On double absolute factorable matrix summability

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Abstract

In this article a new result on $|A,p_m,q_n;\delta|_k$ summability of doubly infinite lower triangular matrix has been establised which generalizes a theorem of E. Savas and B.E. Rhoades and subsequently a theorem of Paikray et al. on summability factor of double infinite weighted mean matrix.

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1 Introduction

A doubly infinite matrix $A=(a_{mnjk})$ is said to be doubly triangular if $a_{mnjk}=0$ for $j>m$ or $k>n$. The mn^{th} th term of the A-transform of a double sequence $\{s_{mn}\}$ is defined by $T_{mn}=\sum_{\mu=0}^{n}\sum_{\nu=0}^{n}a_{mn\mu\nu}s_{\mu\nu}$.

For any double sequence u_{mn}, Δ_{11} is defined by $\Delta_{11}u_{mn}=u_{mn}-u_{m+1,n}-u_{m,n+1}+u_{m+1,n+1}$.

For any fourfold sequence v_{mnij}, $\Delta_{11}v_{mnij}=v_{mnij}-v_{m+1,n,i,j}-v_{m,n+1,i,j}+v_{m+1,n+1,i,j}$, $\Delta_{ij}v_{mnij}=v_{mnij}-v_{m,n,i+1,j}-v_{m,n,i,j+1}+v_{m,n,i+1,j+1}$, $\Delta_{0j}v_{mnij}=v_{mnij}-v_{m,n,i,j+1}$ and

$$\Delta_{i0}v_{mnij}:=v_{mnij}-v_{m,n,i+1,j}. \tag{1.1}$$

A double series $\sum\sum b_{mn}$, is said to be summable $|A|_k$, $k\geq 1$, [3] if

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{k-1}|\Delta_{11}T_{m-1,n-1}|^k<\infty \tag{1.2}$$

and is said to be summable $|A;\delta|_k$, $k\geq 1$, [2] if

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{\delta k+k-1}|\Delta_{11}T_{m-1,n-1}|^k<\infty. \tag{1.3}$$

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By taking $a_{mnjk} = \frac{p_{ij}}{P_{mn}}$, then the A transform of a double sequence reduces to the mn^{th} term of the double weighted mean transform of a double sequence $\{s_{mn}\}$ by

$$t_{mn} = \frac{1}{P_{mn}} \sum_{i=0}^{m} \sum_{j=0}^{n} p_{ij} s_{ij}, \ where \ P_{mn} = \sum_{i=0}^{m} \sum_{j=0}^{n} p_{ij}.$$

Further, a double infinite weighted mean matrix is said to be factorable [1], if there exist sequences $(p_m), (q_n)$ such that $p_{mn} = p_m q_n$ for every m and n.

A double series $\sum \sum b_{mn}$ is said to be summable $|\bar{N}, p_m, q_n|_k, k \geq 1$, [3] if

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left(\frac{mnp_m q_n}{P_m Q_n} \right)^{k-1} |\Delta_{11} t_{m-1,n-1}|^k < \infty \tag{1.4}$$

and the series $\sum \sum b_{mn}$ is summable $|A, p_m, q_n|_k, k \geq 1$, [4] if

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left(\frac{mnp_m q_n}{P_m Q_n} \right)^{k-1} |\Delta_{11} T_{m-1,n-1}|^k < \infty. \tag{1.5}$$

Similarly we define a double series $\sum \sum b_{mn}$ is said to be summable $|\bar{N}, p_m, q_n; \delta|_k, k \geq 1$, [2] if

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left(\frac{mnp_m q_n}{P_m Q_n} \right)^{\delta k + k - 1} |\Delta_{11} t_{m-1,n-1}|^k < \infty, \tag{1.6}$$

and the series $\sum \sum b_{mn}$ is summable $|A, p_m, q_n; \delta|_k, k \geq 1$, if

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left(\frac{mnp_m q_n}{P_m Q_n} \right)^{\delta k + k - 1} |\Delta_{11} T_{m-1,n-1}|^k < \infty. \tag{1.7}$$

Clearly, by taking $a_{mnij} = \frac{p_i q_j}{P_i Q_j}$, the $|A, p_m, q_n; \delta|_k$ summability reduces to$|\bar{N}, p_m, q_n; \delta|_k$ summability.

Associate with the matrix A, we consider two doubly triangular matrices $\bar{A}$ and $\hat{A}$ as follows:

$$\bar{a}_{mnij} = \sum_{\mu=i}^{m} \sum_{\nu=j}^{n} a_{mn\mu\nu} \ and \ \hat{a}_{m,n,i,j} = \Delta_{11} \bar{a}_{m-1,n-1,i,j} \quad m, n = 1, 2, ... \tag{1.8}$$

Note that $\hat{a}_{0000} = \bar{a}_{0000} = a_{0000}$.

Let y_{mn} denote the mn^{th} term of the A-transform of a factored doubly series $\sum_{\mu=0}^{m} \sum_{\nu=0}^{n} b_{\mu\nu\lambda_{\mu\nu}}$. Then we have

$$y_{mn} = \sum_{\mu=0}^{m}\sum_{\nu=0}^{n} a_{mn\mu\nu}\sum_{i=0}^{\mu}\sum_{j=0}^{\nu} b_{ij}\lambda_{ij} = \sum_{i=0}^{m}\sum_{j=0}^{n} b_{ij}\lambda_{ij}\sum_{\mu=i}^{m}\sum_{\nu=j}^{n} a_{mn\mu\nu} = \sum_{i=0}^{m}\sum_{j=0}^{n} b_{ij}\lambda_{ij}\bar{a}_{mnij},$$

