenga index, mean residual life and mean waiting time. The rth incomplete moment of X can be determined as

$$m_r(y) = \int_{-\infty}^{y} x^r\, f(x)\, dx = \sum_{u,k=0}^{\infty} \int_{-\infty}^{R(y)} Q_R(u)^n\, u^k\, du \tag{4.3}$$

The last integral can be computed for most baseline G distributions. A general expression for the first incomplete moment of X can be obtained from (3.1) as $m_1(y) = \sum_{u,k=0}^{\infty} a_{u+k}\, J_{u+k}(y)$, where $J_{u+k}(z) = \int_{-\infty}^{y} x\, h_{u+k}(x)\, dx = \int_{-\infty}^{R(y)} Q_r(u)\, u^k\, du$ is the basic quantity to compute the first incomplete moment of the exp-R distribution.

For a given probability p, the Lorenz and Bonferroni curves are given by $L(\pi) = \frac{m_1(q)}{\mu_1'}$ and $B(\pi) = \frac{m_1(q)}{\pi\,\mu_1'}$, respectively, where $\mu_1' = E(X)$ and $q = Q_X(\pi)$ is the qf of X at π given by (2.4).

The mean deviations about the mean $\delta_1 = E(|X - \mu_1'|)$ and about the median $\delta_2 = E(|X - M|)$ of X can be expressed as $\delta_! = 2\,\mu_1'\,F(\mu_1') - 2\,m_1(\mu_1')$ and $\delta_2 = \mu_1' - 2\,m_1(M)$, respectively, where $\mu_1' = E(X)$, $M = Median(X) = Q_X(0.5)$ is the median, $F(\mu_1')$ is easily evaluated from (2.1) and $m_1(z)$ is the first incomplete moment given by (4.3) with $r = 1$.

5 Stochastic ordering

The concept of stochastic ordering are frequently used to show the ordering mechanism in life time distributions. For more detail about stochastic ordering see (Shaked et al ,(1994)). A random variable is said to be stochastically greater ($X \leq_{st} Y$) than Y if $F_X(x) \leq F_Y(x)$ for all x. In the simillar way, X is said to be stochastically greater ($X \leq_{st} Y$) than Y in the

- stochastic order ($X \leq_{st} Y$) if $F_X(x) \geq F_Y(x)$ for all x.
- hazard rate order ($X \leq_{hr} Y$) if $h_X(x) \geq h_Y(x)$ for all x.
- mean residual order ($X \leq_{mrl} Y$) if $m_X(x) \geq m_Y(x)$ for all x.
- likelihood ratio order ($X \leq_{hr} Y$) if $f_X(x) \geq f_Y(x)$ for all x.
- reversed hazard rate order ($X \leq_{rhr} Y$) if $\frac{F_X(x)}{F_Y(x)}$ is decreasing for all x.

The stochastic orders defined above are related to each other, as the following implications

$$X \leq_{rhr} Y \Leftarrow X \leq_{lr} Y \Rightarrow X \leq_{hr} Y \Rightarrow X \leq_{st} Y \Rightarrow X \leq_{mrl} Y \tag{5.1}$$

Let $X_1 \sim MOBXG(\theta, \lambda, \alpha_1)$ and $X_2 \sim MOBXG(\theta, \lambda, \alpha_2)$. Then according to the definition of likelihood ratio ordering $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]$.

$$\begin{aligned}
\frac{f(x)}{g(x)} &= \left(\frac{1-\bar{\alpha}_2\xi}{1-\bar{\alpha}_1\xi}\right)^2 \\
\text{Let } \xi &= \left\{1 - \left[1 - e^{-\{\lambda H(x)\}^2}\right]^\theta\right\} \\
\text{and } \xi' &= -2\lambda^2\theta h(x)H(x)e^{-[\lambda H(x)]^2}\left\{1 - e^{-[\lambda H(x)]^2}\right\}^{\theta-1}
\end{aligned}$$

Since, $\alpha_1 < \alpha_2$,

$$\frac{d}{dx}\log\left[\frac{f(x)}{g(x)}\right] = \frac{2\xi'\left(\bar{\alpha}_1 - \bar{\alpha}_2\right)}{\left(1-\bar{\alpha}_1\xi\right)\left(1-\bar{\alpha}_2\xi\right)} < 0$$

Hence, $f(x)/g(x)$ is decreasing in x. That is $X \leq_{lr} Y$. The remaining statements follow from the implications (5.1).

6 Stress and strength analysis

In the context of reliability, the stress-strength model defines the life of a element which has a random strength X_1 that is subjected to a accidental stress X_2. The component fails at the instant that the stress applied to it exceeds the strength, and the component will function suitably whenever $X_1 > X_2$. Hence, $R = P(X_2 < X_1)$ is a measure of component reliability. It has many applications especially in the area of engineering. We derive the reliability R when X_1 and X_2 have independent MOBX$(\lambda_1, \theta_1, \alpha)$ and MOBX$(\lambda_2, \theta_2, \alpha)$ distributions with the common shape parameter and scale parameter. From equations (2.1) and (2.2), the reliability reduces to

$$R = P(X_1 < X_2) = \int_0^\infty f_1(x)\, F_2(x) dx. \tag{6.1}$$

From equations (3.1) and (3.2), we have

$$R = P(X_1 < X_2) = \sum_{u+k=0}^{\infty} \sum_{n+p=0}^{\infty} b_{u+k} b_{n+p+1} \int_0^\infty h_{u+k}(x) H_{n+p+1}(x) dx \tag{6.2}$$

where $G(x)$ and $g(x)$ be the cdf and pdf of any base line distribution.

7 Order statistics

Order statistics make their appearance in many areas of statistical theory and practice. Let $X_1, X_2, ..., X_n$ be a random sample from MOBX family of distributions. Let $X_{i:n}$ denote the ith order statistic, then the pdf of $X_{i:n}$ is

$$f(X_{i:n}) = \frac{1}{\beta\,(i, n-i+1)}, \sum_{j=0}^{n-i} \left(\begin{array}{c} n-i \\ j \end{array} \right) (-1)^j\, f(x)\, F(x)^{j+i-1}.$$

Using equations (3.1) and (3.2), we get

$$\begin{aligned} f_{i:n}(x) &= \frac{1}{\beta\,(i, n-i+1)} \sum_{j=0}^{n-i} \left(\begin{array}{c} n-i \\ j \end{array} \right) (-1)^i \Big[\sum_{n,p=0}^{\infty} b_{n+p+1} R^{n+p+1}(x) \Big]^{i+j-1} \\ &\times \Big[\sum_{u,k=0}^{\infty} a_{u+k}\,(u+k+1)\, r(x)\, R^{u+k}(x) \Big] \end{aligned} \tag{7.1}$$

Using power series expansion(see Granshteyn-Ryzhik (2007) pages [17,18])

$$\left(\sum_{i=0}^{\infty} a_i\, x^i \right)^n = \sum_{i=0}^{\infty} c_i\, x^i$$

$c_0 = a_0^i$ and $c_m = (m\, a_0)^{-1} \sum_{k=0}^{m} \big(k(n+1) - m\big)\, a_k\, c_{m-k}$

we can write equation (7.1) as

$$f_{i:n}(x) = \sum_{u,k=0}^{\infty} \sum_{n,p=0}^{\infty} V_{u,k,n,p}\, h_{u+k+n+p+1}(x), \tag{7.2}$$

where

$$V_{u,k,n,p} = \frac{\sum_{j=0}^{n-i} \binom{n-i}{j} (-1)^i\, a_{u+k}\, c_{n+p}\,(u+k+1)}{(u+k+n+p+2)\,\beta\,(i, n-i+1)}$$

and

$$h_m(x) = (m+1)\, r(x)\, R^m(x).$$

The result in equation (7.2) is the final result of this section.

