Fuzzy inner product spaces and fuzzy orthogonality

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Abstract

In this paper, we present two new fuzzy inner product spaces and investigate some basic properties of these spaces. Specially, we prove parallelogram law for two and three vectors. Also, we introduce a suitable notion of orthogonality.

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1 Introduction

The concept of fuzzy set, initially has been introduced by Zadeh [22]. Later, the fuzzy metric space was introduced by Kramosil and Michalek [14]. Also, George and Veeramani [9] modified the notion of fuzzy metric spaces. After that, the notions of fuzzy normed spaces and fuzzy numbers are studied in [19] and [18], respectively. Schweizer et al in [20], introduced the notion of probabilistic inner product spaces which can be considered as the generalization of inner product spaces. This notion has been modified in [1] and [23]. Induced norms of these spaces are important because they have many applications in quantum particle physics specially in connections with string and E-infinity theories; see [5], [6], [7] and [8]. Also, Su et al. [21] introduced the concept of a probabilistic Hilbert space in a special case. The fuzzy inner product spaces, or briefly, FIP-spaces are very close to the above mentioned spaces. For the first time, Biswas [3] and El-Abyad et al.,[4] simultaneously defined FIP-spaces. In [3], the authors showed that the fuzzy intersection of two FIP-spaces is again a FIP-space. Then, Kohli and Kumar [13] modified the definition of inner product space and fuzzy norm functions by Biswas. In fact, they showed that the definition of a FIP-space (fuzzy norm) in terms of the conjugate of a vector is redundant. They also introduced fuzzy co-inner product spaces and fuzzy co-norm functions. Other definitions of FIP-spaces can be found in [2], [12], [15], [16] and [17].

In this paper, we first bring some definitions and results from the theory of fuzzy normed spaces and FIP-spaces and then correct the definition of fuzzy inner product space which is introduced by Goudarzi and Vaezpour in [10]. Furthermore, we define an inner product on FIP-spaces and study some basic properties of these spaces. Finally, we introduce a suitable notion of orthogonality and investigate some important properties of this notion.

2 Notation and preliminary results

In this section, we state the usual terminology, notations and conventions of the theory of fuzzy normed spaces and FIP-spaces. We firstly recall some definitions in the fuzzy setting.

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Definition 2.1. ([22]) Let X be an arbitrary set. A fuzzy set M in X is a function with domain X and values in $[0,1]$.

Definition 2.2. ([20]) A binary opration $*:[0,1]\times[0,1]\longrightarrow[0,1]$ is said to be a *continuous triangular norm* (briefly, a continuous t-norm) if $*$ satisfies the following conditions:

(i) $*$ is commutative, associative and continuous;

(ii) $a*1=a$ for all $a\in[0,1]$;

(iii) $a*b\le c*d$ whenever $a\le c$ and $b\le d$ for all $a,b,c,d\in[0,1]$.

Also a t-norm $*$ is called *strong* if it has the following extra property: $a*b>0$, for all $a,b\in(0,1)$.

Typical examples of continuous t-norms are as follows:

- $T_M(a,b)$=min$\{a,b\}$ (minimum t-norm);
- $T_P(a,b)=ab$ (product t-norm).

Definition 2.3. [19] The 3-tuple $(X,N,*)$ is said to be a *fuzzy normed space* if X is a vector space, $*$ is a continuous t-norm and N is a fuzzy set on $X\times(0,\infty)$ such that the following conditions are satisfied for all $x,y\in X$ and $t,s>0$.

(i) $N(x,t)>0$;

(ii) $N(x,t)=1$ if and only if $x=0$;

(iii) $N(\alpha x,t)=N(x,\frac{t}{|\alpha|})$ for all $\alpha\neq 0$;

(iv) $N(x,t)*N(y,s)\le N(x+y,t+s)$;

(v) $N(x,\cdot):(0,\infty)\longrightarrow[0,1]$ is continuous;

(vi) $\lim_{t\longrightarrow\infty}N(x,t)=1$.

From now on, we consider the function H as $H(t)=\begin{cases}1 & \text{if } t>0\\ 0 & \text{if } t\le 0.\end{cases}$

Definition 2.4. [10] The 3-tuple $(X,F,*)$ is said to be a *fuzzy inner product space* (FIP-space) if X is a real vector space, $*$ is a continuous t-norm and F is a fuzzy set on $X^2\times\mathbb{R}$ such that the following conditions are satisfied for all $x,y,z\in X$ and $s,t,r\in\mathbb{R}$.

(FIP-1) $F(x,x,0)=0$ and $F(x,x,t)>0$, for each $t>0$;

(FIP-2) $F(x,y,t)=F(y,x,t)$;

(FIP-3) $F(x,x,t)\neq H(t)$ for some $t\in\mathbb{R}\Leftrightarrow x\neq 0$

(FIP-4) For any real number α,

$$F(\alpha x,y,t)=\begin{cases}F(x,y,\frac{t}{\alpha}) & \text{if } \alpha>0\\ H(t) & \text{if } \alpha=0\\ 1-F(x,y,\frac{t}{-\alpha}) & \text{if } \alpha<0\end{cases}$$

(FIP-5) $F(x,x,t) * F(y,y,s) \leq F(x+y,x+y,t+s)$;

(FIP-6) $\sup_{s+r=t}(F(x,z,s) * F(y,z,r)) = F(x+y,z,t)$;

(FIP-7) $F(x,y,\cdot): \mathbb{R} \longrightarrow [0,1]$ is continuous on $\mathbb{R}\backslash\{0\}$;

(FIP-8) $\lim_{t\longrightarrow\infty} F(x,y,t) = 1$.

Remark 2.1. One should note that the condition (FIP-4) in Definition 2.4 is not correct for $\alpha < 0$. So, we were unable to validate the results of [10] by using this definition. In other words, if $\alpha < 0$, then the condition (FIP-4) is true for $x, y \in X$. Let $x = y = 0$. By condition (FIP-3) we have $F(\alpha x, y, t) = H(t) = \begin{cases} 1 & \text{if } t > 0 \\ 0 & \text{if } t \leq 0 \end{cases}$. On the other hand, $1 - F(x, y, \frac{t}{-\alpha}) = 1 - H(\frac{t}{-\alpha}) = \begin{cases} 0 & \text{if } t > 0 \\ 1 & \text{if } t \leq 0 \end{cases}$ which is a contradiction. Thus, in view of the foregoing discussion, we improve the definition of $F(\alpha x, y, t)$ for condition (FIP-4) as follows:

(FIP-4) For any real number α,

$$F(\alpha x, y, t) = \begin{cases} F(x, y, \frac{t}{\alpha}) & \text{if } \alpha > 0 \\ H(t) & \text{if } \alpha = 0 \\ 1 - F(x, y, \frac{t}{\alpha}) & \text{if } \alpha < 0 \end{cases}$$

By this modification, all results in [10] are valid. This can be seen at a glance.