and consequently we have,

$$\begin{aligned}
\Delta_{11}y_{m-1,n-1} &= y_{m-1,n-1} - y_{m,n-1} - y_{m-1,n} + y_{mn} \\
&= \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} b_{ij}\lambda_{ij}\bar{a}_{m-1,n-1,i,j} - \sum_{i=0}^{m}\sum_{j=0}^{n-1} b_{ij}\lambda_{ij}\bar{a}_{m,n-1,i,j} \\
&\qquad - \sum_{i=0}^{m-1}\sum_{j=0}^{n} b_{ij}\lambda_{ij}\bar{a}_{m-1,n,i,j} + \sum_{i=0}^{m}\sum_{j=0}^{n} b_{ij}\lambda_{ij}\bar{a}_{mnij} \\
&= \sum_{i=0}^{m}\sum_{j=0}^{n} b_{ij}\lambda_{ij}\hat{a}_{m,n,i,j} - \sum_{j=0}^{n-1} b_{mj}\lambda_{mj}\bar{a}_{m-1,n-1,m,j} \\
&\qquad - \sum_{i=0}^{m-1} b_{in}\lambda_{in}\bar{a}_{m-1,n-1,i,n} + \sum_{i=0}^{m} b_{in}\lambda_{in}\bar{a}_{m,n-1,i,n} + \sum_{j=0}^{n} b_{mn}\lambda_{mj}\bar{a}_{m-1,n,m,j} \\
&= \sum_{i=0}^{m}\sum_{j=0}^{n} b_{ij}\lambda_{ij}\hat{a}_{mnij}.
\end{aligned}$$

Since $\bar{a}_{m-1,n-1,m,j} = \bar{a}_{m-1,n-1,i,n} = \bar{a}_{m,n-1,i,n} = \bar{a}_{m-1,n,m,n} = 0$ and $b_{mn} = s_{m-1,n-1} - s_{m-1,n} - s_{m,n-1} + s_{mn}$, we have

$$\begin{aligned}\Delta_{11}y_{m-1,n-1} &= \sum_{i=0}^{m}\sum_{j=0}^{n}\hat{a}_{mnij}\lambda_{ij}(s_{i-1,j-1}-s_{i-1,j}-s_{i,j-1}+s_{ij})\\ &= \sum_{i=0}^{m-1}\sum_{j=0}^{n-1}\hat{a}_{m,n,i+1,j+1}\lambda_{i+1,j+1}s_{ij}-\sum_{i=0}^{m-1}\sum_{j=0}^{n}\hat{a}_{m,n,i+1,j+1}\lambda_{i+1,j}s_{ij}\\ &\qquad -\sum_{i=0}^{m}\sum_{j=0}^{n-1}\hat{a}_{m,n,i,j+1}\lambda_{i,j+1}s_{ij}+\sum_{i=0}^{m}\sum_{j=0}^{n}\hat{a}_{mnij}\lambda_{ij}s_{ij}\\ &= \sum_{i=0}^{m-1}\sum_{j=0}^{n-1}\Delta_{ij}(\hat{a}_{mnij}\lambda_{ij})s_{ij}-\sum_{i=0}^{m-1}\hat{a}_{m,n,i+1,n}\lambda_{i+1,n}s_{in}\\ &\qquad -\sum_{j=0}^{n-1}\hat{a}_{m,n,m,j+1}\lambda_{m,j+1,n+1}s_{mj}+\sum_{i=0}^{n}\hat{a}_{mnmj}\lambda_{m,j}s_{mj}+\sum_{i=0}^{m-1}\hat{a}_{mnin}\lambda_{in}s_{in}\\ &= \sum_{i=0}^{m-1}\sum_{j=0}^{n-1}\Delta_{ij}(\hat{a}_{mnij}\lambda_{ij})s_{ij}+\sum_{i=0}^{m-1}(\Delta_{i0}\hat{a}_{mnin}\lambda_{in})s_{in}\\ &\qquad +\sum_{j=0}^{n-1}(\Delta_{0j}\hat{a}_{mnmj}\lambda_{mj})s_{mj}+\hat{a}_{mnmn}\lambda_{mn}s_{mn}. \qquad (1.9)\end{aligned}$$

Further, we have,

$\Delta_{i0}\hat{a}_{mnin}\lambda_{in}=\lambda_{in}\Delta_{i0}\hat{a}_{mnin}+\hat{a}_{m,n,i+1,n}\Delta_{i0}\lambda_{in}$
and
$\Delta_{0j}\hat{a}_{mnmj}\lambda_{mj}=\lambda_{mj}\Delta_{0j}\hat{a}_{mnmj}+\hat{a}_{m,n,m,j+1}\Delta_{0j}\lambda_{mj}$.

Therefore,

$$\begin{aligned}\sum_{i=0}^{m-1}(\Delta_{i0}\hat{a}_{mnin}\lambda_{in})s_{in}+\sum_{j=0}^{n-1}(\Delta_{0j}\hat{a}_{mnmj}\lambda_{mj})s_{mj} &= \sum_{i=0}^{m-1}[\lambda_{in}\Delta_{i0}\hat{a}_{mnin}+\hat{a}_{m,n,i+1,n}\Delta_{i0}\lambda_{in}]s_{in}\\ &\qquad +\sum_{j=0}^{n-1}[\lambda_{mj}\Delta_{0j}\hat{a}_{mnmj}+\hat{a}_{m,n,m,j+1}\Delta_{0j}\lambda_{mj}]s_{mj}. \qquad (1.10)\end{aligned}$$

Lemma 1. let $\{u_{ij}\},\{v_{ij}\}$ be two double sequences. Then

$$\Delta_{ij}(u_{ij}v_{ij})=v_{ij}\Delta_{ij}u_{ij}+(\Delta_{0j}u_{i+1,j})(\Delta_{i0}v_{ij})+(\Delta_{i0}u_{i,j+1})(\Delta_{0j}v_{ij})+u_{i+1,j+1}\Delta_{ij}v_{ij} \qquad (1.11)$$

Proof. By simply expanding the right-hand side of (1.8) the result will be obtained.

2 Known result

E. Savaş and B.E. Rhoades [2] has proved the following result for $|\bar{N}, p_m, q_n|_k$ summability of double infinity series.