7.1 Estimation of parameters

Let $X = (x_1, ..., x_n)$ be a random sample of size n form a MOBX family of distributions. Let $\Theta = (\theta, \alpha, \lambda, \xi)^T$ is the parameter vector. Then the log-likelihood function of Θ is

$$\begin{aligned} l &= n \log(2\,\lambda^2\,\theta\,\alpha) + \sum_{i=1}^{n} \log h_R(x) + \sum_{i=1}^{n} \log H_R(x) - \lambda^2 \sum_{i=1}^{n} [H_R(x)]^2 \\ &+ (\theta-1) \sum_{i=1}^{n} \log z\, - 2 \sum_{i=1}^{n} \log\left[1 - \bar{\alpha}\left\{1 - z^\theta\right\}\right] \end{aligned} \tag{7.3}$$

where $z = 1 - e^{-\lambda^2\,[H_R(x)]^2}$.

The component of score vector are

$$\begin{aligned} U_\theta &= \frac{n}{\theta} - \sum_{i=1}^{n} \log z - 2 \sum_{i=1}^{n} \left[\frac{\bar{\alpha} z^\theta\, \log z}{1 - \bar{\alpha}(1 - z^\theta)}\right] = 0, \\ U_\alpha &= \frac{n}{\alpha} - 2 \sum_{i=1}^{n} \left[\frac{(1 - z^\theta)}{1 - \bar{\alpha}(1 - z^\theta)}\right] = 0, \\ U_\lambda &= \frac{2n\lambda}{\lambda^2} + (\theta - 1) \sum_{i=1}^{n} \frac{z'_{i:\lambda}}{z} - 2 \sum_{i=1}^{n} \left[\frac{\bar{\alpha}\,\theta\, z^{\theta-1} z'_{i:\lambda}}{1 - \bar{\alpha}(1 - z^\theta)}\right] = 0, \\ U_\xi &= \sum_{i=1}^{n} \left[\frac{h_R^\xi(x)}{h_R(x)}\right] + \sum_{i=1}^{n} \left[\frac{h_R(x)\, h_R^\xi(x)}{H_R(x)}\right] - 2\,\lambda^2 \sum_{i=1}^{n} \left[H_R(x)\, h_R(x)\, h_R^\xi(x)\right] = 0, \\ &+ (\theta - 1) \sum_{i=1}^{n} \left[\frac{z_{i:\xi}'}{z}\right] - 2 \sum_{i=1}^{n} \left[\frac{\bar{\alpha}\theta z^{\theta-1}}{1 - \bar{\alpha}(1 - z^\theta)} z_{i:\xi}'\right] = 0. \end{aligned}$$

where $h^\xi(x)$ means the derivative of the function h with respect to ξ.

8 Censoring

In the failure censoring scheme, the n experimental units are placed under observation in a typical life test and the number of uncensored observations r is predetermined. The data consist of observations are $x_{(1)}, x_{(2)}, ..., x_{(r)}$ the ordered lifetimes of these life testing items, this means that we have no information about $n-r$ survival item except that their lifetimes are greater than $x_{(r)}$. The experiment is terminated when the r^{th} item fails and remaining $n-r$ items are regarded as censored data. The likelihood function for $x_{(1)}, x_{(2)}, ..., x_{(r)}$ failed observations, (Cohen , 1965) is,

$$L = \frac{n!}{(n-r)!} \prod_{i=1}^{r} g(x_{(i)}) \ \left[\bar{G}(x_{(0)})\right]^{n-r}, \tag{8.1}$$

where T , r is time and number of survival and $g(x_{(i)})$, $\bar{G}(x_{(0)})$ are the pdf and sf of the base line distribution.

If T is fixed and r is a random variable the censoring is said to be single censored type I. And if "r" is fixed and "T" is a random variable the censoring is said to be single censored type II. When $x_{(0)} = T$ and $x_{(0)} = x_{(r)}$ in equation (8.1) we get the likelihood functions for censoring type I and type II respectively, if $r = n$ equation (8.1) turns out to be likelihood function for complete samples. Substituting the equations (2.2) and (2.1) in equation (8.1) the log likelihood function is.

$$\begin{aligned} L &= \log\frac{n!}{(n-r)!} + r\log 2 + r2\log\lambda + r\log\theta + \sum_{i=1}^{r}\log z' + \sum_{i=1}^{r}\log z \\ &- \sum_{i=1}^{r}\lambda^2 z^2 + (\theta-1)\sum_{i=1}^{r}\log\left(1-e^{-\lambda^2 z^2}\right) - 2\sum_{i=1}^{r}\log\left[1-\bar{\alpha}\left\{1-\left(1-e^{-\lambda^2 z^2}\right)^{\theta}\right\}\right] \\ &+ (n-r)\log\alpha + (n-r)\log\left[1-\left(1-e^{-\lambda^2 z_0^2}\right)^{\theta}\right] - (n-r)\log\left[1-\bar{\alpha}\left\{1-\left(1-e^{-\lambda^2 z_0^2}\right)^{\theta}\right\}\right] \end{aligned}$$

Taking derivative with respect to α, λ and θ we get following results.

$$\frac{\partial}{\partial\alpha}\log L = \frac{1}{\alpha} - 2\sum_{i=1}^{r}\left\{\frac{\left\{1-\left(1-e^{-\lambda^2 z_i^2}\right)^{\theta}\right\}}{\left[1-\bar{\alpha}\left\{1-\left(1-e^{-\lambda^2 z_i^2}\right)^{\theta}\right\}\right]}\right\} - (n-r)\left\{\frac{\left[1-\left(1-e^{-\lambda^2 z_0^2}\right)^{\theta}\right]}{\left\{1-\bar{\alpha}\left[1-\left(1-e^{-\lambda^2 z_0^2}\right)^{\theta}\right]\right\}}\right\}$$

$$\begin{aligned}\frac{\partial}{\partial\theta}\log L &= \frac{r}{\theta} + \sum_{i=1}^{r}\log\left(1-e^{-\lambda^2 z_i^2}\right) - 2\bar{\alpha}\sum_{i=1}^{r}\left\{\frac{\left(1-e^{-\lambda^2 z_i^2}\right)^{\theta}\log\left(1-e^{-\lambda^2 z_i^2}\right)}{1-\bar{\alpha}\left[1-\left(1-e^{-\lambda^2 z_i^2}\right)^{\theta}\right]}\right\} \\ &+ (n-r)\left\{\frac{\left(1-e^{-\lambda^2 z_0^2}\right)^{\theta}\log\left(1-e^{-\lambda^2 z_0^2}\right)}{1-\bar{\alpha}\left[1-\left(1-e^{-\lambda^2 z_0^2}\right)^{\theta}\right]}\right\} - (n-r)\bar{\alpha}\left\{\frac{\left(1-e^{-\lambda^2 z_0^2}\right)^{\theta}\log\left(1-e^{-\lambda^2 z_0^2}\right)}{1-\bar{\alpha}\left[1-\left(1-e^{-\lambda^2 z_0^2}\right)^{\theta}\right]}\right\}\end{aligned}$$