Let $(X, F, *)$ be a FIP-space. For each $z \in X$, put $\xi_z = \{F(x, z, \cdot) | x \in X\}$. Consider the addition and scalar multipication on ξ_z by

$$\alpha \odot F(x,z,t) = F(\alpha x, z, t) \quad F(x,z,t) \oplus F(y,z,t) = F(x+y,z,t) \quad (\alpha, t \in \mathbb{R}).$$

Theorem 2.2. [10] Let $(X, F, *)$ be a *FIP*-space. Then F is a continuous function on $X^2 \times \mathbb{R}\backslash\{0\}$.

Let $(X, F, *)$ be a FIP-space and N be its induced fuzzy norm defined by

$$N(x,t) = \begin{cases} F(x,x,t^2) & \text{if } t > 0 \\ 0 & \text{if } t \leq 0. \end{cases}$$

The sequence $\{u_n\} \subset X$ is called τ_F*-convergent* to $u_0 \in X$ (we write $u_n \xrightarrow{\tau_F} u_0$) if for given $\varepsilon > 0$ and $0 < \lambda < 1$, there exists a positive integer $N_0 = N_0(\varepsilon, \lambda)$ such that $N(u_n - u_0, \varepsilon) > 1 - \lambda$, whenever $n \geq N_0$. $(X, F, *)$ is called τ_F*-complete* if every τ_F*-Cauchy* sequence is convergent.

Definition 2.5. [11] Let $(X, F, *)$ be a FIP-space and $u, v \in X$. Then we say u and v are *fuzzy orthogonal* if $F(u, v, t) = H(t)$, for all $t \in \mathbb{R}$ and it is denoted by $u \perp v$.

We will use the next fundamental result frequently in this paper which is taken from [11].

Theorem 2.3. Let $(X, F, *)$ be a *FIP*-space. The orthogonality has the following properties:

(i) $0 \perp u \quad (\forall u \in X)$;

(ii) If $u \perp v$, then $v \perp u$ (symmetric);

(iii) If $u \perp u$, then $u = 0$;

(iv) If $u \perp u_i$ $(i = 1, 2, \cdots, n)$, then $u \perp (\sum_{i=1}^{n} u_i)$;

(v) If $u \perp v$, then for any $\alpha \in \mathbb{R}$, $u \perp \alpha v$;

(vi) If F is F-continuous, then $u_n \xrightarrow{\tau_F} u$ and $v \perp u_n$ $(n = 1, 2, \cdots)$ imply that $v \perp u$.

Lemma 2.6. [10] Let $(X, F, *)$ be a FIP-space. Then $F(x, y, t)$ is non decreasing with respect to t, for each $x, y \in X$.

Definition 2.7. [11] Let $(X, F, *)$ be a FIP-space and $M \subseteq X$. *The fuzzy orthogonal complement of* M, denoted by $M^{\perp}$, is the set of vectors in X, fuzzy orthogonal to every vector in M, i.e.

$$M^{\perp} := \{x \in X : x \perp m, \forall\, m \in M\} = \{x \in X : F(x, m, t) = H(t),\ \forall\, m \in M, \forall t \in \mathbb{R}\}.$$

Theorem 2.4. [10] Let $(X, F, *)$ be a *FIP*-space, where $*$ is a strong t-norm and for each $x, y \in X$, $\sup\{t \in \mathbb{R} : F(x, y, t) < 1\} < \infty$. Then $(X, \langle\cdot,\cdot\rangle)$ is an inner product space in which the function $\langle\cdot,\cdot\rangle : X \times X \longrightarrow \mathbb{R}$ is defined via

$$\langle x, y\rangle = \sup\{t \in \mathbb{R} : F(x, y, t) < 1\}.$$

It is shown in [10, Corollary 3.2.] that if we consider $\|x\| = \langle x, x\rangle^{\frac{1}{2}}$, then $(X, \|.\|)$ is a normed space.

3 Fuzzy inner product space

In this section, we define two new notions of inner product and investigate some properties in FIP-spaces and also prove the parallelogram law. In the upcoming resultant, in analogy with Theorem 2.4, we introduce a different inner product on a FIP-space and show that it is an inner product space.

Theorem 3.1. Let $(X, F, *)$ be a FIP-space with a continuous t-norm $*$ satisfying $t * t \geq t$ for all $t \in [0, 1]$. Also, suppose that for each $x, y \in X$, $\inf\{t \in \mathbb{R} : F(x, y, t) \geq \alpha\} < \infty$, for all $\alpha \in (0, 1)$ and $F(x, x, t) > 0$, for all $t > 0$ imply $x = 0$. Define $\langle\cdot,\cdot\rangle_\alpha : X \times X \longrightarrow \mathbb{R}$ by

$$\langle x, y\rangle_\alpha = \inf\{t \in \mathbb{R} : F(x, y, t) \geq \alpha\}.$$

Then, $(X, \langle\cdot,\cdot\rangle_\alpha)$ is an inner product space.

Proof. We show that $\langle\cdot,\cdot\rangle_\alpha$ satifies in inner product conditions. By (FIP-1) and Lemma 2.6, we have $F(x, x, t) = 0$ for all $t \leq 0$. Hence, for each $t \leq 0$ we get $0 = F(x, x, t) < \alpha$. It follows from the last relation that $\langle x, x\rangle_\alpha \geq 0$ for all $\alpha \in (0, 1)$ and all $x \in X$. If for $\alpha \in (0, 1)$, $\langle x, x\rangle_\alpha = 0$, then for each $t > 0$ we have $F(x, x, t) \geq \alpha > 0$ which implies that $x = 0$. Now, let $x = 0$. Then $F(x, x, t) = H(t)$. So,

$$\langle x, x\rangle_\alpha = \inf\{t \in \mathbb{R} : H(t) \geq \alpha\} = \inf\{t \in \mathbb{R} : H(t) = 1\} = \inf\{t \in \mathbb{R} : t > 0\} = 0.$$