Theorem 1. Let $(p_m), (q_n)$ be sequence of positive numbers satisfying

$$(i)\ O(mnp_mq_n) = P_mQ_n\ as\ m,n \to \infty.$$

Let X_{mn} be a given double sequence of positive numbers and suppose that $s_{mn} = O(X_{mn})$, as $m,n \to \infty$. If λ_{mn} is a double sequence of complex numbers satisfying

$$(ii)\ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{p_mq_n}{P_mQ_n}(|\lambda_{mn}|X_{mn})^k = O(1),$$

$$(iii)\ \sum_{i=0}^{m-1}\sum_{j=0}^{n-1}|\Delta_{0j}\lambda_{ij}|X_{ij} = O(1),$$

$$(iv)\ \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}|\Delta_{i0}\lambda_{ij}|X_{ij} < \infty,$$

$$(v)\ \sum_{i=0}^{m-1}\sum_{j=0}^{n-1}|\Delta_{ij}\lambda_{ij}|X_{ij} = O(1),$$

$$(vi)\ \sum_{i=0}^{m}\sum_{j=0}^{n}(|\lambda_{ij}|X_{ij})^k = O(1),$$

then the series $\sum\sum b_{mn}\lambda_{mn}$ is summable $|\bar{N}, p_m, q_n|_k, k \geq 1$.

Extending theorem-1 for double absolute factorable matrix summability, Paikray et al. [4] established the following theorem:

Theorem 2. Let A be a doubly triangular matrix with non-negative entries satisfying the conditions

$$(i)\ \Delta_{11}a_{m-1,n-1,i,j} \geq 0$$

$$(ii)\ \sum_{v=0}^{n} a_{mniv} = \sum_{v=0}^{n-1} a_{m,n-1,i,v} = b(m,i),\ and\ \sum_{\mu=0}^{m} a_{mn\mu,j} = \sum_{\mu=0}^{m-1} a_{m-1,n,\mu,j} = a(n,j),$$

$$(iii)\ a_{mnij} \geq max\{a_{m,n+1,i,j}a_{m+1,n,i,j}\}\ for\ m \geq i, n \geq j,\ and\ i,j = 0,1,...,$$

$$(iv)\ \sum_{i=0}^{m}\sum_{j=0}^{n} a_{mnij} = O(1),$$

$$(v)\ \frac{mnp_mq_n}{P_mQ_n}a_{mnmn} = O(1),$$

Further, let $\{X_{mn}\}$ be a given double sequence of positive numbers and suppose that $\{s_{mn}\} = O(X_{mn})$ as $m, n \to \infty$. If $\{\lambda_{mn}\}$ is a double sequence of complex numbers satisfying

$$(vi)\ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mnmn}(|\lambda_{mn}|X_{mn})^k < \infty,$$

$$(vii)\ \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} |\Delta_{0j}\lambda_{ij}|X_{ij} = O(1),$$

$$(viii)\ \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} |\Delta_{i0}\lambda_{ij}|X_{ij} < \infty,$$

$$(ix)\ \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} |\Delta_{ij}\lambda_{ij}|X_{ij} = O(1),$$

$$(x)\ \sum_{i=0}^{m}\sum_{j=0}^{n} (|\lambda_{ij}|X_{ij})^k = O(1),$$

then the series $\sum\sum b_{mn}\lambda_{mn}$ is summable $|A, p_m, q_n|_k$, $k \geq 1$.

3 Main result

The aim of this article is to generalize theorem-2 for double absolute factorable matrix summability method $|A, p_m, q_n; \delta|_k$,$k \geq 1$.

Theorem 3. Let A be a doubly triangular matrix with non-negative entries satisfying the conditions

$$(i)\ \Delta_{11}a_{m-1,n-1,i,j} \geq 0,$$

$$(ii)\ \sum_{v=0}^{n} a_{mniv} = \sum_{v=0}^{n-1} a_{m,n-1,i,v} = b(m,i),\ and\ \sum_{\mu=0}^{m} a_{mn\mu,j} = \sum_{\mu=0}^{m-1} a_{m-1,n,\mu,j} = a(n,j),$$

$$(iii)\ \frac{mnp_mq_n}{P_mQ_n}a_{mnmn} = O(1)$$

$$(iv)\ a_{mnij} \geq max\{a_{m,n+1,i,j}a_{m+1,n,i,j}\}\ for\ m \geq i, n \geq j,\ and\ i,j = 0,1,...,$$

$$(v)\ \sum_{i=0}^{m}\sum_{j=0}^{n} a_{mnij} = O(1),$$

$$(vi)\ \sum_{m=i+1}^{M+1}\sum_{n=j+1}^{N+1} (mn)^{\delta k}|\Delta_{ij}\hat{a}_{mnij}| = O((ij)^{\delta k}a_{ijij}),$$

Further, let $\{X_{mn}\}$ be a given double sequence of positive numbers and suppose that $\{s_{mn}\} = O(X_{mn})$ as $m,n \to \infty$. Let $\{\lambda_{mn}\}$ be a double sequence of complex numbers such that

$$(vii)\ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{p_m q_n}{P_m Q_n}\right)^{\delta k} a_{mnmn}(|\lambda_{mn}|X_{mn})^k < \infty,$$

$$(viii)\ \sum_{i=0}^{m-1}\sum_{j=0}^{n-1}\left(\frac{p_m q_n}{P_m Q_n}\right)^{\delta k} |\Delta_{0j}\lambda_{ij}|X_{ij} = O(1),$$

$$(ix)\ \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\left(\frac{p_m q_n}{P_m Q_n}\right)^{\delta k} |\Delta_{i0}\lambda_{ij}|X_{ij} < \infty,$$

$$(x)\ \sum_{i=0}^{m-1}\sum_{j=0}^{n-1}\left(\frac{p_m q_n}{P_m Q_n}\right)^{\delta k} |\Delta_{ij}\lambda_{ij}|X_{ij} = O(1),$$

$$(xi)\ \sum_{i=0}^{m}\sum_{j=0}^{n}\left(\frac{p_m q_n}{P_m Q_n}\right)^{\delta k} (|\lambda_{ij}|X_{ij})^k = O(1).$$

Then the series $\sum\sum b_{mn}\lambda_{mn}$ is summable $|A, p_m, q_n; \delta|_k$, $k \geq 1, \delta \geq 0$.