$$\begin{aligned}\frac{\partial}{\partial\lambda}\log L &= \frac{2r}{\lambda} - 2\lambda\sum_{i=1}^{r} z_i^2 + 2(\theta-1)\lambda\sum_{i=1}^{r}\left(\frac{z_i^2 e^{-\lambda^2 z_i^2}}{1-e^{-\lambda^2 z_i^2}}\right) - 4\bar{\alpha}\lambda\theta\sum_{i=1}^{r}\left\{\frac{z_i^2 e^{-\lambda^2 z_i^2}\left(1-e^{-\lambda^2 z_i^2}\right)^{\theta}}{1-\bar{\alpha}\left[1-\left(1-e^{-\lambda^2 z_i^2}\right)^{\theta}\right]}\right\} \\ &- (n-r)\left\{\frac{2\theta z_0^2 e^{-\lambda^2 z_0^2}\left(1-e^{-\lambda^2 z_0^2}\right)^{\theta}}{\left[1-\left(1-e^{-\lambda^2 z_0^2}\right)^{\theta}\right]}\right\} - (n-r)\left\{\frac{2\bar{\alpha}\lambda\theta z_0^2 e^{-\lambda^2 z_0^2}\left(1-e^{-\lambda^2 z_0^2}\right)^{\theta}}{1-\bar{\alpha}\left[1-\left(1-e^{-\lambda^2 z_0^2}\right)^{\theta}\right]}\right\}\end{aligned}$$

where $z_i = \log(1-R(x_i))$ and $z_0 = \log(1-R(x_0))$. Setting these equations equal to zero and solving these equations simultaneously yields the the maximum likelihood estimates.

9 Special sub-models

In this section, we present some special member of Marshall-Olkin Burr-X family (MOBX), namely the Marshall-Olkin Burr X-Weibull (MOBXW) , Marshall-Olkin Burr X-Frechet (MOBXFr), Marshall-Olkin Burr X-Burr(MOBXB) and Marshall-Olkin Burr X-Lomax (MOBXLx). We provide plots of the density and hazard rate functions for some parametric values to illustrate the flexibility of family of the distributions. One special models in the family is described with some details.Further we letting scale parametr $\lambda = 1$ for each submodel.

9.1 MOBX-Modified Weibull distribution

Let Modified Weibull distribution is the first parent distribution with cdf and pdf as $R(x) = 1 - \exp(-a\,x - b\,x^c)$ and $r(x) = (a + b\,c\,x^{c-1})\exp(-a\,x - b\,x^c)$ respectively. Then the cdf of MOBX-MW distribution is as under.

$$F(x) = \frac{\left\{1 - exp\left[-\left(a\,x + b\,x^c\right)^2\right]\right\}^{\theta}}{1-\bar{\alpha}\left\{1 - \left[1 - exp\left\{-(a\,x+b\,x^c)^2\right\}\right]^{\theta}\right\}} \tag{9.1}$$

The pdf corresponding to (9.1) is

$$f(x) = \frac{2\,a\,b\,\alpha\,\theta\,(a + b\,c\,x^{c-1})\,e^{-(a\,x + b\,x^c)^2}\,\left\{1 - exp\left[-\,(a\,x + b\,x^c)^2\right]\right\}^{\theta-1}}{\left\{1 - \bar{\alpha}\left[1 - \{1 - exp\,[-(a\,x + b\,x^c)^2]\}^{\theta}\right]\right\}^2} \tag{9.2}$$

where α is Marshall Olkin parameter, a , b is scale and c , θ are the shape parameters. A random variable with density (9.2) is denoted by $X \sim MOBXMW(\theta, \alpha, a, b, c)$.

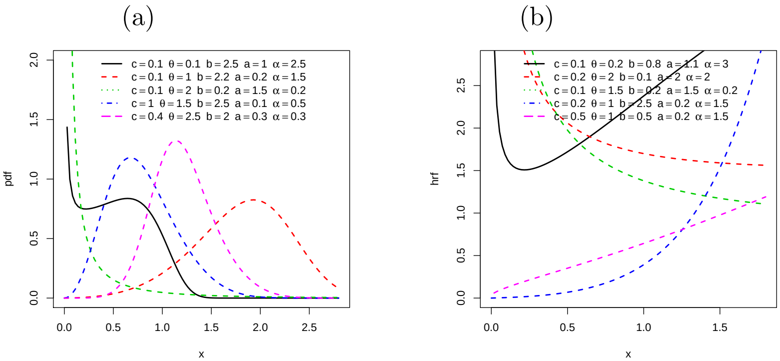


FIGURE 1. Plots of pdf and hrf for MOBXW distribution.

9.2 MOBX-Burr distribution

Let Burr distribution is the second parent distribution with cdf and pdf as $R(x) = 1 - (1 + x^a)^{-b}$ and $r(x) = a\,b\,x^{a-1}\,(1 + x^a)^{-b-1}$ respectively. Then the cdf of MOBXB is as under

$$F(x) = \frac{\left\{1 - e^{-[\log(1+x^a)^{-b}]^2}\right\}^{\theta}}{1 - \bar{\alpha}\left\{1 - e^{-[\log(1+x^a)^{-b}]^2}\right\}^{\theta}} \tag{9.3}$$

The pdf corresponding to (9.3) is as under

$$f(x) = \frac{2\,a\,b\,x^{a-1}\,\alpha\,\theta\,\log(1 + x^a)^{-b}\,e^{-[\log(1+x^a)^{-b}]^2}\,\left\{1 - e^{-[\log(1+x^a)^{-b}]^2}\right\}^{\theta-1}}{(1 + x^a)\,\left\{1 - \bar{\alpha}\left[1 - e^{-\{\log(1+x^a)^{-b}\}^2}\right]^{\theta}\right\}^2} \tag{9.4}$$

where α is Marshall Olkin parameter, a , b and $\theta$is the shape parameters. A random variable with density (9.4) is denoted by $X \sim MOBXB(\theta, \alpha, a, b)$.

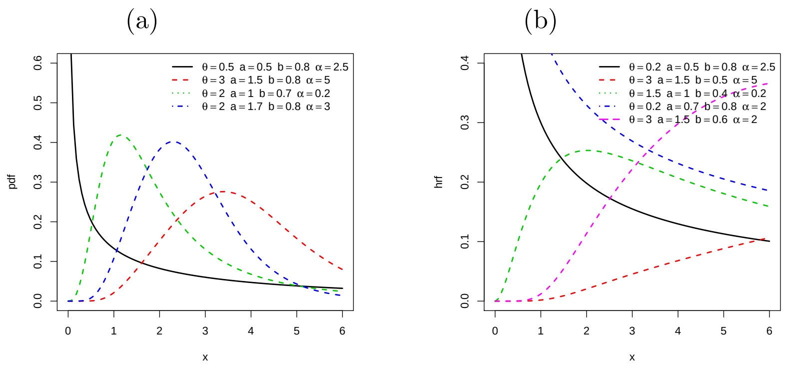


FIGURE 2. Plots of pdf and hrf for MOBXB distribution.