Obviously, $\langle x,y\rangle_\alpha = \langle y,x\rangle_\alpha$. Let $\alpha\in(0,1)$ and $\varepsilon>0$. By (FIP-6) and assumptions, we obtain

$$\alpha \le \alpha*\alpha \le F(x,z,\langle x,z\rangle_\alpha + \frac{\varepsilon}{2}) * F(y,z,\langle y,z\rangle_\alpha + \frac{\varepsilon}{2}) \le F(x+y,z,\langle x,z\rangle_\alpha + \langle y,z\rangle_\alpha + \varepsilon).$$

Hence

$$\begin{aligned}\langle x+y,z\rangle_\alpha &= \inf\{t\in I\!R : F(x+y,z,t)\ge\alpha\}\\ &\le \{\langle x,z\rangle_\alpha + \langle y,z\rangle_\alpha + \varepsilon : F(x+y,z,\langle x,z\rangle_\alpha + \langle y,z\rangle_\alpha + \varepsilon)\ge\alpha\}.\end{aligned}$$

Since ε is arbitrary, the above inequality shows that

$$\langle x+y,z\rangle_\alpha \le \langle x,z\rangle_\alpha + \langle y,z\rangle_\alpha. \tag{3.1}$$

Also, let $A = 1-[B*C]$, where $B = 1-F(x,z,\langle x,z\rangle_\alpha - \frac{\varepsilon}{2})$ and $C = 1-F(y,z,\langle y,z\rangle_\alpha - \frac{\varepsilon}{2})$. By (FIP-4) and (FIP-6), we have

$$\begin{aligned}A &= 1-\Big[\Big(1-(1-F(-x,z,\frac{\varepsilon}{2}-\langle x,z\rangle_\alpha))\Big) * \Big(1-(1-F(-y,z,\frac{\varepsilon}{2}-\langle y,z\rangle_\alpha))\Big)\Big]\\ &\ge 1-F(-x-y,z,\varepsilon-\langle x,z\rangle_\alpha-\langle y,z\rangle_\alpha) = F(x+y,z,\langle x,z\rangle_\alpha+\langle y,z\rangle_\alpha-\varepsilon).\end{aligned} \tag{3.2}$$

Since $F(x,z,\langle x,z\rangle_\alpha - \frac{\varepsilon}{2}) < \alpha$, we have $F(y,z,\langle y,z\rangle_\alpha - \frac{\varepsilon}{2}) < \alpha$. Consequently, $B > 1-\alpha$, $C > 1-\alpha$ and hence $B*C > (1-\alpha)*(1-\alpha) \ge (1-\alpha)$. Therefore

$$A = 1-(B*C) < 1-(1-\alpha) = \alpha. \tag{3.3}$$

It follows from (3.2), (3.3)

$$F(x+y,z,\langle x,z\rangle_\alpha + \langle y,z\rangle_\alpha - \varepsilon) \le A < \alpha. \tag{3.4}$$

By (3.4), we get

$$\langle x+y,z\rangle_\alpha \ge \langle x,z\rangle_\alpha + \langle y,z\rangle_\alpha - \varepsilon.$$

Due to the arbitrarily of ε, we have

$$\langle x+y,z\rangle_\alpha \ge \langle x,z\rangle_\alpha + \langle y,z\rangle_\alpha. \tag{3.5}$$

The relations (3.1), (3.5) imply that

$$\langle x+y,z\rangle_\alpha = \langle x,z\rangle_\alpha + \langle y,z\rangle_\alpha. \tag{3.6}$$

We now show that $\langle kx,y\rangle_\alpha = k\langle x,y\rangle_\alpha$ for all $k\in\mathbb{R}$.
case 1: $k>0$.

$$\begin{aligned}\langle kx,y\rangle_\alpha &= \inf\{t\in\mathbb{R} : F(kx,y,t)\ge\alpha\} = \inf\{t\in\mathbb{R} : F(x,y,\frac{t}{k})\ge\alpha\}\\ &= \inf\{kt_1\in\mathbb{R} : F(x,y,t_1)\ge\alpha\} = k\langle x,y\rangle_\alpha.\end{aligned}$$

case 2: $k = 0$. By (FIP-4) for $\alpha \in (0,1)$, we have

$$\begin{aligned}\langle kx, y\rangle_\alpha &= \inf\{t \in \mathbb{R} : H(t) \geq \alpha\} = \inf\{t \in \mathbb{R} : H(t) = 1\}\\ &= \inf\{t \in \mathbb{R} : t > 0\} = 0 = k\langle x, y\rangle_\alpha.\end{aligned}$$

case 3: $k < 0$. By case 1, case 2 and (3.6), we get

$$0 = \langle 0, y\rangle_\alpha = \langle kx - kx, y\rangle_\alpha = \langle kx, y\rangle_\alpha + \langle -kx, y\rangle_\alpha.$$

So, $-\langle kx, y\rangle_\alpha = \langle -kx, y\rangle_\alpha = -k\langle x, y\rangle_\alpha$ and thus $\langle kx, y\rangle_\alpha = k\langle x, y\rangle_\alpha$. Therefore, $(X, \langle\cdot,\cdot\rangle_\alpha)$ is an inner product space. This finishes the proof. Q.E.D.

We note that in Theorem 3.1, minimum t-norm satisfy in the condition $t * t \geq t$ for all $t \in [0,1]$. On the other hand, the above theorem shows that there are plenty of inner products in term of $\alpha \in (0,1)$. The following corollary is an immediate consequence of Theorem 3.1.

Corollary 3.1. By hypotheses of Theorem *3.1*, if we define $\|x\|_\alpha = \langle x, x\rangle_\alpha^{\frac{1}{2}}$, then $(X, \|.\|_\alpha)$ is a normed space.

Theorem 3.2. Let $(X, F, *)$ be a FIP-space with a continuous t-norm $*$ satisfying $t * t \geq t$ for all $t \in [0,1]$. Suppose that

(i) $\inf\{t \in \mathbb{R} : F(x_{11}, x_{12}, t) * F(x_{21}, x_{22}, t) * \cdots * F(x_{n1}, x_{n2}, t) \geq \alpha\} < \infty$ for all $x_{ij} \in X$, $(i = 1, \cdots, n,\ j = 1, 2)$ and all $\alpha \in (0,1)$;

(ii) for all $\alpha \in (0,1)$, $F(x, x, t) * F(x, x, t) * \cdots * F(x, x, t) > 0$ necessities that $x = 0$.