Proof. In order to prove the theorem, using (1.7), it is necessary to show that

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{p_m q_n}{P_m Q_n}\right)^{\delta k+k-1} |\Delta_{11}y_{mn}| < \infty$$

From (1.11) we have ,

$$\Delta_{ij}(\hat{a}_{mnij}\lambda_{ij}) = \lambda_{ij}\Delta_{ij}(\hat{a}_{mnij}) + (\Delta_{0j}\hat{a}_{m,n,i+1,j})(\Delta_{i0}\lambda_{ij}) \\ (\Delta_{i0}\hat{a}_{m,n,i,j+1})(\Delta_{0j}\lambda_{ij}) + \hat{a}_{m,n,i+1,j+1}\Delta_{ij}\lambda_{ij}. \quad (3.1)$$

Then

$$\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}\Delta_{ij}(\hat{a}_{mnij}\lambda_{ij})s_{ij} = \sum_{i=0}^{m-1}\sum_{j=0}^{n-1}[\lambda_{ij}(\Delta_{ij}\hat{a}_{mnij}) + (\Delta_{0j}\hat{a}_{m,n,i+1,j})(\Delta_{i0}\lambda_{ij}) \\ +(\Delta_{i0}\hat{a}_{m,n,i,j+1})(\Delta_{0j}\lambda_{ij}) + \hat{a}_{m,n,i+1,j+1}(\Delta_{ij}\lambda_{ij})]s_{ij}. \quad (3.2)$$

Therefore, using (1.9), (1.10) and (3.2), we may, write $\Delta_{11}y_{m-1,n-1} = \sum_{r=1}^{9} T_r$.

From Minkowski's inequality, it is sufficient to show that

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{mnp_m q_n}{P_m Q_n}\right)^{\delta k+k-1} |T_r|^k < \infty,\ for\ r = 1, 2, ..., 9.$$

Using Hölder's inequality,

$$\begin{aligned} I_1 &= \sum_{m=1}^{M+1}\sum_{n=1}^{N+1}\left(\frac{mnp_mq_n}{P_mQ_n}\right)^{\delta k+k-1}|T_1|^k \\ &= O(1)\sum_{m=1}^{M+1}\sum_{n=1}^{N+1}\left(\frac{mnp_mq_n}{P_mQ_n}\right)^{\delta k+k-1}\left(\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}|\Delta_{ij}\hat{a}_{mnij}||\lambda_{ij}|X_{ij}\right)^k \\ &= O(1)\sum_{m=1}^{M+1}\sum_{n=1}^{N+1}\left(\frac{mnp_mq_n}{P_mQ_n}\right)^{\delta k+k-1}\left(\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}|\Delta_{ij}\hat{a}_{mnij}||\lambda_{ij}|^k|X_{ij}|^k\right)\left(\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}|\Delta_{ij}\hat{a}_{mnij}|\right)^{k-1}. \end{aligned}$$

From (1.8),

$$\begin{aligned} \hat{a}_{mnij} &= \Delta_{11}\bar{a}_{m-1,n-1,i,j} = \bar{a}_{m-1,n-1,i,j} - \bar{a}_{m,n-1,i,j} - \bar{a}_{m-1,n,i,j} + \bar{a}_{mnij} \\ &= \sum_{\mu=i}^{m-1}\sum_{\nu=j}^{n-1}a_{m-1,n-1,\mu,\nu} - \sum_{\mu=i}^{m}\sum_{\nu=j}^{n-1}a_{m,n-1,\mu,\nu} - \sum_{\mu=i}^{m-1}\sum_{\nu=j}^{n}a_{m-1,n,\mu,\nu} + \sum_{\mu=i}^{m}\sum_{\nu=j}^{n}a_{mn\mu\nu}. \end{aligned}$$

Since $a_{m-1,n,m,\nu} = a_{m,n-1,\mu,n} = 0$

Using (1.1) and property (ii)

$$\begin{aligned} \hat{a}_{mnij} &= \sum_{\mu=i}^{m}\sum_{\nu=j}^{n}(a_{m-1,n-1,\mu,\nu} - a_{m,n-1,\mu,\nu} - a_{m-1,n,\mu,\nu} + a_{m,n,\mu,\nu}) \\ &= \sum_{\mu=i}^{m-1}[b(m-1,\mu) - \sum_{\nu=0}^{j-1}a_{m-1,n-1,\mu,\nu} - b(m,\mu) + \sum_{\nu=0}^{j-1}a_{m,n-1,\mu,\nu} \\ &\quad -b(m-1,\mu) + \sum_{\nu=0}^{j-1}a_{m-1,n,\mu,\nu} + b(m,\mu) - \sum_{\nu=0}^{j-1}a_{m,n,\mu,\nu}] \\ &= \sum_{\mu=i}^{m-1}\sum_{\nu=j}^{n-1}(-a_{m-1,n-1,\mu,\nu} + a_{m,n-1\mu,\nu} + a_{m-1,n,\mu,\nu} - a_{m,n,\mu,\nu}) \\ &= \sum_{\nu=0}^{j-1}\sum_{\mu=i}^{m-1}(-a_{m-1,n-1,\mu,\nu} + a_{m,n-1\mu,\nu} + a_{m-1,n,\mu,\nu} - a_{m,n,\mu,\nu}) \\ &= \sum_{\nu=0}^{j-1}[-a(m-1,\nu) + \sum_{\mu=0}^{j-1}a_{m-1,n-1,\mu,\nu} + a(m,\nu) \\ &\quad -\sum_{\mu=0}^{i-1}a_{m,n-1,\mu,\nu} + a(m-1,\nu) - \sum_{\mu=0}^{i}a_{m-1,n,\mu,\nu} - a(m,\nu) + \sum_{\mu=0}^{i}a_{m,n,\mu,\nu}] \\ &= \sum_{\mu=0}^{i-1}\sum_{\nu=0}^{j-1}\Delta_{11}a_{m-1,n-1,\mu,\nu} \geq 0. \end{aligned} \tag{3.3}$$