9.3 MOBX-Frechet distribution

Let Frechet distribution is the third parent distribution with cdf and pdf as $R(x) = \exp\left[-\left(\frac{x}{b}\right)^{-a}\right]$ and $r(x) = \frac{a}{b^a}\, x^{-a-1} \exp\left[-\left(\frac{x}{b}\right)^{-a}\right]$ respectively. Then the cdf of Marshall Olkin Burr X Frchet distribution is as under

$$F(x) = \frac{\left\{1 - e^{-\left[\log\left\{1-\exp\left[-\left(\frac{x}{b}\right)^{-a}\right]\right\}\right]^2}\right\}^{\theta}}{1 - \bar{\alpha}\left\{1 - e^{-\left[\log\left\{1-\exp\left[-\left(\frac{x}{b}\right)^{-a}\right]\right\}\right]^2}\right\}^{\theta}} \tag{9.5}$$

The pdf corresponding to (9.5) is becomes

$$\begin{aligned} f(x) \;&=\; \frac{2\,\theta\,\alpha\,a\,x^{-a-1}\,e^{\left[-\left(\frac{x}{b}\right)^{-a}\right]}\;\left[\log\left\{1 - \exp\left[-\left(\frac{x}{b}\right)^{-a}\right]\right\}\right]}{b^a\,\left\{1 - \exp\left[-\left(\frac{x}{b}\right)^{-a}\right]\right\}\,e^{\left[\log\left\{1-\exp\left[-\left(\frac{x}{b}\right)^{-a}\right]\right\}\right]^2}} \\ &\times\; \frac{\left\{1 - e^{-\left[\log\left\{1-\exp\left[-\left(\frac{x}{b}\right)^{-a}\right]\right\}\right]^2}\right\}^{\theta}}{1 - \bar{\alpha}\left\{1 - e^{-\left[\log\left\{1-\exp\left[-\left(\frac{x}{b}\right)^{-a}\right]\right\}\right]^2}\right\}^{\theta}} \end{aligned} \tag{9.6}$$

where α is a Marshall Olkin parameter,b is scale parameter and a and θ are the shape parameters. A random variable with density (9.6) is denoted by $X \sim MOBXFr(\theta, \alpha, a, b)$.

9.4 MOBX-Lomax distribution

Let Lomax distribution is the fourth parent distribution with cdf and pdf as $R(x) = 1 - (1 + \frac{x}{b})^{-a}$ and $r(x) = \frac{a}{b}\,(1 + \frac{x}{b})^{-a-1}$ respectively. Then the cdf of Marshall Olkin Burr X Lomax distribution

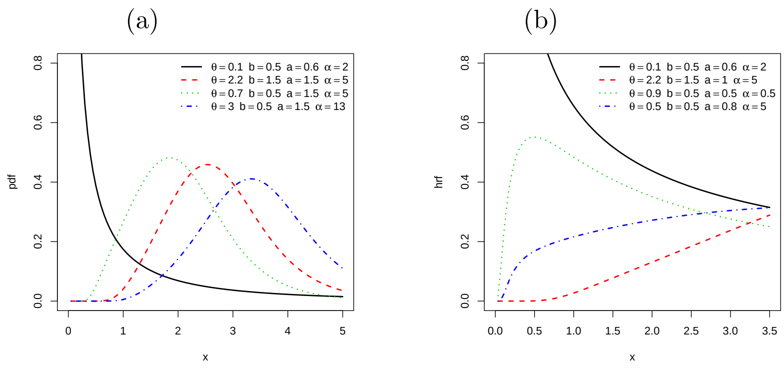


FIGURE 3. Plots of pdf and hrf for MOBXFr distribution.

is as under

$$F(x) = \frac{\left\{1 - e^{-\left[a\,\log(1+\frac{x}{b})\right]^2}\right\}^{\theta}}{1 - \bar{\alpha}\left\{1 - e^{-\left[a\,\log(1+\frac{x}{b})\right]^2}\right\}^{\theta}} \tag{9.7}$$

The pdf corresponding to (9.7) is given as

$$f(x) = \frac{2\,\theta\,\alpha\,a\,\log\left(1+\frac{x}{b}\right)\,e^{-\left[a\,\log(1+\frac{x}{b})\right]^2}\,\left\{1 - e^{-\left[a\,\log(1+\frac{x}{b})\right]^2}\right\}^{\theta-1}}{b\,(1+\frac{x}{b})\,\left\{1 - \bar{\alpha}\left[1 - e^{-\{a\,\log(1+\frac{x}{b})\}^2}\right]^{\theta}\right\}^2} \tag{9.8}$$

where α is a Marshall Olkin parameter,b is scale parameter and a and θ are the shape parameters. A random variable with density (9.8) is denoted by $X \sim MOBXLx(\theta, \alpha, a, b)$.

The hazard rate function can be obtained as

$$\gamma(x;\alpha) = \frac{2\,\theta\;a\,\log(1+\frac{x}{b})\,e^{-\left[a\,\log(1+\frac{x}{b})\right]^2}\;\left\{1 - e^{-\left[a\,\log(1+\frac{x}{b})\right]^2}\right\}^{\theta-1}}{(1+\frac{x}{b})\left\{1 - \left[1 - e^{-\{a\,\log(1+\frac{x}{b})\}^2}\right]^{\theta}\right\}\left\{1 - \bar{\alpha}\left[1 - \left\{1 - e^{-\left[a\,\log(1+\frac{x}{b})\right]^2}\right\}^{\theta}\right]\right\}} \tag{9.9}$$

The mixture representation of MOBXLx from equations (3.1) and (3.2) is

$$f(x) = \sum_{u,k=0}^{\infty} a_{u+k}(u+k+1)\frac{a}{b}\left(1+\frac{x}{b}\right)^{-a-1}\left[1 - \left(1+\frac{x}{b}\right)^{-a}\right]^{u+k}$$

$$F(x) = \sum_{u,k=0}^{\infty} a_{u+k+1}\left[1 - \left(1+\frac{x}{b}\right)^{-a}\right]^{u+k+1}$$

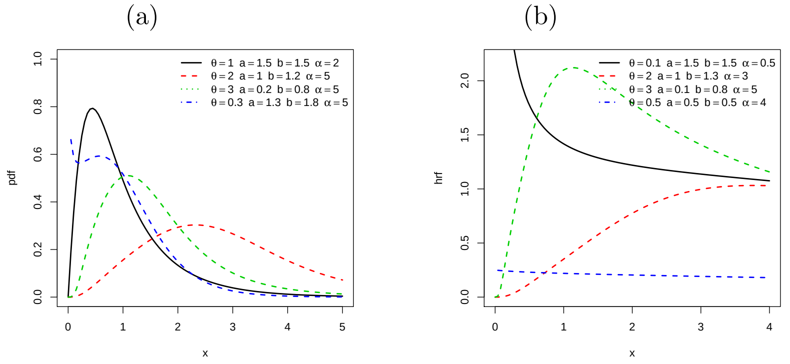


FIGURE 4. Plots of pdf and hrf for MOBXLx distribution.