Define

$$\begin{aligned}&\langle x_{11}, x_{12}\rangle_\alpha + \langle x_{21}, x_{22}\rangle_\alpha + \cdots + \langle x_{n1}, x_{n2}\rangle_\alpha\\ &\qquad = \inf\{t \in \mathbb{R} : F(x_{11}, x_{12}, t) * F(x_{21}, x_{22}, t) * \cdots * F(x_{n1}, x_{n2}, t) \geq \alpha\}.\end{aligned}$$

and

$$\begin{aligned}&\langle x_{11}, x_{12}\rangle_\alpha - \langle x_{21}, x_{22}\rangle_\alpha - \cdots - \langle x_{n1}, x_{n2}\rangle_\alpha\\ &\qquad = \inf\{t \in \mathbb{R} : F(x_{11}, x_{12}, t) * F(-x_{21}, x_{22}, t) * \cdots * F(-x_{n1}, x_{n2}, t) \geq \alpha\}.\end{aligned}$$

Then, $(X, \langle\cdot,\cdot\rangle_\alpha + \langle\cdot,\cdot\rangle_\alpha + \cdots + \langle\cdot,\cdot\rangle_\alpha)$ is an inner product space.

Proof. We check the properties of being inner product for $\langle\cdot,\cdot\rangle_\alpha + \langle\cdot,\cdot\rangle_\alpha$. The case of n-tuple is similar.

(a) We show that for each $\alpha \in (0,1)$ and $x \in X$, $\langle x, x\rangle_\alpha + \langle x, x\rangle_\alpha \geq 0$ and also $\langle x, x\rangle_\alpha + \langle x, x\rangle_\alpha = 0$ if and only if $x = 0$. By (FIP-1) and lemma 2.6, we get $F(x, x, t) = 0$ for all $t \leq 0$. Hence, $0 = F(x, x, t) < \alpha$ for all $t \leq 0$. So, $F(x, x, t) * F(x, x, t) < \alpha * \alpha < \alpha * 1 = \alpha$ for all $t \leq 0$. Thus,

$$\langle x, x\rangle_\alpha + \langle x, x\rangle_\alpha = \inf\{t \in \mathbb{R} : F(x, x, t) * F(x, x, t) \geq \alpha\} \geq 0.$$

Assume that for $\alpha \in (0,1)$, $\langle x, x\rangle_\alpha + \langle x, x\rangle_\alpha = 0$. So, $F(x, x, t) * F(x, x, t) \geq \alpha > 0$ for all $t \leq 0$. Therefore $x = 0$. Now let $x = 0$.

$$\begin{aligned}\langle x, x\rangle_\alpha + \langle x, x\rangle_\alpha &= \inf\{t \in \mathbb{R} : H(t) * H(t) \geq \alpha\} = \inf\{t \in \mathbb{R} : H(t) * H(t) = 1\}\\ &= \inf\{t \in \mathbb{R} : H(t) = 1\} = \inf\{t \in \mathbb{R} : t > 0\} = 0.\end{aligned}$$

(b) It is clear that $\langle x,y\rangle_\alpha + \langle z,w\rangle_\alpha = \langle y,x\rangle_\alpha + \langle w,z\rangle_\alpha$.

(c) We prove that

$$\langle x_1+x_2,y\rangle_\alpha + \langle x_3+x_4,y\rangle_\alpha = \langle x_1,y\rangle_\alpha + \langle x_2,y\rangle_\alpha + \langle x_3,y\rangle_\alpha + \langle x_4,y\rangle_\alpha.$$

We have

$$\begin{aligned}&\langle x_1,y\rangle_\alpha + \langle x_2,y\rangle_\alpha + \langle x_3,y\rangle_\alpha + \langle x_4,y\rangle_\alpha\\ &\qquad = \inf\{\tfrac{t}{2}\in\mathbb{R} : F(x_1,y,\tfrac{t}{2}) * F(x_2,y,\tfrac{t}{2}) * F(x_3,y,\tfrac{t}{2}) * F(x_4,y,\tfrac{t}{2}) \geq \alpha\}.\end{aligned}$$

By (FIP-6), we obtain

$$\alpha \leq F(x_1,y,\frac{t}{2}) * F(x_2,y,\frac{t}{2}) * F(x_3,y,\frac{t}{2}) * F(x_4,y,\frac{t}{2}) \leq F(x_1+x_2,y,t) * F(x_3+x_4,y,t). \tag{3.7}$$

By lemma 2.6 and inequality (3.7), we arrive at

$$\begin{aligned}&\inf\{\tfrac{t}{2}\in\mathbb{R} : F(x_1,y,\tfrac{t}{2}) * F(x_2,y,\tfrac{t}{2}) * F(x_3,y,\tfrac{t}{2}) * F(x_4,y,\tfrac{t}{2}) \geq \alpha\}\\ &\qquad \geq \inf\{t\in\mathbb{R} : F(x_1+x_2,y,t) * F(x_3+x_4,y,t) \geq \alpha\}\end{aligned}$$

So,

$$\langle x_1,y\rangle_\alpha + \langle x_2,y\rangle_\alpha + \langle x_3,y\rangle_\alpha + \langle x_4,y\rangle_\alpha \geq \langle x_1+x_2,y\rangle_\alpha + \langle x_3+x_4,y\rangle_\alpha. \tag{3.8}$$

On the other hand,

$$\begin{aligned}&\langle x_1+x_2,y\rangle_\alpha - \langle x_1,y\rangle_\alpha + \langle x_3+x_4,y\rangle_\alpha - \langle x_3,y\rangle_\alpha\\ &= \inf\{\frac{t}{2}\in\mathbb{R} : [F(x_1+x_2,y,\frac{t}{2}) * F(-x_1,y,\frac{t}{2})] * [F(x_3+x_4,y,\frac{t}{2}) * F(-x_3,y,\frac{t}{2})] \geq \alpha\}\end{aligned}$$

By (FIP-6), we get

$$\begin{aligned}\alpha &\leq [F(x_1+x_2,y,\frac{t}{2}) * F(-x_1,y,\frac{t}{2})] * [F(x_3+x_4,y,\frac{t}{2}) * F(-x_3,y,\frac{t}{2})]\\ &\leq F(x_2,y,t) * F(x_4,y,t).\end{aligned} \tag{3.9}$$

By lemma 2.6 and the inequality (3.9), we deduce that

$$\begin{aligned}&\inf\{\frac{t}{2}\in\mathbb{R} : [F(x_1+x_2,y,\frac{t}{2}) * F(-x_1,y,\frac{t}{2})] * [F(x_3+x_4,y,\frac{t}{2}) * F(-x_3,y,\frac{t}{2})] \geq \alpha\}\\ &\quad \geq \inf\{t\in\mathbb{R} : F(x_2,y,t) * F(x_4,y,t) \geq \alpha\}.\end{aligned}$$