Then using (1.1) and (3.3) we get,

$$\begin{aligned}\Delta_{ij}\hat{a}_{mnij} &= \left(\sum_{\mu=0}^{i-1}\sum_{\nu=0}^{j-1} - \sum_{\mu=0}^{i}\sum_{\nu=0}^{j-1} - \sum_{\mu=0}^{i-1}\sum_{\nu=0}^{j} + \sum_{\mu=0}^{i}\sum_{\nu=0}^{j}\right)\Delta_{11}a_{m-1,n-1,\mu,\nu} \\ &= -\sum_{\nu=0}^{j-1}\Delta_{11}a_{m-1,n-1,i,\nu} + \sum_{\nu=0}^{j}\Delta_{11}a_{m-1,n-1,i,\nu} = \Delta_{11}a_{m-1,n-1,i,j}. \qquad (3.4)\end{aligned}$$

Using condition (ii), we obtain

$$\begin{aligned}\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}\Delta_{ij}\hat{a}_{mnij} &= \sum_{i=0}^{m-1}\sum_{j=0}^{n-1}(a_{m-1,n-1,i,j} - a_{m,n-1,i,j} - a_{m-1,n,i,j} + a_{mnij}) \\ &= \sum_{i=0}^{m-1}(b(m-1,i) - b(m,i) - b(m-1,i) + a_{m-1,n,i,n} + b(m,i) - a_{mnin}) \\ &= \sum_{i=0}^{m-1}(a_{m-1,n,i,n} - a_{mnin}) = a(n,n) - a(n,n) + a_{mnmn}.\end{aligned}$$

Consequently using (iii) we get ,

$$\begin{aligned}I_1 &= O(1)\sum_{i=1}^{M}\sum_{j=1}^{N}(|\lambda_{ij}|X_{ij})^k \sum_{m=i+1}^{M+1}\sum_{n=j+1}^{N+1}\left(\frac{mnp_mq_n}{P_mQ_n}a_{mnmn}\right)^{k-1}\left(\frac{mnp_mq_n}{P_mQ_n}\right)^{\delta k}|\Delta_{ij}\hat{a}_{mnij}|. \\ &= O(1)\sum_{i=1}^{M}\sum_{j=1}^{N}(|\lambda_{ij}|X_{ij})^k\left(\frac{mnp_iq_j}{P_iQ_j}\right)^{\delta k}\sum_{m=i+1}^{M+1}\sum_{n=j+1}^{N+1}|\Delta_{ij}\hat{a}_{mnij}|.\end{aligned}$$

Thus finally, using condition(v) and (vi),

$$I_1 = O(1)\sum_{i=0}^{M}\sum_{j=0}^{N}\left(\frac{mnp_iq_j}{P_iQ_j}\right)^{\delta k} a_{ijij}(|\lambda_{ij}|X_{ij})^k = O(1).$$

Next, using Hölder's inequality,

$$\begin{aligned}I_2 &= \sum_{m=1}^{M+1}\sum_{n=1}^{N+1}\left(\frac{mnp_mq_n}{P_mQ_n}\right)^{\delta k+k-1}|T_2|^k \\ &= \sum_{m=1}^{M+1}\sum_{n=1}^{N+1}\left(\frac{mnp_mq_n}{P_mQ_n}\right)^{\delta k+k-1}\left|\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}(\Delta_{0j}\hat{a}_{m,n,i+1,j})(\Delta_{i0}\lambda_{ij})s_{ij}\right|^k \\ &= O(1)\sum_{m=1}^{M+1}\sum_{n=1}^{N+1}\left(\frac{mnp_mq_n}{P_mQ_n}\right)^{\delta k+k-1}\left[\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}|\Delta_{0j}\hat{a}_{m,n,i+1,j}|\Delta_{i0}\lambda_{ij}|X_{ij}\right] \\ &\quad\times\left[\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}|\Delta_{0j}\hat{a}_{m,n,i+1,j}|\Delta_{i0}\lambda_{ij}|X_{ij}\right]^{k-1}.\end{aligned}$$

By using (3.3) and property (ii) we have ,

$$\begin{aligned}
0 \leq \hat{a}_{m,n,i+1,j} &= \sum_{\mu=0}^{i}\sum_{\nu=0}^{j-1} \Delta_{11} a_{m-1,n-1,\mu,\nu} \\
&\leq \sum_{\mu=0}^{m-1}\sum_{\nu=0}^{n-1} (a_{m-1,n-1,\mu,\nu} - a_{m,n-1,\mu,\nu} - a_{m-1,n,\mu,\nu} + a_{m,n,\mu,\nu}) \\
&= \sum_{\mu=0}^{m-1} (b(m-1,\mu) - b(m,\mu) - b(m-1,\mu) + a_{m-1,n,\mu,n} + b(m,\mu) - a_{mn\mu\nu}) \\
&= \sum_{\mu=0}^{m-1} (a_{m-1,n,\mu,n} - a_{mn\mu\nu}) \\
&= a(n,n) - a(n,n) + a_{mnmn}.
\end{aligned}$$

Since $|\Delta_{0j}\hat{a}_{m,n,i+1,j}| \leq \hat{a}_{m,n,i+1,j} + \hat{a}_{m,n,i+1,j+1}$, using properties (viii) we get ,

$$\begin{aligned}
I_2 &= O(1) \sum_{m=1}^{M+1}\sum_{n=1}^{N+1} \left(\frac{mnp_m q_n}{P_m Q_n} a_{mnmn}\right)^{k-1} \left(\frac{mnp_m q_n}{P_m Q_n}\right)^{\delta k} \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} |\Delta_{0j}\hat{a}_{m,n,i+1,j}|\Delta_{i0}\lambda_{ij}|X_{ij} \\
&= O(1) \sum_{m=1}^{M}\sum_{n=1}^{N} |\Delta_{i0}\lambda_{ij}|X_{ij} \sum_{m=i+1}^{M+1}\sum_{n=j+1}^{N+1} \left(\frac{mnp_m q_n}{P_m Q_n}\right)^{\delta k} |\Delta_{0j}\hat{a}_{m,n,i+1,j}| \\
&= O(1) \sum_{m=1}^{M}\sum_{n=1}^{N} |\Delta_{i0}\lambda_{ij}|X_{ij} \sum_{m=i+1}^{M+1}\sum_{n=j+1}^{N+1} \left(\frac{mnp_m q_n}{P_m Q_n}\right)^{\delta k} (\hat{a}_{m,n,i+1,j} + \hat{a}_{m,n,i+1,j+1}) \\
&= O(1).
\end{aligned}$$

Similarly, we can prove that $I_3 = O(1)$.