The quantile function of MOBXLx from equations (2.4) is

$$Q_x(u) = b\left[(1-t)^{-\frac{1}{a}} - 1\right]$$

and $t = 1 - \exp\left\{-\ln\left[1 - \left(\frac{u\alpha}{1-\bar{\alpha}u}\right)^{\frac{1}{\theta}}\right]^{\frac{1}{2}}\right\}$.
The r^{th} moment of MOBXLx from equations (4.1)

$$\mu_r' = \sum_{u,k=0}^{\infty} a_{u+k}(u+k+1)\frac{a}{b}\sum_{j=0}^{\infty}\binom{u+k}{j}(-1)^j b^{r+1}\beta(r+1, a(j+1)-r).$$

The moment generating function can be obtained by equation (4.2)

$$M(t) = \sum_{u,k=0}^{\infty} a_{u+k}(u+k+1)\frac{a}{b}\sum_{j=0}^{\infty}\binom{u+k}{j}\sum_{I=0}^{\infty}\binom{a(j+1)-i}{i}\frac{(-1)^{2i+j+1}}{b^i}\frac{\Gamma(i+1)}{t^{i+1}}.$$

The rth Incomplete moments can be obtained by equation (4.3)

$$\mu_r^I = \sum_{u,k=0}^{\infty} a_{u+k}(u+k+1)\frac{a}{b}\sum_{j=0}^{\infty}\binom{u+k}{j}(-1)^j b^{r+1}\beta_{\frac{x}{b}}(r+1, a(j+1)-r).$$

The first Incomplete moment is obtain by Setting $r = 1$ we get

$$\mu_1^I = \sum_{u,k=0}^{\infty} a_{u+k}(u+k+1)\frac{a}{b}\sum_{j=0}^{\infty}\binom{u+k}{j}(-1)^j b^2\beta_{\frac{x}{b}}(2, a(j+1)-1)$$

or

$$\mu_r^I = \sum_{j=0}^{\infty} b_j\beta_{\frac{x}{b}}(2, a(j+1)-1).$$

Mean deviations about mean and median can be obtained as

$$D(\mu) = \mu\, F(\mu) - 2\sum_{j=0}^{\infty} b_j \beta_{\frac{\mu}{b}}(2, a(j+1)-1)$$

$$D(M) = \mu - 2\sum_{j=0}^{\infty} b_j \beta_{\frac{M}{b}}(2, a(j+1)-1)$$

Let b is the common parameter between two MOBXLx distributions such as MOBXLx$(\theta_1, \alpha_1, a_1, b)$ and MOBXLx$(\theta_2, \alpha_2, a_2, b)$. Then from equations (6.2) we have reliability.

$$R = \sum_{u,k=0}^{\infty}\sum_{n,p=0}^{\infty} a_{u+k}\, b_{n+p+1}\,(u+k+1)\, a_1 \sum_{i,j=0}^{\infty}\binom{u+k}{j}\binom{n+p+1}{i}(-1)^{i+j}\,\frac{b}{a_2\,(j+1)+a_1\, i}$$

We have the pdf of i^{th} order statistic from equation (7.1).

$$f_{i:n}(x) = \sum_{u,k=0}^{\infty}\sum_{n,p=0}^{\infty} V_{u,k,n,p}\,(u,k,n,p+1)\,\frac{a}{b}\sum_{q=0}^{\infty}\binom{u+k+n+p+1}{q}(-1)^q\left(1+\frac{x}{b}\right)^{-a\,(q+1)-1}$$

Let $X_1, X_2, ..., X_n$ be a sample of size n from X$\sim$MOBXLx, then the log-likelihood function from (7.3) can be expressed as

$$\begin{aligned}
\ell \;&=\; n\,\log\left(2\,\theta\,\alpha\,a^2\right) + \sum_{i=1}^{n}\log\left[\log\left(1+\frac{x}{b}\right)\right] - \sum_{i=1}^{n}\left(a\,\log\left(1+\frac{x}{b}\right)\right)^2 \\
&-\;(\theta-1)\sum_{i=1}^{n}\log z - n\,\log b - \sum_{i=1}^{n}\log\left(1+\frac{x}{b}\right) \\
&-\;2\sum_{i=1}^{n}\log\left[1-\bar{\alpha}z^{\theta}\right]
\end{aligned}$$

The components of score vector are.

$$\begin{aligned}
U_\theta \;&=\; \frac{n}{\theta} - \sum_{i=1}^{n}\log z - 2\sum_{i=1}^{n}\left[\frac{-\bar{\alpha}z^{\theta}\log z}{1-\bar{\alpha}z^{\theta}}\right] \\
U_\alpha \;&=\; \frac{n}{\alpha} - 2\sum_{i=1}^{n}\left[\frac{-z^{\theta}}{1-\bar{\alpha}z^{\theta}}\right] \\
U_a \;&=\; \frac{2\,n}{a} - 2a\sum_{i=1}^{n}\left[\lambda\log\left(1+\frac{x}{b}\right)\right]^2 - (\theta-1)\sum_{i=1}^{n}\frac{z'}{z} - 2\sum_{i=1}^{n}\frac{\theta z^{\theta} z'_{i:a}}{1-\bar{\alpha}z^{\theta}} \\
U_b \;&=\; \sum_{i=1}^{n}\left[\frac{\left(1+\frac{x}{b}\right)^{-1}}{b^2\,\log\left(1+\frac{x}{b}\right)}\right] - 2(a)^2\sum_{i=1}^{n}\frac{\left[\log\left(1+\frac{x}{b}\right)\right]^2}{b\left(1+\frac{x}{b}\right)} + \frac{n}{b^3}\sum_{i=1}^{n}\left(1+\frac{x}{b}\right)^{-1} - 2\sum_{i=1}^{n}\frac{\theta z^{\theta} z'_{i:b}}{1-\bar{\alpha}z^{\theta}}
\end{aligned}$$

where $z = 1 - e^{-\left(a\,\log\left(1+\frac{x}{b}\right)\right)^2}$

10 Simulation of the MOBXLx distribution

In this section, we study the performance of the method of MLE of MOBXLx as a submodel of the MOBX family by conducting various simulations for different sizes (n=50,100,250 and 500) by using R-Language. We simulate 1000 samples for the true parameters values I:a=1,b=1.5, θ=1, α=5, II :a=2,b=2, θ=1, α=1 and II :a=0.5,b=0.5, θ=1, α=1 in order to obtain average estimates (AEs), biases and mean square errors (MSEs) of the parameters. They are listed in Table 1. The MSE decreases as the sample size increases.The results indicate that the maximum likelihood method performs quite well in estimating the model parameters of the proposed distribution.

Table 1. Estimated AEs, Biases, and MSEs of the MLEs of parameters of MOBXLx distribution based on 1000 simulations of with n=50,100,n=250 and n=500.