Thus

$$\langle x_1,y\rangle_\alpha + \langle x_2,y\rangle_\alpha + \langle x_3,y\rangle_\alpha + \langle x_4,y\rangle_\alpha \leq \langle x_1+x_2,y\rangle_\alpha + \langle x_3+x_4,y\rangle_\alpha. \tag{3.10}$$

It follows from (3.8) and (3.10) that

$$\langle x_1,y\rangle_\alpha + \langle x_2,y\rangle_\alpha + \langle x_3,y\rangle_\alpha + \langle x_4,y\rangle_\alpha = \langle x_1+x_2,y\rangle_\alpha + \langle x_3+x_4,y\rangle_\alpha.$$

(d) Now, for every $k_1, k_2 \in \mathbb{R}$

$$\begin{aligned}\langle k_1x_1, y\rangle_\alpha + \langle k_2x_2, y\rangle_\alpha &= \inf\{t \in \mathbb{R} : F(k_1x_1, y, t) * F(k_2x_2, y, t) \geq \alpha\}\\ &= \inf\{t \in \mathbb{R} : k_1 \odot F(x_1, y, t) * k_2 \odot F(x_2, y, t) \geq \alpha\}\\ &= k_1 \odot \langle x_1, y\rangle_\alpha + k_2 \odot \langle x_2, y\rangle_\alpha.\end{aligned}$$

This completes the proof. Q.E.D.

One of important theorems in inner product spaces is parallelogram law. We will generalize this theorem to fuzzy inner product spaces for two and three vectors. For this purpose, at first we present two following Lemmas.

Lemma 3.2. Let $(X, F, *)$ be a FIP-space with a continuous t-norm $*$ satisfying $t * t \geq t$ for all $t \in [0, 1]$. Then, $F(x - y, x - y, t) * F(x + y, x + y, t) = 2 \odot F(x, x, t) * F(y, y, \frac{t}{2})$ for all $x, y \in X$.

Proof. Put $A = F(x - y, x - y, t) * F(x + y, x + y, t)$ and $B = 2 \odot F(x, x, t) * F(y, y, \frac{t}{2})$ whih are in $[0, 1]$. By (FIP-2) and (FIP-4), we have

$$F(-x, -x, t) = 1 - F(x, -x, -t) = 1 - (1 - F(x, x, t)) = F(x, x, t). \tag{3.11}$$

for all $x \in X$. By (FIP-2), (FIP-4), (FIP-5) and (3.11), we get

$$A \leq A * A \leq F(2x, 2x, 2t) * F(2y, 2y, 2t) = F(x, 2x, t) * F(y, 2y, t) = B.$$

On the other hand, by (FIP-5) and (3.11), we have

$$\begin{aligned}B \leq B * B &= [2 \odot F(x, x, t) * F(-y, -y, \frac{t}{2})] * [2 \odot F(x, x, t) * F(y, y, \frac{t}{2})]\\ &= [F(2x, x, t) * F(-y, -y, \frac{t}{2})] * [F(2x, x, t) * F(y, y, \frac{t}{2})]\\ &= [F(x, x, \frac{t}{2}) * F(-y, -y, \frac{t}{2})] * [F(x, x, \frac{t}{2}) * F(y, y, \frac{t}{2})]\\ &\leq F(x - y, x - y, t) * F(x + y, x + y, t) = A.\end{aligned}$$

So, the proof is complete. Q.E.D.

Lemma 3.3. Let $(X, F, *)$ be a FIP-space equipped to a continuous t-norm $*$ satisfying $t * t \geq t$ for all $t \in [0, 1]$. Then

$$\begin{aligned}F(x + y + z, x + y + z, t) &* F(x + y - z, x + y - z, t)\\ &* F(x - y + z, x - y + z, t) * F(x - y - z, x - y - z, t)\\ &= 4 \odot F(x, x, t) * F(y, y, \frac{t}{4}) * F(z, z, \frac{t}{4}).\end{aligned}$$

for all $x, y, z \in X$.

Proof. Take $A = F(x+y+z, x+y+z, t) * F(x+y-z, x+y-z, t) * F(x-y+z, x-y+z, t) * F(x-y-z, x-y-z, t) \in [0,1]$ and $B = 4 \odot F(x,x,t) * F(y,y,\frac{t}{4}) * F(z,z,\frac{t}{4}) \in [0,1]$. By (FIP-2), (FIP-4) and (3.11), for all $x, y, z \in X$,

$$\begin{aligned} A \le A * A \le A * A * A &= [F(x+y+z, x+y+z, t) * F(-x-y+z, -x-y+z, t)] \\ &* [F(x-y+z, x-y+z, t) * F(x+y-z, x+y-z, t)] * [F(x+y+z, x+y+z, t) \\ &* F(-x+y-z, -x+y-z, t)] * [F(x-y-z, x-y-z, t) * F(x+y+z, x+y+z, t)] \\ &* [F(x+y-z, x+y-z, t) * F(-x+y+z, -x+y+z, t)] * [F(x-y+z, x-y+z, t) \\ &* F(-x+y+z, -x+y+z, t)] \le F(2x, 2x, 2t) * F(2y, 2y, 2t) * F(2z, 2z, 2t) * F(2x, 2x, 2t) \\ &* F(2y, 2y, 2t) * F(2z, 2z, 2t) = 4 \odot F(x, 4x, 4t) * F(y, 4y, t) * F(z, 4z, t) = B. \end{aligned}$$

Now, we prove that $B \le A$. Let $t > 0$. By the relation (3.11), Lemma 2.6 and (FIP-5), we get

$$\begin{aligned} B \le B * B \le B * B * B * B &\le F(x+y+z, x+y+z, \frac{3t}{4}) * F(x+y-z, x+y-z, \frac{3t}{4}) \\ &* F(x-y+z, x-y+z, \frac{3t}{4}) * F(x-y-z, x-y-z, \frac{3t}{4}) \le F(x+y+z, x+y+z, t) \\ &* F(x+y-z, x+y-z, t) * F(x-y+z, x-y+z, t) * F(x-y-z, x-y-z, t) = A. \end{aligned}$$

If $t \le 0$, then by assumption, (FIP-1) and Lemma 2.6, we obtain

$$B = F(x, x, \frac{t}{4}) * F(y, y, \frac{t}{4}) * F(z, z, \frac{t}{4}) = 0 * 0 * 0 \le 0 * 0 * 0 * 0 = A.$$

Q.E.D.