Using Hölder's inequality,

$$\begin{aligned}
I_4 &= \sum_{m=1}^{M+1}\sum_{n=1}^{N+1} \left(\frac{mnp_m q_n}{P_m Q_n}\right)^{\delta k+k-1} |T_4|^k \\
&= O(1) \sum_{m=1}^{M+1}\sum_{n=1}^{N+1} \left(\frac{mnp_m q_n}{P_m Q_n}\right)^{\delta k+k-1} \left(\sum_{i=0}^{m-1}\sum_{j=0}^{n-1} |\hat{a}_{m,n,i+1,j+1}||\Delta_{ij}\lambda_{ij}|X_{ij}\right)^k \\
&= O(1) \sum_{m=1}^{M+1}\sum_{n=1}^{N+1} \left(\frac{mnp_m q_n}{P_m Q_n}\right)^{\delta k+k-1} \left[\sum_{i=0}^{m-1}\sum_{j=0}^{n-1} |\hat{a}_{m,n,i+1,j+1}||\Delta_{ij}\lambda_{ij}|X_{ij}\right] \\
&\qquad \times \left[\sum_{i=0}^{m-1}\sum_{j=0}^{n-1} |\hat{a}_{m,n,i+1,j+1}||\Delta_{ij}\lambda_{ij}|X_{ij}\right]^{k-1}.
\end{aligned}$$

From (3.3) and property (ii) we have ,

$$
\begin{aligned}
0 \leq \hat{a}_{m,n,i+1,j+1} &= \sum_{\mu=0}^{i}\sum_{\nu=0}^{j}\Delta_{11}a_{m-1,n-1,\mu,\nu}\\
&\leq \sum_{\mu=0}^{m-1}\sum_{\nu=0}^{n-1}(a_{m-1,n-1,\mu,\nu}-a_{m,n-1,\mu,\nu}-a_{m-1,n,\mu,\nu}+a_{m,n,\mu,\nu})\\
&= \sum_{\mu=0}^{m-1}(b(m-1,\mu)-b(m,\mu)-b(m-1,\mu)+a_{m-1,n,\mu,n}+b(m,\mu)-a_{mn\mu\nu})\\
&= \sum_{\mu=0}^{m-1}(a_{m-1,n,\mu,n}-a_{mn\mu\nu}).\\
&= a(n,n)-a(n,n)+a_{mnmn}.
\end{aligned}
$$

Thus using properties (ii),(iv) and (x) we get,

$$
\begin{aligned}
I_4 &= O(1)\sum_{m=1}^{M+1}\sum_{n=1}^{N+1}\left(\frac{mnp_mq_n}{P_mQ_n}a_{mnmn}\right)^{k-1}\left(\frac{mnp_mq_n}{P_mQ_n}\right)^{\delta k}\left[\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}|\hat{a}_{m,n,i+1,j+1}||\Delta_{ij}\lambda_{ij}|X_{ij}\right]\\
&\qquad\times\left[\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}|\Delta_{ij}\lambda_{ij}|X_{ij}\right]^{k-1}\\
&= O(1)\sum_{m=1}^{M+1}\sum_{n=1}^{N+1}\left(\frac{mnp_mq_n}{P_mQ_n}\right)^{\delta k}\left[\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}|\hat{a}_{m,n,i+1,j+1}||\Delta_{ij}\lambda_{ij}|X_{ij}\right]\\
&= O(1)\sum_{i=0}^{M}\sum_{j=0}^{N}|\Delta_{ij}\lambda_{ij}|X_{ij}\sum_{m=i+1}^{M+1}\sum_{n=j+1}^{N+1}\left(\frac{mnp_mq_n}{P_mQ_n}\right)^{\delta k}|\hat{a}_{m,n,i+1,j+1}|\\
&= O(1)\sum_{i=0}^{m-1}\sum_{j=0}^{N}\left(\frac{mnp_mq_n}{P_mQ_n}\right)^{\delta k}|\Delta_{ij}\lambda_{ij}|X_{ij}\\
&= O(1).
\end{aligned}
$$

Further,using (1.10) and Hölder's inequality,

$$\begin{aligned}
I_5 &= \sum_{m=1}^{M+1}\sum_{n=1}^{N+1}\left(\frac{mnp_mq_n}{P_mQ_n}\right)^{\delta k+k-1}|T_5|^k\\
&= \sum_{m=1}^{M+1}\sum_{n=1}^{N+1}\left(\frac{mnp_mq_n}{P_mQ_n}\right)^{\delta k+k-1}\left|\sum_{i=0}^{m-1}\lambda_{in}\Delta_{i0}\hat{a}_{mnin}s_{in}\right|^k\\
&= O(1)\sum_{m=1}^{M+1}\sum_{n=1}^{N+1}\left(\frac{mnp_mq_n}{P_mQ_n}\right)^{\delta k+k-1}\left(\sum_{i=0}^{m-1}\lambda_{in}|\Delta_{i0}\hat{a}_{mnin}|X_{in}\right)^k\\
&= O(1)\sum_{m=1}^{M+1}\sum_{n=1}^{N+1}\left(\frac{mnp_mq_n}{P_mQ_n}\right)^{\delta k+k-1}\left[\sum_{i=0}^{m-1}|\Delta_{i0}\hat{a}_{mnin}|(|\lambda_{in}|X_{in})^k\right]\times\left[\sum_{i=0}^{m-1}|\Delta_{i0}\hat{a}_{mnin}|\right]^{k-1}
\end{aligned}$$