		I			II			III		
n	parameters	A.E	Bias	MSE	A.E	Bias	MSE	A.E	Bias	MSE
50	a	1.351	0.351	2.128	2.176	0.176	0.356	0.557	0.057	0.024
	b	0.754	1.746	2.201	2.316	0.316	0.514	0.649	0.149	0.171
	θ	1.129	0.129	0.259	1.036	0.051	0.086	1.059	0.079	0.114
	α	4.551	0.449	1.034	1.143	0.143	0.544	1.085	0.085	0.168
100	a	0.820	0.180	0.419	2.080	0.080	0.159	0.518	0.018	0.009
	b	0.273	1.227	1.922	2.132	0.132	0.244	0.580	0.080	0.091
	θ	1.102	0.102	0.184	1.051	0.036	0.052	1.079	0.059	0.102
	α	4.640	0.360	0.021	1.128	0.128	0.434	1.004	0.019	0.118
250	a	0.629	0.071	0.152	2.028	0.028	0.060	0.507	0.008	0.003
	b	0.115	1.085	1.826	2.030	0.030	0.116	0.546	0.046	0.033
	θ	1.155	0.155	0.109	1.003	0.006	0.020	1.003	0.050	0.036
	α	4.703	0.297	0.014	1.106	0.106	0.173	0.995	0.005	0.051
500	a	0.616	0.054	0.146	1.992	0.008	0.032	0.492	0.007	0.001
	b	0.113	1.007	1.725	1.988	0.012	0.076	0.503	0.003	0.020
	θ	1.086	0.086	0.054	1.006	0.003	0.010	1.050	0.003	0.031
	α	4.728	0.272	0.007	1.061	0.061	0.105	0.981	0.004	0.045

11 Application

In this section, the flexibility of the MOBXLx model is illustrated by means of three real life data sets. The required numerical computations are carried out using the R software. The TTT and kernel density curve are plotted to identify the shape of the hazard function and density.

The first data set deals with the fatigue life of 6061-T6 aluminum coupons cut parallel with the direction of rolling and oscillated at 18 cycles per second given in Birnbaum and Saunders (1969). The data set consists of 101 observations with maximum stress per cycle 31,000 psi. The data are: 70, 90, 96, 97, 99, 100, 103, 104, 104, 105, 107, 108, 108, 108, 109, 109, 112, 112,113, 114, 114, 114, 116, 119, 120, 120, 120, 121, 121, 123, 124, 124, 124, 124, 124,128, 128, 129, 129, 130, 130, 130, 131, 131, 131, 131, 131, 132, 132, 132, 133, 134,134, 134, 134, 134, 136, 136, 137, 138, 138, 138, 139, 139, 141, 141, 142, 142, 142,142, 142, 142, 144, 144, 145, 146, 148, 148, 149, 151, 151, 152, 155, 156, 157, 157,157, 157, 158, 159, 162, 163, 163, 164, 166, 166, 168, 170, 174, 196, 212. This data have recently been used by Bourguignon et al.(2014), Mahmoudi (2014), Singh (2015), and Ghosh (2015).

We compare MOBXLx to Weibull log-logistic (WLL), Weibull-Lomax (WLx), Beta-inverted exponential (BIE), Gamma-Kumaraswamy (GKw), exponentiated Weibull-logarithmic (EWL), Burr type X (BX) and Lomax models. Their corresponding densities are:

$$\begin{aligned}
f_{WLL}(x) &= \frac{\lambda\theta\alpha\; a^{-\alpha}x^{\alpha-1}e^{-\lambda\left[\left(1+\left(\frac{x}{a}\right)^{\alpha}\right)-1\right]^{\theta}}\left[\left(1+\left(\frac{x}{a}\right)^{\alpha}\right)-1\right]^{\theta-1}}{1+\left(\frac{x}{a}\right)^{\alpha}}\\
f_{WLo}(x) &= \frac{\alpha a}{b}\left[1+\frac{x}{b}\right]^{a-1}\left[\left(1+\frac{x}{b}\right)^{a}-1\right]^{\alpha-1} exp\left[-\left[\left(1+\frac{x}{b}\right)^{a}-1\right]^{\alpha}\right]\\
f_{BIE}(x) &= \frac{\theta}{\beta(a,b)\,x^{2}}\; e^{-\frac{a\theta}{x}}\left[1-e^{-\frac{\theta}{x}}\right]^{b-1}\\
f_{EWL}(x) &= \frac{a\,\theta\,\alpha\,b^{\alpha}x^{\alpha-1}e^{-b\,x^{\alpha}}\left[1-e^{-bx^{\alpha}}\right]^{a-1}}{\log\left(1-\theta\right)\left[\theta\left(1-e^{-bx^{\alpha}}\right)^{a}-1\right]}
\end{aligned}$$

The second data set represents the annual flood discharge rates for the 39 years (1935-1973) at Floyd River located in James, Iowa, USA. The Floyd River data were reported by Mudholkar(1993) and Akinsete et al(2008). The data are: 1460, 4050, 3570, 2060, 1300, 1390, 1720, 6280, 1360, 7440, 5320, 1400, 3240, 2710, 4520, 4840, 8320, 13900, 71500, 6250, 2260, 318,1330, 970, 1920, 15100, 2870, 20600, 3810, 726, 7500, 7170, 2000, 829, 17300,4740, 13400, 2940, 5660. This data have also been analysed by Alzaatreh et al.(2015) and Tahir et al. (2015).
We compare MOBXLx with Kumaraswamy-Half-Cauchy (KwHC), beta -half-Cauchy (BHC), Poisson power Cauchy (PPC), Gamma-half-Cauchy (GH), half-Cauchy (HC) models their corresponding densities are:

$$\begin{aligned}
f_{KHC}(x) &= \frac{2^{a}\,a\,b}{\alpha\pi^{a}}\left[1+\left(\frac{x}{\alpha}\right)^{2}\right]^{-1}\left[\tan^{-1}\left(\frac{x}{\alpha}\right)\right]^{a-1}\left[1-\left(2\pi^{-1}\tan^{-1}\left(\frac{x}{\alpha}\right)\right)^{a}\right]^{b-1}\\
f_{BHC}(x) &= \frac{2^{a}}{\alpha\pi^{a}\beta(a,b)}\left[1+\left(\frac{x}{\alpha}\right)^{2}\right]^{-1}\left[\tan^{-1}\left(\frac{x}{\alpha}\right)\right]^{a-1}\left[1-2\pi^{-1}\tan^{-1}\left(\frac{x}{\alpha}\right)\right]^{b-1}\\
f_{HC}(x) &= \frac{2}{\pi\alpha}\left[1+\left(\frac{x}{\alpha}\right)^{2}\right]^{-1}\\
f_{GHC}(x) &= \frac{2}{\pi\alpha\Gamma(\lambda)\,\theta^{\lambda}}\left[1+\left(\frac{x}{\alpha}\right)^{2}\right]^{-1}\left[-\log\left[1-2\pi^{-1}\tan^{-1}\left(\frac{x}{\alpha}\right)\right]^{\lambda-1}\right]\\
&\times\left[1-2\pi^{-1}\tan^{-1}\left(\frac{x}{\alpha}\right)\right]^{\frac{1}{\theta}-1}\\
f_{PPC}(x) &= \frac{2\lambda\theta\left(\frac{x}{\alpha}\right)^{\theta-1}\left(2\pi^{-1}\tan^{-1}\left(\frac{x}{\alpha}\right)\right)^{\lambda-1}e^{-\left(2\pi^{-1}\tan^{-1}\left(\frac{x}{\alpha}\right)\right)^{\lambda}}}{\alpha\pi(1-e^{-1})\left[1+\left(\frac{x}{\alpha}\right)^{2\theta}\right]}
\end{aligned}$$