Theorem 3.3. (Parallelogram law) Let $(X, F, *)$ be a FIP-space with a continuous t-norm $*$ satisfying $t * t \ge t$ for all $t \in [0,1]$. Suppose that for all $x, y, z, w \in X$,

$$\inf\{t \in \mathbb{R} : F(x,y,t) * F(z,w,t) \ge \alpha\} < \infty, \ \alpha \in (0,1)$$

and $F(x,x,t) > 0$, for all $t > 0$ imply $x = 0$. Then

$$\|x-y\|_\alpha^2 + \|x+y\|_\alpha^2 = 2 \odot (\|x\|_\alpha^2 + \|y\|_\alpha^2).$$

for all $x, y \in X$.

Proof. By Lemma 3.2, we have

$$F(x-y, x-y, t) * F(x+y, x+y, t) = 2 \odot F(x,x,t) * F(y,y,\frac{t}{2}).$$

for all $x, y \in X$ and $\alpha \in (0,1)$. Thus

$$\begin{aligned} \langle x-y, x-y\rangle_\alpha + \langle x+y, x+y\rangle_\alpha &= \inf\{t \in \mathbb{R} : 2 \odot F(x,x,t) * F(y,y,\frac{t}{2}) \ge \alpha\} \\ &= \inf\{t \in \mathbb{R} : 2 \odot F(x,x,t) * 2 \odot F(y,y,t) \ge \alpha\} = 2 \odot (\langle x,x\rangle_\alpha + \langle y,y\rangle_\alpha). \end{aligned}$$

The above relation implies that $\|x-y\|_\alpha^2 + \|x+y\|_\alpha^2 = 2 \odot (\|x\|_\alpha^2 + \|y\|_\alpha^2)$. Q.E.D.

Corollary 3.4. By the assumptions of Theorem *3.3*, we have

$$\|x-y\|_\alpha^2 + \|x+y-2z\|_\alpha^2 = 2 \odot (\|x-z\|_\alpha^2 + \|y-z\|_\alpha^2).$$

for all $x, y, z \in X$.

Proof. Replacing x, y by $x-z$ and $y-z$, respectively in Theorem 3.3, one can obtain the desired result. Q.E.D.

Theorem 3.4. (Generalized Parallelogram law for three vectors) Let $(X, F, *)$ be a FIP-space with a continuous t-norm $*$ satisfying $t * t \geq t$ for all $t \in [0,1]$. Suppose that for each $x_{ij} \in X$ $(i = 1, \cdots, n \text{ and } j = 1, 2)$,

$$\inf\{t \in \mathbb{R} : F(x_{11}, x_{12}, t) * F(x_{21}, x_{22}, t) * \cdots * F(x_{n1}, x_{n2}, t) \geq \alpha\} < \infty,$$

for all $\alpha \in (0,1)$ and $F(x,x,t) * F(x,x,t) * \cdots * F(x,x,t) > 0$, for all $t > 0$ implies $x = 0$. Then, for all $x, y, z \in X$,

$$\|x+y+z\|_\alpha^2 + \|x+y-z\|_\alpha^2 + \|x-y+z\|_\alpha^2 + \|x-y-z\|_\alpha^2 = 4 \odot (\|x\|_\alpha^2 + \|y\|_\alpha^2 + \|z\|_\alpha^2).$$

Proof. By Lemma 3.3, we have

$$\langle x+y+z, x+y+z\rangle_\alpha + \langle x+y-z, x+y-z\rangle_\alpha + \langle x-y+z, x-y+z\rangle_\alpha + \langle x-y-z, x-y-z\rangle_\alpha$$

$$= \inf\{t \in \mathbb{R} : 4 \odot F(x,x,t) * F(y,y,\tfrac{t}{4}) * F(z,z,\tfrac{t}{4}) \geq \alpha\}$$

$$= \inf\{t \in \mathbb{R} : 4 \odot F(x,x,t) * 4 \odot F(y,y,t) * 4 \odot F(z,z,t) \geq \alpha\} = 4 \odot (\langle x,x\rangle_\alpha + \langle y,y\rangle_\alpha + \langle z,z\rangle_\alpha).$$

It follows the above relation that

$$\|x+y+z\|_\alpha^2 + \|x+y-z\|_\alpha^2 + \|x-y+z\|_\alpha^2 + \|x-y-z\|_\alpha^2 = 4 \odot (\|x\|_\alpha^2 + \|y\|_\alpha^2 + \|z\|_\alpha^2).$$

Q.E.D.

4 Fuzzy orthogonality

In this section, we study the properties of fuzzy orthogonality for vectors and subsets according to the definition of inner product in Theorem 3.1.

Theorem 4.1. Let $(X, F, *)$ be a FIP-space satisfying in the hypothesises of Theorems *2.4* and *3.1.* If for each $x, y \in X$, $F(x, y, 0) = 0$, then the following assertions are equivalent.

(i) $\langle x, y\rangle = 0$;

(ii) $x \perp y$;

(iii) $\langle x, y\rangle_\alpha = 0$, for all $\alpha \in (0,1)$;

(iv) $\|x + \beta y\| = \|x - \beta y\|$ for all $\beta \in \mathbb{R}$;

(v) $\|x+y\| = \|x-y\|$;

(vi) $\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\| = \|\frac{x}{\|x\|} - \frac{y}{\|y\|}\| \quad (x, y \neq 0)$;

(vii) $\|x+\beta y\|^2 = \|x\|^2 + \beta^2\|y\|^2$ for all $\beta \in \mathbb{R}$;

(iix) $\|x\| \leq \|x+\beta y\|$ for all $\beta \in \mathbb{R}$.

Also, the assertions (iv)-(iix) are equivalent for $\|\cdot\|_\alpha$ instead of $\|\cdot\|$.