From (1.8) we have,

$$\begin{aligned}
\Delta_{i0}\hat{a}_{mnin} &= \Delta_{i0}(\Delta_{11}\bar{a}_{m-1,n-1,i,n})\\
&= \Delta_{i0}(\bar{a}_{m-1,n-1,i,n}-\bar{a}_{m,n-1,i,n}-\bar{a}_{m-1,n,i,n}+\bar{a}_{mnin})\\
&= \Delta_{i0}\left(-\sum_{\mu=i}^{m-1}a_{m-1,n,v,n}+\sum_{\mu=i}^{m}a_{mn\mu n}\right)\\
&= a_{m-1,n,i,n}+a_{mnin}\leq 0.
\end{aligned}$$

Then, using property (ii) we get,

$$\begin{aligned}
\sum_{i=0}^{m-1}|\Delta_{i0}\hat{a}_{mnin}| &= \sum_{i=0}^{m-1}(a_{m-1,n-1,i,n}-a_{mnin})\\
&= a(n,n)-a(n,n)+a_{mnmn}.
\end{aligned}$$

Thus using property (iii), (vi) and (ix),

$$\begin{aligned}
I_5 &= O(1)\sum_{m=1}^{M+1}\sum_{n=1}^{N+1}\left(\frac{mnp_mq_n}{P_mQ_n}a_{mnmn}\right)^{k-1}\left(\frac{mnp_mq_n}{P_mQ_n}\right)^{\delta k}\left[\sum_{i=0}^{m-1}|\Delta_{i0}\hat{a}_{mnin}|(|\lambda_{in}|X_{in})^k\right]\\
&= O(1)\sum_{m=1}^{M+1}\sum_{n=1}^{N+1}\left(\frac{mnp_mq_n}{P_mQ_n}\right)^{\delta k}\left(\sum_{i=0}^{m-1}|\Delta_{i0}\hat{a}_{mnin}|(|\lambda_{in}|X_{in})^k\right)\\
&= O(1)\sum_{n=1}^{N+1}\sum_{i=0}^{M}(|\lambda_{in}|X_{in})^k\left(\sum_{i=0}^{m-1}\left(\frac{mnp_mq_n}{P_mQ_n}\right)^{\delta k}|\Delta_{i0}\hat{a}_{mnin}|\right)\\
&= O(1).
\end{aligned}$$

Again using Hölder's inequality

$$
\begin{aligned}
I_6 &= \sum_{m=1}^{M+1}\sum_{n=1}^{N+1}\left(\frac{mnp_mq_n}{P_mQ_n}\right)^{\delta k+k-1}|T_6|^k \\
&= \sum_{m=1}^{M+1}\sum_{n=1}^{N+1}\left(\frac{mnp_mq_n}{P_mQ_n}\right)^{\delta k+k-1}\left|\sum_{i=0}^{m-1}\hat{a}_{m,n,i+1,n}(\Delta_{i0}\lambda_{in})s_{in}\right|^k \\
&= O(1)\sum_{m=1}^{M+1}\sum_{n=1}^{N+1}\left(\frac{mnp_mq_n}{P_mQ_n}\right)^{\delta k+k-1}\left(\sum_{i=0}^{m-1}|\hat{a}_{m,n,i+1,n}||(\Delta_{i0}\lambda_{in})|X_{in}\right)^k \\
&= O(1)\sum_{m=1}^{M+1}\sum_{n=1}^{N+1}\left(\frac{mnp_mq_n}{P_mQ_n}\right)^{\delta k+k-1} \\
&\qquad \left[\sum_{i=0}^{m-1}|\hat{a}_{m,n,i+1,n}||(\Delta_{i0}\lambda_{in})|X_{in}\right]\left[\sum_{i=0}^{m-1}|\hat{a}_{m,n,i+1,n}||(\Delta_{i0}\lambda_{in})|X_{in}\right]^{k-1}
\end{aligned}
$$

Using (1.8), and condition (ii),

$$
\begin{aligned}
\hat{a}_{m,n,i+1,n} &= \bar{a}_{m-1,n-1,i+1,n}-\bar{a}_{m,n-1,i+1,n}-\bar{a}_{m-1,n,i+1,n}+\bar{a}_{m,n,i+1,n} \\
&= -\sum_{\mu=i+1}^{m-1}a_{m-1,n,\mu,n}+\sum_{\mu=i+1}^{m}a_{m,n,\mu,n} \\
&= -a(n,n)+\sum_{\mu=0}^{i}a_{m-1,n,\mu,n}+a(n,n)-\sum_{\mu=0}^{i}a_{m,n,\mu,n}\geq 0 \\
&\leq \sum_{\mu=0}^{m-1}(a_{m-1,n,\mu,n}-a_{m,n,\mu,n}) \\
&= a(n,n)-a(n,n)+a_{mnmn}.
\end{aligned}
$$

Thus, using condition (iii), (vii) and (ix)

$$\begin{aligned} I_6 &= O(1)\sum_{m=1}^{M+1}\sum_{n=1}^{N+1}\left(\frac{mnp_mq_n}{P_mQ_n}a_{mnmn}\right)^{k-1}\left(\frac{mnp_mq_n}{P_mQ_n}\right)^{\delta k}\left[\sum_{i=0}^{m-1}|\hat{a}_{m,n,i+1,n}||(\Delta_{i0}\lambda_{in})|X_{in}\right] \\ &\quad \times\left[\sum_{i=0}^{m-1}|\Delta_{i0}\lambda_{in}|X_{in}\right]^{k-1} \\ &= O(1)\sum_{m=1}^{M+1}\sum_{n=1}^{N+1}\left(\frac{mnp_mq_n}{P_mQ_n}\right)^{\delta k}\left[\sum_{i=0}^{m-1}|\hat{a}_{m,n,i+1,n}||(\Delta_{i0}\lambda_{in})|X_{in}\right] \\ &= O(1)\sum_{m=1}^{M}\sum_{n=1}^{N+1}|\Delta_{i0}\lambda_{in}|X_{in}\sum_{m=i+1}^{M+1}\left(\frac{mnp_mq_n}{P_mQ_n}\right)^{\delta k}|\hat{a}_{m,n,i+1,n}| \\ &= O(1)\sum_{m=1}^{M}\sum_{n=1}^{N+1}\left(\frac{mnp_mq_n}{P_mQ_n}\right)^{\delta k}|\Delta_{i0}\lambda_{in}|X_{in} \\ &= O(1). \end{aligned}$$