The third data set given by Abouammoh et al.(1994) which represent 40 patients suffering from leukemia. The data are: 115 ,181, 255, 418, 441, 461, 516, 739, 743 ,789 ,807, 865, 924, 983, 1024,1062, 1063,1165, 1191,1222,1251,1277, 1290 ,1357,1369, 1408 ,1455, 1478, 1222 1549, 1578,

1578, 1599, 1603, 1605, 1696, 1735, 1799, 1815 ,1852. This data have been used by Elbatal and Hiba (2014) and Elbatal (2013).
We compare MOBXLx with exponentiated generalized inverse Weibull (EGIW), Generalized inverse Weibull (GIW), Weibull Burr XII (WBXII), transmuted generalized linear exponential (TGLED), Kumaraswamy Lomax(KLx) models. Their corresponding densities are:

$$
\begin{aligned}
f_{TGLED}(x) &= \theta\,(a+bx)\left(ax+\frac{b}{2}x^2\right)^{\theta-1} e^{-\left(ax+\frac{b}{2}x^2\right)^{\theta}}\left[1-\lambda+2\lambda e^{-\left(ax+\frac{b}{2}x^2\right)^{\theta}}\right]\\
f_{KwLo}(x) &= \frac{a\,\lambda\,\theta}{b}\left(1+\frac{x}{b}\right)^{-a-1}\left(1-\left(1+\frac{x}{b}\right)^{-a}\right)^{\lambda-1}\left[1-\left(1-\left(1+\frac{x}{b}\right)^{-a}\right)^{\lambda}\right]^{\theta-1}\\
f_{WBXII}(x) &= \frac{\lambda\theta\,a\,b^{\alpha}x^{\alpha-1}}{1+\left(\frac{x}{b}\right)^{\alpha}}\left(\left(1+\frac{x^{\alpha}}{b}\right)^{a}-1\right)^{\theta-1}\exp\left[-\lambda\left(\left(1+\frac{x^{\alpha}}{b}\right)^{a}-1\right)^{\theta}\right]\\
f_{EGIW}(x) &= a\,b\,\theta\,\lambda^{\theta}x^{-\theta-1}e^{-\left(\frac{\lambda}{x}\right)^{\theta}}\left(1-e^{-\left(\frac{\lambda}{x}\right)^{\theta}}\right)^{a-1}\left[1-\left(1-e^{-\left(\frac{\lambda}{x}\right)^{\theta}}\right)^{a}\right]^{b-1}
\end{aligned}
$$

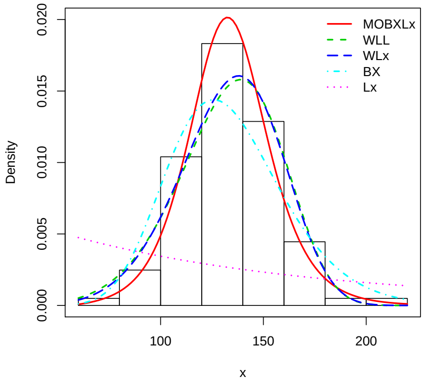
TABLE 2. MLEs and their standard errors (in parentheses) for data set 1.

Distribution	λ	θ	α	a	b
MOBXLx	-	149.089	147.088	3.694	42.565
	-	(59.900)	(46.890)	(9.693)	(26.594)
WLL	19.950	0.378	235.96	15.845	-
	(13.72)	(0.233)	(27.32)	(9.79)	-
WLx	-	-	9.804	0.360	24.413
	-	3.034	0.161	24.721	-
BIE	-	630.667	-	1.833	173.402
	-	(1.357)	-	(0.124)	(4.618)
EWL	3.05	0.138	-	5.14	0.009
	(1.355)	(1.199)	-	(6.305)	(0.002)
BX	-	-	-	9.273	0.0123
	-	-	-	1.663	0.00046

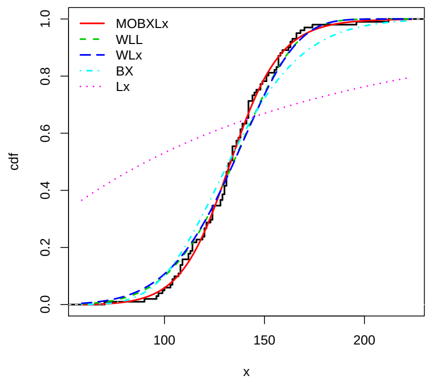
12 Concluding remarks

In this paper, the Marshall-Olkin Burr-X family of distributions is proposed and some of its mathematical properties are studied. One special models(MOBXLx) in the family is studies in detail. The maximum likelihood method is employed for estimating the model parameters. We fit the sub model MOBXLx to three real data sets to demonstrate its usefulness. The new model provides consistently a better fit than other competing models. We hope that the proposed model will attract wider applications in areas such as engineering, survival and lifetime data,hydrology, economics, among others.

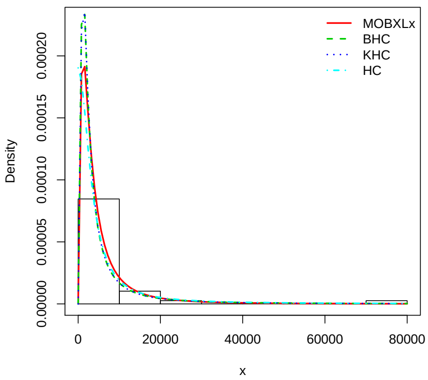
(a) Estimated pdf for data set 1.



(b) Estimated cdf for data set 1.



(c) Estimated pdf for data set 2.



(d) Estimated cdf for data set 2.

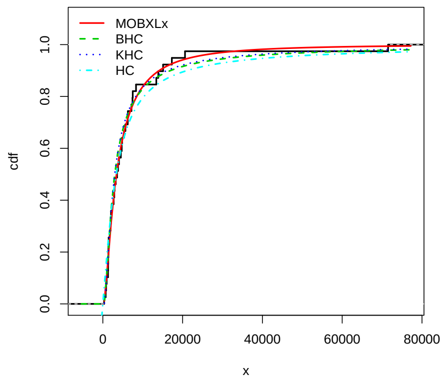
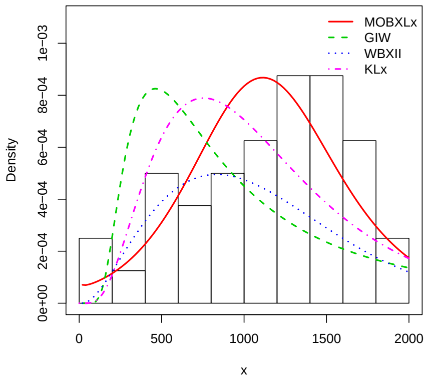


FIGURE 5. Estimated pdf's and cdf's for data set 1 and 2 .