Proof. (i)$\Rightarrow$(ii) We have $0 = \langle x, y\rangle = \sup\{t \in \mathbb{R} : F(x,y,t) < 1\}$. Thus, for all $t > 0$, $F(x,y,t) = 1$. Since $F(x,y,0) = 0$, by Lemma 2.6, $F(x,y,t) = 0$ for all $t \leq 0$. Hence $F(x,y,t) = H(t)$, i.e., $x \perp y$.
(ii)$\Rightarrow$(iii) Let $x \perp y$. Then $F(x,y,t) = H(t)$ for all $t \in \mathbb{R}$. We have

$$\begin{aligned}\langle x, y\rangle_\alpha &= \inf\{t \in \mathbb{R} : \ F(x,y,t) \geq \alpha\} = \inf\{t \in \mathbb{R} : \ H(t) \geq \alpha\} \\ &= \inf\{t \in \mathbb{R} : \ H(t) = 1\} = \inf\{t \in \mathbb{R} : \ t > 0\} = 0.\end{aligned}$$

Thus $\langle x, y\rangle_\alpha = 0$ for all $\alpha \in (0,1)$.
(iii)$\Rightarrow$(i) We note that

$$0 = \langle x, y\rangle_\alpha = \inf\{t \in \mathbb{R} : F(x,y,t) \geq \alpha\} = \inf\{t \in \mathbb{R} : F(x,y,t) = 1\}. \tag{4.1}$$

By Lemma 2.6, the relation $F(x,y,0) = 0$ and equality (4.1), we get

$$F(x,y,t) = \begin{cases} 1 & \text{if } t > 0 \\ 0 & \text{if } t \leq 0. \end{cases}$$

Therefore

$$\langle x, y\rangle = \sup\{t \in \mathbb{R} : F(x,y,t) < 1\} = \sup\{t \in \mathbb{R} : F(x,y,t) = 0\} = \sup\{t \in \mathbb{R} : t \leq 0\} = 0.$$

(i)$\Rightarrow$(iv) Since $\langle x, y\rangle = 0$, we obtain

$$\begin{aligned}\|x+\beta y\|^2 &= \langle x+\beta y, x+\beta y\rangle = \langle x, x\rangle + 2\beta\langle x, y\rangle + \beta^2\langle y, y\rangle = \langle x, x\rangle - 2\beta\langle x, y\rangle + \beta^2\langle y, y\rangle \\ &= \langle x-\beta y, x-\beta y\rangle = \|x-\beta y\|^2 \Rightarrow \|x+\beta y\| = \|x-\beta y\|.\end{aligned}$$

(iv)$\Rightarrow$(v) It is trivial.
(v)$\Rightarrow$(vi) For each $0 \neq x, y \in X$, we have

$$\begin{aligned}&\|x+y\|^2 = \|x-y\|^2 \Leftrightarrow \langle x+y, x+y\rangle = \langle x-y, x-y\rangle \Leftrightarrow 2\langle x, y\rangle = -2\langle x, y\rangle \\ &\Leftrightarrow \frac{2\langle x, y\rangle}{\|x\|\cdot\|y\|} = \frac{-2\langle x, y\rangle}{\|x\|\cdot\|y\|} \Leftrightarrow \frac{\langle x, x\rangle}{\|x\|^2} + \frac{\langle y, y\rangle}{\|y\|^2} + \frac{2\langle x, y\rangle}{\|x\|\cdot\|y\|} = \frac{\langle x, x\rangle}{\|x\|^2} + \frac{\langle y, y\rangle}{\|y\|^2} + \frac{-2\langle x, y\rangle}{\|x\|\cdot\|y\|} \\ &\Leftrightarrow \langle \frac{x}{\|x\|} + \frac{y}{\|y\|}, \frac{x}{\|x\|} + \frac{y}{\|y\|}\rangle = \langle \frac{x}{\|x\|} - \frac{y}{\|y\|}, \frac{x}{\|x\|} - \frac{y}{\|y\|}\rangle \Leftrightarrow \left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\| = \left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\|.\end{aligned}$$

(vi)$\Rightarrow$(i) For any $0 \neq x, y \in X$, we get

$$\left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \right\|^2 = \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\|^2 \Leftrightarrow \langle \frac{x}{\|x\|} + \frac{y}{\|y\|}, \frac{x}{\|x\|} + \frac{y}{\|y\|} \rangle = \langle \frac{x}{\|x\|} - \frac{y}{\|y\|}, \frac{x}{\|x\|} - \frac{y}{\|y\|} \rangle$$
$$\Leftrightarrow \frac{\langle x, x\rangle}{\|x\|^2} + \frac{\langle y, y\rangle}{\|y\|^2} + \frac{2\langle x, y\rangle}{\|x\| \cdot \|y\|} = \frac{\langle x, x\rangle}{\|x\|^2} + \frac{\langle y, y\rangle}{\|y\|^2} + \frac{-2\langle x, y\rangle}{\|x\| \cdot \|y\|} \Leftrightarrow \langle x, y\rangle = 0.$$

(i)$\Rightarrow$(vii) It is obvious.
(vii)$\Rightarrow$(iix) Since $\beta^2\|y\|^2 \geq 0$, the result is clear.
(iix)$\Rightarrow$(i) By assumtion, we have $\|x\|^2 \leq \|x + \beta y\|^2 = \langle x, x\rangle + 2\beta\langle x, y\rangle + \beta^2\langle y, y\rangle$. This necessities $-2\beta\langle x, y\rangle \leq \beta^2\langle y, y\rangle$, and hence

$$\frac{-2}{\beta}\langle x, y\rangle \leq \langle y, y\rangle. \tag{4.2}$$

On the other hand, $\sup\{t \in \mathbb{R} : \ F(x, y, t) \leq 1\} < \infty$. for all $x, y \in X$. Hence, $\langle x, y\rangle < \infty$, and $\langle y, y\rangle < \infty$. Since (4.2) is true for all $\beta \neq 0$ and also $\langle y, y\rangle \geq 0$, we deduce that $\langle x, y\rangle = 0$. Q.E.D.

In the next theorem, we study other properties of the fuzzy orthogonality.

Theorem 4.2. Let $(X, F, *)$ be a FIP-space. Then

(i) If $u \perp v$, then $\alpha u \perp \beta v$. for all $\alpha, \beta \in \mathbb{R}$;

(ii) If $u \perp v$, then $< u > \cap < v >= \{0\}$ where $< u >= \{\alpha u : \ \alpha \in \mathbb{R}\}$.