Using Hölder's inequality,

$$\begin{aligned} I_7 &= \sum_{m=1}^{M+1}\sum_{n=1}^{N+1}\left(\frac{mnp_mq_n}{P_mQ_n}\right)^{\delta k+k-1}|T_7|^k \\ &= \sum_{m=1}^{M+1}\sum_{n=1}^{N+1}\left(\frac{mnp_mq_n}{P_mQ_n}\right)^{\delta k+k-1}\left|\sum_{j=0}^{n-1}\lambda_{mj}(\Delta_{0j}\hat{a}_{mnmj})s_{mj}\right|^k \\ &= O(1)\sum_{m=1}^{M+1}\sum_{n=1}^{N+1}\left(\frac{mnp_mq_n}{P_mQ_n}\right)^{\delta k+k-1}\left(\sum_{j=0}^{n-1}|\lambda_{mj}||(\Delta_{0j}\hat{a}_{mnmj})|X_{mj}\right)^k \\ &= O(1)\sum_{m=1}^{M+1}\sum_{n=1}^{N+1}\left(\frac{mnp_mq_n}{P_mQ_n}\right)^{\delta k+k-1}\left[\sum_{j=0}^{n-1}|\Delta_{0j}\hat{a}_{mnmj}(|\lambda_{mj}||X_{mj})^k\right]\left[\sum_{j=0}^{n-1}|\Delta_{0j}\hat{a}_{mnmj}\right]^{k-1}. \end{aligned}$$

From (1.8),

$$\begin{aligned} \hat{a}_{mnmj} &= \bar{a}_{m-1,n-1,m,j}-\bar{a}_{m,n-1,m,j}-\bar{a}_{m-1,n,m,j}+\bar{a}_{m,n,m,j} \\ &= -\sum_{v=j}^{n-1}a_{m,n-1,m,j}+\sum_{v=j}^{n}a_{m,n,m,j}. \end{aligned}$$

Therefore,

$$\Delta_{0j}\hat{a}_{mnmj} = -a_{m,n-1,m,j}+a_{m,m,m,j},$$

and using properties (iv) and (ii),

$$\sum_{j=0}^{n-1}|\Delta_{0j}\hat{a}_{mnmj}| = \sum_{j=0}^{n-1}(a_{m,n-1,m,j} - a_{m,n,m,j})$$
$$= b(m,m) - b(m,m) + a_{mnmn}.$$

Using properties (iii), (vi) and (xi),

$$I_7 = O(1)\sum_{m=1}^{M+1}\sum_{n=1}^{N+1}\left(\frac{mnp_mq_n}{P_mQ_n}a_{mnmn}\right)^{k-1}\left(\frac{mnp_mq_n}{P_mQ_n}\right)^{\delta k}\sum_{j=0}^{n-1}|\Delta_{0j}\hat{a}_{mnmj}|(|\lambda_{mj}|X_{mj})^k$$
$$= O(1)\sum_{m=1}^{M+1}\sum_{n=1}^{N+1}(|\lambda_{mj}|X_{mj})^k\sum_{n=j+1}^{N+1}\left(\frac{mnp_mq_n}{P_mQ_n}\right)^{\delta k}|\Delta_{0j}\hat{a}_{mnmj}|$$
$$= O(1).$$

Using Hölder inequality,

$$I_8 = \sum_{m=1}^{M+1}\sum_{n=1}^{N+1}\left(\frac{mnp_mq_n}{P_mQ_n}\right)^{\delta k+k-1}|T_8|^k$$
$$= \sum_{m=1}^{M+1}\sum_{n=1}^{N+1}\left(\frac{mnp_mq_n}{P_mQ_n}\right)^{\delta k+k-1}\left|\sum_{j=0}^{n-1}\hat{a}_{m,n,m,j+1}(\Delta_{0j}\lambda_{mj})s_{mj}\right|^k$$
$$= O(1)\sum_{m=1}^{M+1}\sum_{n=1}^{N+1}\left(\frac{mnp_mq_n}{P_mQ_n}\right)^{\delta k+k-1}\left(\sum_{j=0}^{n-1}\hat{a}_{m,n,m,j+1}(\Delta_{0j}\lambda_{mj})X_{mj}\right)^k$$
$$= O(1)\sum_{m=1}^{M+1}\sum_{n=1}^{N+1}\left(\frac{mnp_mq_n}{P_mQ_n}\right)^{\delta k+k-1}\left[\sum_{j=0}^{n-1}\hat{a}_{m,n,m,j+1}(\Delta_{0j}\lambda_{mj})X_{mj}\right]$$
$$\times\left[\sum_{j=0}^{n-1}\hat{a}_{m,n,m,j+1}(\Delta_{0j}\lambda_{mj})X_{mj}\right]^{k-1}.$$

Using similar argument to that for the proof of I_6, and using properties (iii), (vii), and (ix), we get

$I_8 = O(1)$.

Finally using (1.7), properties (ii), (iii), (v) and (viii), and we that $\hat{a}_{mnmn} = a_{mnmn}$,

$$
\begin{aligned}
I_9 &= \sum_{m=1}^{M+1}\sum_{n=1}^{N+1}\left(\frac{mnp_mq_n}{P_mQ_n}\right)^{\delta k+k-1}|T_9|^k \\
&= O(1)\sum_{m=1}^{M+1}\sum_{n=1}^{N+1}\left(\frac{mnp_mq_n}{P_mQ_n}\right)^{\delta k+k-1}(a_{mnmn}|\lambda_{mn}|X_{mn})^k \\
&= O(1)\sum_{m=1}^{M+1}\sum_{n=1}^{N+1}(mna_{mnmn})^{k-1}\left(\frac{mnp_mq_n}{P_mQ_n}\right)^{\delta k}a_{mnmn}(|\lambda_{mn}|X_{mn})^k \\
&= O(1).
\end{aligned}
$$

This completes proof of theorem 2.

Conclusion. Taking $\delta = 0$, Theorem-3 reduces to Theorem-2 and in addition to this taking $p_m = 1$ and $q_n = 1$, Theorem-3 reduces to Theorem-2.

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