TABLE 3. The AIC, A*, W* and K-S values for data set 1.

Distribution	$\hat{\ell}$	AIC	A	W	K-S	p-value(K-S)
MOBXLx	455.13	920.27	0.2492	0.0362	0.050	0.956
WLL	462.43	933.2	0.924	0.141	0.099	0.272
WLx	461.08	928.17	0.779	0.118	0.090	0.375
BIE	456.15	918.31	0.2813	0.0668	0.758	0.756
EWL	456.60	921.204	0.3417	0.0707	0.6942	0.665
BX	461.12	926.25	0.4262	0.0724	0.110	0.168

(e) Estimated pdf for data set 3.



(f) Estimated cdf for data set 3.

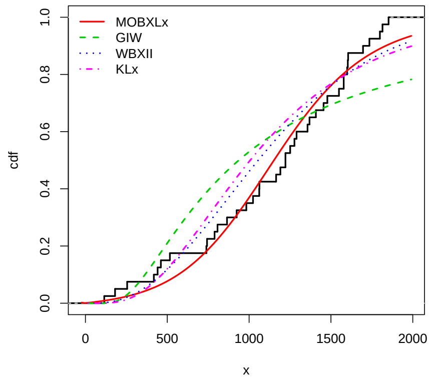


FIGURE 6. Estimated pdf's and cdf's for data set 3 .

TABLE 4. MLEs and their standard errors (in parentheses) for data set 2

Distribution	λ	θ	α	a	b
MOBXLx	-	8.878	1.916	0.016	23.020
	-	(2.532)	(3.009)	(0.010)	(10.615)
KHC	-	-	32.8645	0.4837	27.4076
	-	-	(9.5156)	(0.1112)	(7.8963)
BHC	-	-	27.7847	0.4196	21.3848
	-	-	(8.4903)	(0.0878)	(6.5972)
PPC	59.9368	0.8812	47.2306	-	-
	(1.7063)	(0.1108)	(1.5768)	-	-
GHC	45.9778	0.1554	4.4487	-	-
	(101.8881)	(0.1758)	(34.8257)	-	-
HC	-	-	3262.2630	-	-
	-	-	(661.1149)	-	-

Table 5. The AIC, CAIC , BIC, A* values for data set 2

Distribution	$\hat{\ell}$	AIC	A^*	W^*	K-S	P-value
MOBXLx	376.3736	762.7472	0.1698	0.0207	0.0646	0.9936
KHC	389.8590	785.7180	0.2852	0.0374	0.2886	0.0022
BHC	392.0631	790.1262	0.2650	0.0347	0.2796	0.0034
PPC	378.3070	762.6140	0.4023	0.0571	0.0816	0.9387
GHC	376.3683	775.8736	0.5121	0.0733	0.0648	0.9932
HC	379.6545	761.3090	0.4130	0.0611	0.1388	0.4029

Table 6. MLEs and their standard errors (in parentheses) for data set 3

Distribution	λ	θ	α	a	b
MOBXLx	-	0.954	73.569	0.139	302.791
	-	0.79	33.0	0.97	230.410
EGIW	1.102	0.453	-	0.933	1.09
	0.002	0.001	-	0.0090	3.742
GIW	3.261	0.298	-	-	0.792
	1.85	0.124	-	-	9.586
WBXII	50.531	6.456	0.915	0.127	31.063
	35.278	2.124	0.523	0.079	3.208
KLx	10.63	79.22	0.36	62.86	-
	17.27	91.21	-	0.068	4.631

Table 7. The AIC, A*, W* and K-S values for data set 3

Distribution	$\hat{\ell}$	AIC	A^*	W^*	K-S	p-value(K-S)
MOBXLx	305.3	620.7	0.960	0.149	0.106	0.7590
EGIW	354.7	717.4	3.012	0.525	0.835	0.0070
GIW	397.7	801.5	4.013	0.724	0.733	0.0081
WBXII	323.3	657.6	1.422	0.242	0.143	0.3470
KwLx	311.5	631.0	2.044	0.345	0.178	0.1570

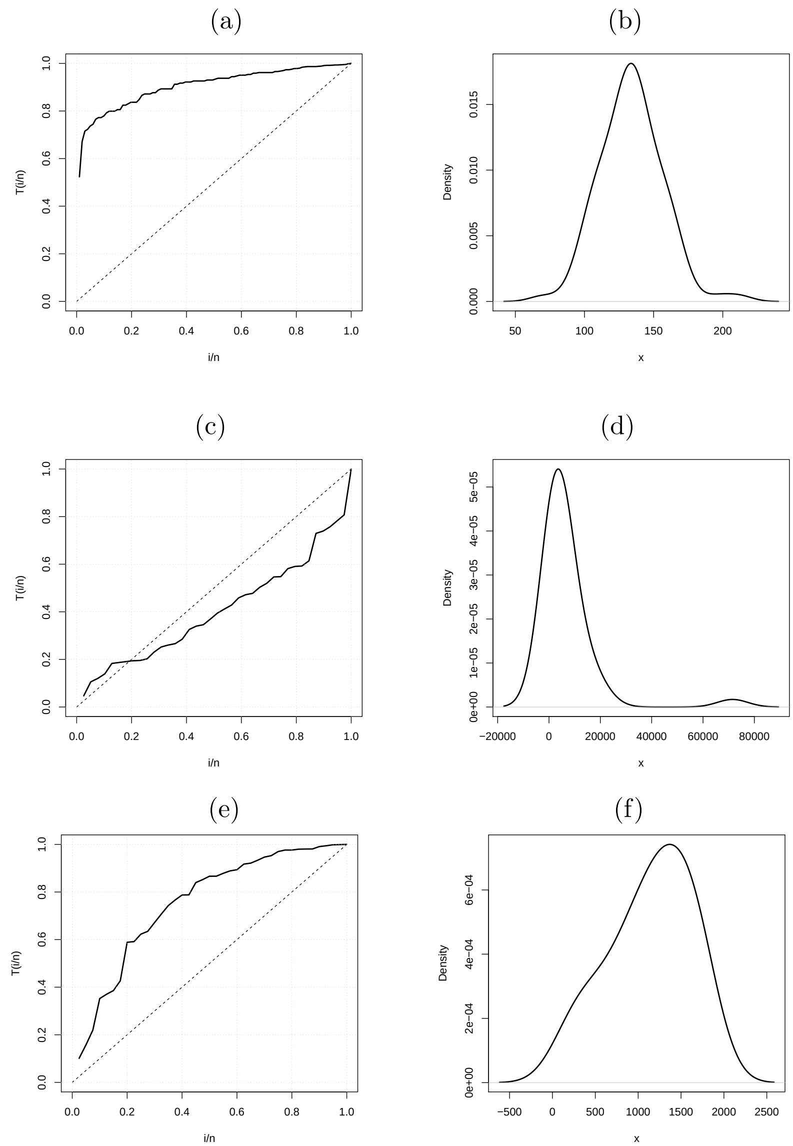


Figure 7. TTT and kernal density plots for data set 1,2 and 3 respectivly.

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