Proof. (i) We divide the proof of this part to four cases.
Case 1: $\alpha > 0, \ \beta > 0$. In this case, we have

$$F(\alpha u, \beta v, t) = F(u, \beta v, \frac{t}{\alpha}) = F(u, v, \frac{t}{\alpha\beta}) = H(\frac{t}{\alpha\beta}) = H(t).$$

Therefore $\alpha u \perp \beta v$.
Case 2: $\alpha\beta = 0$. It follows from $F(\alpha u, \beta v, t) = H(t)$ that $\alpha u \perp \beta v$.
Case 3: $\alpha\beta < 0$. Without loss of the generality, we assume that $\alpha > 0, \ \beta < 0$. So $F(\alpha u, \beta v, t) = F(u, \beta v, \frac{t}{\alpha}) = 1 - F(u, v, \frac{t}{\alpha\beta})$. Since $u \perp v$, we have

$$F(u, v, \frac{t}{\alpha\beta}) = \begin{cases} 0 & \text{if } t > 0 \\ 1 & \text{if } t < 0 \end{cases} \Rightarrow 1 - F(u, v, \frac{t}{\alpha\beta}) = H(t) \Rightarrow F(\alpha u, \beta v, t) = H(t) \Rightarrow \alpha u \perp \beta v.$$

Case 4: $\alpha < 0, \ \beta < 0$. Thus

$$F(\alpha u, \beta v, t) = 1 - F(u, \beta v, \frac{t}{\alpha}) = 1 - [1 - F(u, v, \frac{t}{\alpha\beta})] = F(u, v, \frac{t}{\alpha\beta}) = H(\frac{t}{\alpha\beta}) = H(t)$$

Therefore, $\alpha u \perp \beta v$ for all $\alpha, \beta \in \mathbb{R}$.

(ii) Let $z \in < u > \cap < v >$. Then, $z \in \{\alpha u : \ \alpha \in \mathbb{R}\} \cap \{\beta v : \ \beta \in \mathbb{R}\}$. So, there exist $\alpha, \beta \in \mathbb{R}$ such that $z = \alpha u$ and $z = \beta v$. By (i), $\alpha u \perp \beta v$ and thus $z \perp z$. Theorem 2.3 implies $z = 0$. Therefore, $< u > \cap < v >= \{0\}$. Q.E.D.

Theorem 4.3. Let $(X, F, *)$ be a FIP-space and M, N be subsets of X. Then

(i) $M \subseteq N \Leftrightarrow N^{\perp} \subset M^{\perp}$;

(ii) $M^{\perp} = (\overline{span} M)^{\perp}$;

(iii) $M^{\perp} = \overline{(M^{\perp})} = (\overline{M})^{\perp}$;

(iv) $\overline{M} = M^{\perp\perp}$;

(v) If M, N are subspaces of X, then $(M+N)^{\perp} = M^{\perp} \cap N^{\perp}$;

(vi) $M^{\perp} = M^{\perp\perp\perp}$.

Proof. (i) Let $M \subseteq N$ and $x \in N^{\perp}$. Then, for each $y \in N$, $x \perp y$. Hence, for each $y \in M$, $x \perp y$. Consequently, $x \in M^{\perp}$. Now, we let $N^{\perp} \subset M^{\perp}$ and $x \in M$. Then for each $y \in M^{\perp}$, $x \perp y$. Thus $x \perp y$, for all $y \in N^{\perp}$. Therefore, $x \in N$.

(ii) Since $M \subseteq \overline{\mathrm{span}} M$, by part (i) , $(\overline{\mathrm{span}} M)^{\perp} \subset M^{\perp}$. Let $x \in M^{\perp}$. By Theorem 2.3 we have $x \perp y+z$ and $x \perp \alpha y$ for all $y, z \in M$ and for all $\alpha \in \mathbb{R}$. This implies that $x \perp \mathrm{span} M$, i.e., $x \in (\mathrm{span} M)^{\perp}$. If $(x_n)_n \in \mathrm{span} M$ such that $x_n \to y$. It follows from Theorem 2.2 that $H(t) = F(x_n, x, t) \to F(y, x, t)$ and so $x \perp y$. Since $y \in \overline{\mathrm{span}} M$, we have $x \in (\overline{\mathrm{span}} M)^{\perp}$. Hence $M^{\perp} \subseteq (\overline{\mathrm{span}} M)^{\perp}$.

(iii) $M^{\perp}$ is closed subspace of X and hence $M^{\perp} = \overline{(M^{\perp})}$. Since $M \subseteq \bar{M}$, by (i) we have $(\overline{M})^{\perp} \subseteq M^{\perp}$. Now, assume that $x \in M^{\perp}$ and $x_n \to y$ in which $(x_n)_N \subset M$. By Theorem 2.2, $H(t) = F(x_n, x, t) \to F(y, x, t)$. Thus $x \perp y$. Since $y \in \bar{M}$, we have $x \in (\overline{M})^{\perp}$. Hence, $M^{\perp} \subset (\overline{M})^{\perp}$.

(iv) Let $x \in \overline{M}$. Then, for each $y \in (\overline{M})^{\perp}$, $x \perp y$. By (iii), $(\overline{M})^{\perp} = M^{\perp}$. Hence, $x \perp y$ for all $y \in M^{\perp}$. Therefore, $\overline{M} \subseteq M^{\perp^{\perp}}$. For another side of inclusion, let $M^{\perp\perp} \not\subset \overline{M}$. Since $\overline{M} \cap (\overline{M})^{\perp} = \{0\}$, $M^{\perp\perp} \not\subset \overline{M}$ and $\overline{M} \subseteq M^{\perp\perp}$, there is $z_1 \neq 0$ such that $z_1 \in (\overline{M})^{\perp} \cap M^{\perp\perp}$. By (iii), there is $z_2 \neq 0$ such that $z_2 \in M^{\perp} \cap M^{\perp\perp}$ which contradicts $M^{\perp} \cap M^{\perp\perp} = \{0\}$.

(v) Let $x \in M^{\perp} \cap N^{\perp}$. Then, $x \in M^{\perp}$ and $x \in N^{\perp}$. Thus, $x \perp M$ and $x \perp N$. So, $x \perp m$ and $x \perp n$ for all $m \in M, n \in N$. By Theorem 2.3, $x \perp m+n$ for all $m \in M$ and all $n \in N$. Hence, $M^{\perp} \cap N^{\perp} \subseteq (M+N)^{\perp}$. Now, if $x \in (M+N)^{\perp}$, then $x \perp M+N = \{m+n : m \in M$ and $n \in N\}$. Since M, N be subspaces of X then $0 \in M$, $0 \in N$. Consequently, by placing $m = 0$, we have $x \perp \{m : m \in M\}$. Similarly $x \perp \{n : n \in N\}$. This means that $x \in M^{\perp} \cap N^{\perp}$.

(vi) Since $\overline{M} = M^{\perp\perp}$, we have $(\overline{M})^{\perp} = M^{\perp\perp\perp}$. By (iii), it follows that $M^{\perp} = M^{\perp\perp\perp}$. Q.E.D.

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